Simulation of radiographs

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The resulting intensity of an X-ray of initial intensity $I_0$ passing along line $L$ through a body with linear attenuation coefficient $\mu(x)$ is given by

$$I = I_0 \exp \left\{ - \int_L \mu(x) \, dx \right\}$$

If the body is homogeneous (i.e., $\mu(x)$ is constant inside the body, 0 outside), then this reduces to

$$I = I_0 \exp \left\{ - \mu \|L \cap \text{Body}\| \right\}$$

In particular, we will call $\mu \|L \cap \text{Body}\|$ the **linear attenuation** due to the body.
T-joint with trace of X-ray path.
Length of intersection of line $L$ with circle centered at the origin with radius $R$ is

$$\Phi(L_{\theta, w}) = 2\sqrt{R^2 - w^2}$$
Parameterize line $L$ by

$$(x_1(s), x_2(s)) = L \vec{v}, \vec{w}(s) = s \vec{v} + \vec{w}$$

where $\vec{v}$, $\vec{w}$ satisfy $\|\vec{v}\| = 1$ and $\langle \vec{v}, \vec{w} \rangle = 0$.

We then have the following 4 conditions on $s$:

$$-e_1 \leq x_1(s) = sv_1 + w_1 \leq e_1$$
$$-e_2 \leq x_2(s) = sv_2 + w_2 \leq e_2$$
Solving yields

\[ s_{\text{min}} = \max \left[ \min \left( \frac{\pm e_1 - w_1}{v_1} \right), \min \left( \frac{\pm e_2 - w_2}{v_2} \right) \right] \]

\[ s_{\text{max}} = \min \left[ \max \left( \frac{\pm e_1 - w_1}{v_1} \right), \max \left( \frac{\pm e_2 - w_2}{v_2} \right) \right] \]

The length of the intersection of \( L \) with the rectangle is

\[ \Phi(L_{\vec{v}, \vec{w}}) = \max(s_{\text{max}} - s_{\text{min}}, 0) \]
Illustration of intersection of line $L$ with a rotated and translated rectangle.
Let $U$ be the orthogonal transformation given by

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and let $T$ be the rigid body motion

$$T\vec{w} = U(\vec{w} - \vec{P})$$

Then the line $L_{\vec{v},\vec{w}}$ becomes in the new coordinate system $L_{\vec{v}',\vec{w}'}$, where

$$\vec{v}' = U\vec{v}$$

$$\vec{w}' = T\vec{w} - \langle T\vec{w}, \vec{v}' \rangle \vec{v}'$$
BASE ELEMENT TYPES:

Sphere
Ellipsoid*
Cylinder
Elliptical Cylinder*
Box*
Free Form*
Erehwon

where * indicates those element types supporting cut-planes.
Ellipsoid:

\[
(x_1/e_1)^2 + (x_2/e_2)^2 + (x_3/e_3)^2 \leq 1
\]
Let
\[ \vec{v'} = (v_1/e_1, v_2/e_2, v_3/e_3)^t \]
\[ \vec{w'} = (w_1/e_1, w_2/e_2, w_3/e_3)^t \]

And further define
\[ a = \langle \vec{v'}, \vec{v'} \rangle \]
\[ b = \langle \vec{v'}, \vec{w'} \rangle \]
\[ c = \langle \vec{w'}, \vec{w'} \rangle - 1 \]
\[ \Delta = \sqrt{b^2 - ac} \]

Then the line-ellipsoid intersection points are at
\[ s_1 = \frac{-b - \Delta}{a} \quad s_2 = \frac{-b + \Delta}{a} \]
/******************************* ELLIPSOID *******************************
float ellipsoid(ELEMENT *ell, LINE *ray)
    /* This routine returns the length of the intersection between */
    /* the ellipsoid ell and the line *ray. The object ell definition */
    /* includes the size, density and orientation of the ellipsoid. */
    {
        int i;
        float eff_length,a,b,c,det,t1,t2;
        LINE newray;

        /* Crude out-of-bounds check */
        for(i=0; i<3; i++) if(fabs(ray->offset[i])<=ell->param[i]) break;
        if(i>2) return 0.0;

        /* Convert to elliptical coordinates */
        for(i=0; i<3; i++) {
            newray.dir[i]=ray->dir[i]/ell->param[i];
            newray.offset[i]=ray->offset[i]/ell->param[i];
        }

        a=dot(newray.dir,newray.dir);
        b=dot(newray.dir,newray.offset);
        c=dot(newray.offset,newray.offset)-1.0;

        /* Compute crossing times */
        det=b*b-a*c;
        if(det<TOO_SMALL*TOO_SMALL) return 0.0; /* No intersection! */
        det=sqrt(det);
        t1=(-b-det)/a; t2=(-b+det)/a;

        /* Incorporate "cut plane" restrictions */
        plane_limits(&t1,&t2,ell,ray);

        /* Compute total length, including density */
        if(t2<t1) eff_length=0;
        else eff_length=(t2-t1)*ell->density;
    return eff_length;
    }

Subroutine for calculation of ellipsoid linear attenuation.
Simulated radiograph of an ellipsoid with cut-plane.
ELLIPSOID
1 0 0
0 0.707107 -0.707107
0 0.707107 0.707107
0 100 0
.3
500 100 50
0 0.707107 0.707107 50 1

Data file for ellipsoid with 1 cut-plane.
Triangular prism constructed as a “free form” element.
Simulated radiograph of triangular prism with parallel beam geometry.
Simulated radiograph of triangular prism with cone beam geometry.
Flowchart for radiograph simulation package.
Simulated radiograph of a pipe with internal spherical and cylindrical inclusions.
Data file for pipe simulation with inclusions.
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<td>0 0 1</td>
<td>0 75 0</td>
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</table>

Data file for simulation of experimental sample.
Simulated projections: $0^\circ$, $45^\circ$, $90^\circ$. 
Experimental projections: $0^\circ$, $45^\circ$, $90^\circ$. 
Reconstruction from simulated data, 65 projections.
Reconstruction from simulated data, 315 projections.
Reconstruction from experimental data, 315 projections.
Simulated T-joint, top view.
Data file for T-joint simulation, Page 1/2.
ELLIPTICAL CYLINDER
1.0 0.0 0.0
0.0 1.0 0.0
0.0 0.0 1.0
64 -64 0
0.0075
16 16 100
1 0 0 0.5 -1
0 -1 0 0 -1

BOX
1.0 0.0 0.0
0.0 1.0 0.0
0.0 0.0 1.0
64.5 0 0
-0.0075
2 10 100

ELLIPTICAL CYLINDER
1.0 0.0 0.0
0.0 1.0 0.0
0.0 0.0 1.0
72 -72 0
-0.0075
3 2 100

Data file for T-joint simulation, Page 2/2.
Projections at 45° increments.
Projections at 2.3° increments.
T-joint reconstruction from simulated data, 315 projections.
Reconstruction from simulated data, 20° missing angle.
Reconstruction from simulated data, 20° missing angle, lower 20% of frequency range filled with ideal data.