30 Years of Newest Vertex Bisection

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Abstract. One aspect of adaptive mesh refinement in the finite element method for solving partial differential equations is the method by which elements are refined. In the early 1980's the dominant method for refining triangles was the red-green algorithm of Bank and Sherman. The red refinements are the desired refinements, but will result in an incompatible grid when used alone. The green refinements are used to recover compatibility for stability of the finite element discretization, and are removed before the next adaptive step. Prof. Bob Skeel raised the question as to whether it is possible to perform adaptive refinement of triangles without this complicated patching/unpatching process. As a result, a new triangle refinement method, called newest vertex bisection, was devised as an alternative to red-green refinement in the mid 1980's. The new approach is simpler and maintains compatibility of the grid at all times, avoiding the patching/unpatching of the green refinement. We review the development of the newest vertex bisection method for adaptive refinement, and subsequent extensions of the method.

Keywords: finite elements, adaptive mesh refinement, newest vertex bisection
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NEWEST VERTEX BISECTION OF TRIANGLES

The numerical solution of partial differential equations is the most computationally intense part of many scientific and engineering applications. Consequently, for many years much research has been devoted to improving the speed and accuracy of the algorithms used for this purpose. In the early 1980's much attention was placed on adaptive mesh refinement in the finite element method. The promise of adaptive mesh refinement was to reduce the number of vertices in the grid, and consequently the size of the linear systems to be solved, by using small elements only in the areas where the solution is changing rapidly, and large elements where the solution is fairly constant.

One approach to adaptive refinement of triangles at that time was the so-called red-green refinement of Bank and Sherman [1]. Here a red refinement is a quadrisection of a triangle by connecting the midpoints of the sides, and a green refinement is a bisection by connecting a vertex to the midpoint of the opposite side (see Figure 1). After determining which elements should be refined, those elements are refined via red refinement, most likely resulting in an incompatible grid with hanging nodes, i.e. vertices in the middle of a side of a triangle. Green refinements are then performed to remove the hanging nodes, resulting in a compatible grid. The partial differential equation is discretized with the finite element method on this grid, and an approximate solution is computed. Then the green refinements are removed and the whole process is repeated until the solution is sufficiently accurate.

When the author began his Ph.D. research under the direction of Prof. Bob Skeel at the University of Illinois Urbana-Champaign in 1985, Prof. Skeel asked the question of whether it would be possible to avoid the green refinement/unrefinement in an adaptive mesh refinement algorithm. The ultimate result of this question was the newest vertex bisection method [2, 3]. In this method only bisection refinements are used, and the vertex selected for the bisection and the order in which triangles are bisected are very carefully controlled. When a triangle is bisected, a new vertex is created in the middle of one of its sides. In the two child triangles this vertex is called the newest vertex or peak. If the children are refined, then the bisection is performed from the newest vertex to the opposite side, which is called the base (see Figure 2).

If performed naively, this bisection would result in hanging nodes and further refinements are needed to make the grid compatible. Instead, refinements are performed recursively and in pairs so that the compatibility is *always*

[1] Dedicated to Bob Skeel on the occasion of his career on Numerical Analysis and Applied Mathematics.

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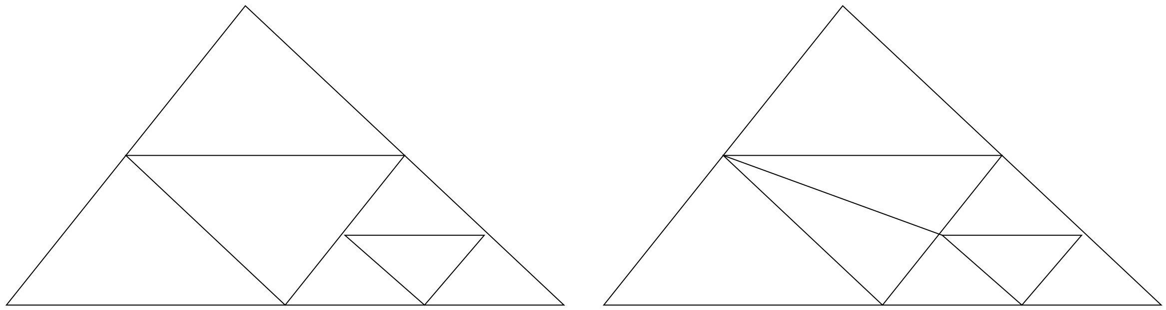

FIGURE 1. Red refinement of a triangle (left) resulting in a hanging node, and green refinement (right) to enforce compatibility.

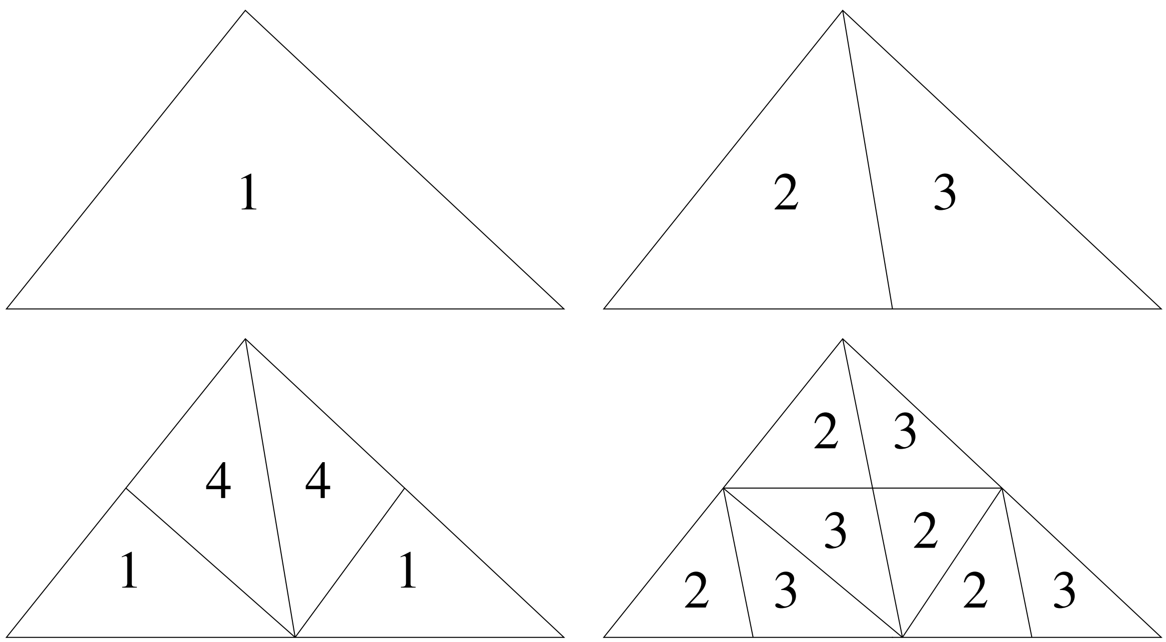

FIGURE 2. Three levels of newest vertex bisection showing the four similarity classes of resulting triangles.

maintained in the grid. A triangle is called compatibly divisible if its base is either the base of the triangle that shares that side or part of the boundary of the domain. If a triangle is compatibly divisible then the triangle and the triangle that shares the base (if it exists), called the mate, are refined simultaneously as a pair. If a triangle is not compatibly divisible, then it is easily seen that after a single refinement of the triangle opposite the peak, it will be. So in this case we recursively refine the neighbor, and then refine the desired triangle and its mate. This recursion is bounded by the generation of the triangle, as each triangle in the recursion chain is of one generation earlier. It was proven in [2] that it is always possible to select peaks in the initial grid such that every triangle is compatibly divisible. A practical algorithm to make that selection was given by Biedl et al. [4] in 2001. Sewell [5] showed that there are only four similarity classes of triangles generated by newest vertex bisection, as shown in Figure 2, and therefore the angles are bounded away from 0 and π.

BISECTION OF TETRAHEDRA AND n-SIMPLICES

Several researchers have extended the newest vertex bisection of triangles to higher dimensions, i.e. the tetrahedron in $\mathbb{R}^3$ and the n-simplex in $\mathbb{R}^n$. In dimensions greater than 2, selection of the newest vertex does not uniquely define the bisection of the simplex. Instead, one usually designates a refinement edge. Some authors use this same point of view with triangles and, in the case of newest vertex bisection, define the refinement edge to be the edge opposite the newest vertex.

In 1984, Rivara [6] considered the bisection of n-simplices, but then presented details only for the triangle. Her term "generalized bisection" refers to choosing the longest edge as the refinement edge. The details for tetrahedra were given in 1992 [7] where they include an empirical study of the minimum angle, but no proof that it is bounded away from 0.

Bänsch [8] presented a tetrahedron bisection method in 1991 that is more in the spirit of newest vertex bisection. In 1994, Kossaczky [9] presented a recursive algorithm that is very similar to newest vertex bisection. This algorithm has some restrictions on the initial mesh.

In 1995 Liu and Joe [10] presented an algorithm that maps an initial tetrahedron to a canonical tetrahedron, performs longest edge bisection in the canonical tetrahedron, and maps the result back to the initial tetrahedron, which is not the same as longest edge bisection in the initial tetrahedron. They prove that a mesh quality metric is bounded away from 0 and there is a finite number of similarity classes.

Also in 1995, Maubach [11] gave a bisection algorithm for n-simplices that is based on newest vertex bisection of triangles. It requires that the order of the vertices in an n-simplex satisfy a particular property with respect to the vertices of the neighboring simplices, and proves that certain grids satisfy that property. He also proves there is a finite number of similarity classes. Two years later, Traxler [12] presented an algorithm for the n-simplex that generates the same tetrahedra as Maubach, but uses a different ordering of the vertices. This method preserves a certain "structural condition" which simplifies and speeds up the algorithm.

Finally in 2000, Arnold, Mukherjee and Pouly [13] presented another tetrahedron bisection algorithm. They show the method is essentially equivalent to those of Bänsch and Liu and Joe, but presented in a simpler way. They also establish the relationship to Maubach's method. They prove there is a finite number of similarity classes, and establish a bound on the number of steps needed to regain compatibility.

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