REGTET: A Program for Computing Regular Tetrahedralizations

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Abstract

REGTET, a Fortran 77 program for computing a regular tetrahedralization for a finite set of weighted points in 3−dimensional space, is discussed. REGTET is based on an algorithm by Edelsbrunner and Shah for constructing regular tetrahedralizations with incremental topological flipping. At the start of the execution of REGTET a regular tetrahedralization for the vertices of an artificial cube that contains the weighted points is constructed. Throughout the execution the vertices of this cube are treated in the proper lexicographical manner so that the final tetrahedralization is correct.

1 Introduction

Let S be a finite set of points in 3−dimensional space ($\mathcal{R}^3$). By a *tetrahedralization* T for S we mean a finite collection of tetrahedra (3-dimensional triangles) with vertices in S, that satisfies the following two conditions.

1. Two distinct tetrahedra in T that are not disjoint, intersect at a common facet, a common edge, or a common vertex.

2. The union of the tetrahedra in T equals the convex hull of S.

For each point p in S let w_p be a real-valued weight assigned to p. Given p in S and a point x in $\mathcal{R}^3$, the *power distance* of x from p, denoted by $\pi_p(x)$, is defined by

$$\pi_p(x) \equiv |xp|^2 - w_p,$$

where $|xp|$ is the Euclidean distance between x and p. Given a tetrahedron t with vertices in S, a point, denoted by $z(t)$, exists in $\mathcal{R}^3$ with the same power distance, denoted by $w(t)$, from all vertices of t. Point $z(t)$ is called the *orthogonal center* of t. Given a tetrahedralization T for S, we then say that T is a *regular tetrahedralization* for S if for each tetrahedron t in T and each point p in S, $\pi_p(z(t)) \geq w(t)$. We observe that T is unique if for each tetrahedron t in T and each point p in S that is not a vertex of t, $\pi_p(z(t)) > w(t)$. If T is unique then the *power diagram* of S [1] is the dual of T. Finally, we observe that if the weights of the points in S are all equal then the power diagram of S is identical to the *Voronoi diagram* of S [10], and the regular and *Delaunay* [4] tetrahedralizations for S coincide.

In this paper we discuss REGTET, a Fortran 77 program for computing regular tetrahedralizations (or Delaunay tetrahedralizations in the absence of weights) with incremental topological flipping [6] and lexicographical manipulations [3]. A copy of REGTET that includes instructions for its execution can be obtained from `http://math.nist.gov/~JBernal`.

2 Topological Flipping

Let T be a tetrahedralization for S, let t be a tetrahedron in T, and let p a point in S that is not a vertex of t. Denote the vertices of t by q_1, q_2, q_3, q_4, and let T_1 and T_2 be the only two possible tetrahedralizations for $\{q_1, q_2, q_3, q_4, p\}$ [9]. Assume t is in T_1, and T_1 is contained in T. A *topological flip* or simply a *flip* on T_1 is an operation that replaces T_1 with T_2 in T.

For each j, $j = 1, \ldots, 4$, denote by f_j the facet of t that does not contain q_j, and by H_j the plane in $\mathcal{R}^3$ that contains f_j. For each j, $j = 1, \ldots, 4$, denote by H_j^+ the open half-space in $\mathcal{R}^3$ determined by H_j that contains q_j, and by H_j^- the open half-space in $\mathcal{R}^3$ determined by H_j that does not contain q_j. Clearly it is by ascertaining which of H_j, H_j^+, H_j^- contains p for each j, $j = 1, \ldots, 4$, that one can identify the tetrahedralizations T_1 and T_2. Accordingly, the following nine configurations of T_1 and T_2 are possible, each configuration depending on which of H_j, H_j^+, H_j^- contains p for each j, $j = 1, \ldots, 4$.

Configuration 1 (possible '1 to 4' flip): p is in $\cap_{j=1}^4 H_j^+$. Denote by t_1, t_2,

t_3, and t_4 the tetrahedra whose vertex sets are $\{q_1, q_2, q_3, p\}$, $\{q_1, q_2, q_4, p\}$, $\{q_1, q_3, q_4, p\}$, and $\{q_2, q_3, q_4, p\}$, respectively. It then follows that T_1 consists exactly of t, and T_2 of t_1, t_2, t_3, and t_4.

Configuration 2 (possible '1 to 3' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H_{j_1} \cap H^+_{j_2} \cap H^+_{j_3} \cap H^+_{j_4}$. Denote by t_1, t_2, and t_3 the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, and $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists exactly of t, and T_2 of t_1, t_2, and t_3.

Configuration 3 (possible '1 to 2' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H_{j_1} \cap H_{j_2} \cap H^+_{j_3} \cap H^+_{j_4}$. Denote by t_1 and t_2 the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$ and $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists exactly of t, and T_2 of t_1 and t_2.

Configuration 4 (possible '2 to 3' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H^-_{j_1} \cap H^+_{j_2} \cap H^+_{j_3} \cap H^+_{j_4}$. Denote by t_1, t_2, t_3, and t' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, and $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t and t', and T_2 of t_1, t_2, and t_3.

Configuration 5 (possible '3 to 2' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H^-_{j_1} \cap H^-_{j_2} \cap H^+_{j_3} \cap H^+_{j_4}$. Denote by t_1, t_2, t', and t'' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t, t', and t'', and T_2 of t_1 and t_2.

Configuration 6 (possible '2 to 2' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H^-_{j_1} \cap H_{j_2} \cap H^+_{j_3} \cap H^+_{j_4}$. Denote by t_1, t_2, and t' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, and $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t and t', and T_2 of t_1 and t_2.

Configuration 7 (possible '4 to 1' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H^-_{j_1} \cap H^-_{j_2} \cap H^-_{j_3} \cap H^+_{j_4}$. Denote by t_1, t', t'', and t''' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, and $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t, t', t'', and t''', and T_2 exactly of t_1.

Configuration 8 (possible '3 to 1' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H^-_{j_1} \cap H^-_{j_2} \cap H_{j_3} \cap H^+_{j_4}$. Denote by t_1, t', and t'' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, and $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t, t', and t'', and T_2 exactly of t_1.

Configuration 9 (possible '2 to 1' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H_{j_1}^- \cap H_{j_2} \cap H_{j_3} \cap H_{j_4}^+$. Denote by t_1 and t' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, and $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t and t', and T_2 exactly of t_1.

3 Lexicographical Manipulations

Program REGTET which is based on an algorithm by Edelsbrunner and Shah [6] computes a regular tetrahedralization for the set S by adding the points in S one at a time into a regular tetrahedralization for the set of previously added points. This implies that before any points in S are added a regular tetrahedralization must be first constructed by REGTET with vertices close to infinity and underlying space equal to $\mathcal{R}^3$. The vertices of this initial tetrahedralization are said to be *artificial*. Throughout the execution of the program artificial points must be treated in the proper lexicographical manner so that the final tetrahedralization does contain a tetrahedralization for S, and this tetrahedralization for S is indeed regular (since the coordinates of the artificial points can be extremely large in absolute value, it is inadvisable to identify them, thus the need to treat artificial points in a lexicographical manner).

Lexicographical manipulations that are employed in REGTET are described below and justified in [3]. At the start of the execution of the implementation a $3-$dimensional cube with vertices close to infinity that contains S in its interior is identified, and a regular tetrahedralization for the set of vertices of the cube (weights set to the same number) is computed. The execution then proceeds with the incremental insertion of points in S as suggested by Edelsbrunner and Shah. However, at all times, because of the lexicographical manipulations employed in the presence of artificial points (the vertices of the cube), the artificial points are assumed to be as close to infinity as the manipulations require.

The lexicographical manipulations are divided in two groups [3]. The first group consists of manipulations for determining the location of a point in S with respect to a facet of a tetrahedron. The second group consists of manipulations for determining which of the only two possible tetrahedralizations for a set of five points is regular. These manipulations are described below.

4 Artificial Points

In what follows we formally define the artificial points as they appear in REGTET.

For each point p in S let w_p be a real valued weight assigned to p. Define real numbers $xmin$, $xmax$, $ymin$, $ymax$, $zmin$, $zmax$ by

$$\begin{aligned}
xmin &\equiv \min\{x : \exists\ y,\ z,\ (x,y,z) \in S\}, \\
xmax &\equiv \max\{x : \exists\ y,\ z,\ (x,y,z) \in S\}, \\
ymin &\equiv \min\{y : \exists\ x,\ z,\ (x,y,z) \in S\}, \\
ymax &\equiv \max\{y : \exists\ x,\ z,\ (x,y,z) \in S\}, \\
zmin &\equiv \min\{z : \exists\ x,\ y,\ (x,y,z) \in S\}, \\
zmax &\equiv \max\{z : \exists\ x,\ y,\ (x,y,z) \in S\}.
\end{aligned}$$

Define a real number $wmin$ by

$$wmin \equiv \min\{w_p :\ p \in S\},$$

real numbers $xctr$, $yctr$,$zctr$ by

$$\begin{aligned}
xctr &\equiv (xmax + xmin)/2, \\
yctr &\equiv (ymax + ymin)/2, \\
zctr &\equiv (zmax + zmin)/2,
\end{aligned}$$

a point $\overline{p}$ in $\mathcal{R}^3$ by

$$\bar{p} \equiv (xctr, yctr, zctr),$$

and finally vectors e_i, $i = 1, \ldots, 8$, by

$$e_1 \equiv (-1, -1,\ 1),$$

$$e_2 \equiv (-1,\ \ 1,\ \ 1),$$

$$e_3 \equiv (\ \ 1,\ \ 1,\ \ 1),$$

$$e_4 \equiv (\ \ 1, -1,\ \ 1),$$

$$\begin{aligned} e_5 &\equiv (-1,-1,-1),\\ e_6 &\equiv (-1,\ \ 1,-1),\\ e_7 &\equiv (\ \ 1,\ \ 1,-1),\\ e_8 &\equiv (\ \ 1,-1,-1). \end{aligned}$$

For any positive real number μ, define the vertices $p_{i\mu}$, $i = 1, \ldots, 8$, of a cube R_μ by

$$p_{i\mu} \equiv \bar{p} + \mu e_i,\ \ i = 1, \ldots, 8.$$

For arbitrarily large μ, R_μ contains S in its interior. Given a positive real number μ, the points $p_{i\mu}$, $i = 1, \ldots, 8$, are the artificial points, and μ is assumed to be as large as the lexicographical manipulations require. In order to be consistent, given a positive real number μ, a number w, $w < wmin$, is selected and assigned as a weight to each of the points $p_{i\mu}$, $i = 1, \ldots, 8$. Since the points $p_{i\mu}$, $i = 1, \ldots, 8$, are the vertices of a cube, it follows that any tetrahedralization for these points is regular. In addition, one such tetrahedralization is not difficult to compute.

5 Redundant Points

For each point q in S let w_q be a real valued weight assigned to q. Without any loss of generality assume that the artificial points are in S. Let p be a point in S that is not artificial, let T be a regular tetrahedralization for $S \setminus \{p\}$, and let t be a tetrahedron in T that contains p. Clearly T is also a tetrahedralization for S although not necessarily regular. Let T_1 and T_2 be the tetrahedralizations as defined in Section 2 with respect to t and p. Clearly T_1 consists exactly of the tetrahedron t so that it is contained in T. If T_2 is not regular it then follows that no regular tetrahedralization for S can have tetrahedra with p as a vertex [6]. Therefore, under this condition, T is also a regular tetrahedralization for S, and the point p is then said to be *redundant in* S.

As mentioned above, REGTET constructs a regular tetrahedralization for the set S by adding the points in S one at a time into a regular tetrahedralization for the set of previously added points. This technique is a generalization of a result for computing incrementally Delaunay triangulations in $\mathcal{R}^2$ [7]. Let p be a point in S and assume that p is a new point that is to be added by REGTET into a regular tetrahedralization T' of the previously added points S'. As p is added, it is first determined whether p is redundant in $S' \cup \{p\}$. If it is then T' is also a regular tetrahedralization for $S' \cup \{p\}$. Otherwise a regular tetrahedralization for $S' \cup \{p\}$ is obtained from T', and points in S' that are redundant in $S' \cup \{p\}$ but not in S' are identified. This is accomplished by REGTET through a finite number of steps, each step involving a decision about whether a certain flip should take place and if so applying the flip. Clearly points found to be redundant in $S' \cup \{p\}$ will continue to be redundant as the rest of the points in S are added.

The first step carried out by REGTET for obtaining a regular tetrahedralization for $S' \cup \{p\}$ from T' involves the determination of whether the point p is redundant in $S' \cup \{p\}$ and if it is not the computation from T' of an initial tetrahedralization for $S' \cup \{p\}$ with p as a vertex of some of its tetrahedra. Let t be a tetrahedron in T' that contains p (the process for identifying t is described below), let T_1 and T_2 be the tetrahedralizations as defined in Section 2 with respect to t and p, and for some integer k, $1 \leq k \leq 9$, let Configuration k be the configuration for T_1 and T_2 (Section 2). Since p is in t it then follows that k can not be larger than 3. REGTET determines the value of k and whether T_2 is regular. If T_2 is not regular, i. e. $\pi_p(z(t)) > w(t)$, then p is marked as being redundant and T' is identified as a regular tetrahedralization for $S' \cup \{p\}$. Otherwise for some positive integer m the tetrahedra $\hat{t}_j$, $j = 1, \ldots, m$, in T' that contain p are identified. Clearly t is one of them, and the value of m depends on that of k (1 if k equals 1, 2 if k equals 2, and greater than or equal to 3 if k equals 3). For each j, $j = 1, \ldots, m$, REGTET then identifies the tetrahedralizations T_1 and T_2 as defined in Section 2 with respect to $\hat{t}_j$ and p, and applies the flip corresponding to Configuration k (Section 2) that replaces T_1 with T_2 in T' (for each j, $j = 1, \ldots, m$, the configuration for T_1 and T_2 is always Configuration k). An initial tetrahedralization not necessarily regular for $S' \cup \{p\}$ with p as a vertex for some of its tetrahedra results.

As just described if p is not redundant in $S' \cup \{p\}$, REGTET first computes from T' a tetrahedralization for $S' \cup \{p\}$ with p as a vertex for some of its tetrahedra. If the new tetrahedralization is not regular other steps follow for the purpose of eventually obtaining one that is. It is through this process that points in S' that are not redundant in S' but that are redundant in

$S' \cup \{p\}$ are identified. The process which involves the flips associated with Configuration 4 through Configuration 9 (Section 2) is described below.

6 Locally Regular Tetrahedra

Let T be a tetrahedralization for S. Given a tetrahedron t in T we denote by $N(t)$ the set of points in $S \setminus t$ that are vertices of tetrahedra in T sharing a facet with t. We then say that t is *locally regular* if for each point q in $N(t)$, $\pi_q(z(t)) \geq w(t)$. By extending results for Delaunay triangulations and tetrahedralizations [8], [9], Edelsbrunner and Shah [6] have proven that if the vertex set of T contains all non-redundant points in S and every tetrahedron in T is locally regular it then follows that T is a regular tetrahedralization for S.

Let p be a point in S that is being added by REGTET into a regular tetrahedralization T' of the previously added points S'. Assume that it has been determined by REGTET that p is not redundant in $S' \cup \{p\}$ and that the program has computed as described above an initial tetrahedralization for $S' \cup \{p\}$ with p as a vertex for some of its tetrahedra. For some positive integer m, REGTET then identifies the tetrahedra $\hat{t}_j$, $j = 1, \ldots, m$, in the initial tetrahedralization with p as a vertex. REGTET then proceeds to transform this initial tetrahedralization through an iterative procedure as follows. For j, $j = 1, \ldots, m+1$, if j equals $m+1$ the procedure terminates. Otherwise REGTET determines whether $\hat{t}_j$ is in the current tetrahedralization (not necessarily equal to the initial tetrahedralization). If it is not then REGTET proceeds to the next value of j. Otherwise REGTET determines whether a tetrahedron t exists in the current tetrahedralization that shares with $\hat{t}_j$ a facet that does not contain p. If it does not then $\hat{t}_j$ is locally regular and REGTET proceeds to the next value of j. Otherwise REGTET determines whether $\pi_p(z(t)) \geq w(t)$. If the inequality holds then $\hat{t}_j$ is locally regular and REGTET proceeds to the next value of j. Otherwise REGTET identifies tetrahedralization T_1 as defined in Section 2 with respect to t and p and determines whether it is contained in the current tetrahedralization. If it is not then REGTET proceeds to the next value of j. Otherwise REGTET identifies tetrahedralization T_2 as defined in Section 2 with respect to t and p, and determines the value of the integer k, $1 \leq k \leq 9$, for which Configuration k is the configuration for T_1 and T_2 (Section 2). Since p is not in t it then follows that k must be larger than 3. REGTET then applies the flip corresponding to Configuration k that replaces T_1 with T_2 in the current tetrahedralization, and marks the tetrahedra in T_1 as not being in the current tetrahedralization (after certain flips the current tetrahedralization may not satisfy the first condition in the definition of a tetrahedralization, however at the end of the iterative procedure the final tetrahedralization will satisfy it). If k is larger than 7 then REGTET identifies the point in S' that is a vertex of both $\hat{t}_j$ and t but not of the one tetrahedron in T_2 and marks this point as being redundant. For some positive integer m', $m' > m$, REGTET then identifies tetrahedra $\hat{t}_j$, $j = m+1, \ldots, m'$, in the current tetrahedralization which are exactly the tetrahedra in T_2 (p is a vertex of each one of these tetrahedra), replaces the value of m by that of m', and proceeds to the next value of j. Clearly when the procedure terminates it then follows that every tetrahedron in the current tetrahedralization with p as a vertex is locally regular [6]. Since all other tetrahedra in the current tetrahedralization are in T', they must also be locally regular. Thus the current tetrahedralization is regular for $S' \cup \{p\}$.

7 Point Location Determination

For arbitrarily large μ, $\mu > 0$, let S' be a proper subset of S that contains the artificial points $p_{i\mu}$ ($\equiv \bar{p} + \mu e_i$), $i = 1, \ldots, 8$, (Section 4), and let T' be a regular tetrahedralization for S'. Given a point p in $S \setminus S'$ and a tetrahedron t in T', we present direct computations and lexicographical manipulations used in REGTET for determining the location of p relative to any given facet of t. This capability allows REGTET to identify the tetrahedralizations T_1 and T_2 as defined in Section 2 with respect to t and p. We do this by cases, each case depending on the number of artificial vertices of the facet of t under consideration. We assume without any loss of generality that S' contains at least one point in S that is not artificial. It then follows that if the vertices of either an edge or a facet of a tetrahedron in T' are all artificial then the edge or facet must be contained in its entirety in the boundary of the cube R_μ (Section 4). In addition, no tetrahedron in T' exists whose vertices are all artificial.

Denote the vertices of t by q_1, q_2, q_3, and q_4, and without any loss of generality assume that the facet under consideration is the facet with vertices q_1, q_2, and q_3. We define a vector v by $v \equiv (q_1 - q_3) \times (q_2 - q_3)$, i. e. the cross product of vectors $(q_1 - q_3)$ and $(q_2 - q_3)$, and assume that q_1, q_2, q_3

are ordered in such a way that $v \cdot (q_4 - q_3)$, i. e. the inner product of v and $(q_4 - q_3)$, is positive. Clearly, the location of p relative to the facet depends on the sign of $v \cdot (p - q_3)$. The solution by cases to the *point location determination problem*, i. e. the problem of determining the sign of $v \cdot (p - q_3)$, follows. This solution is justified in [3].

Case 1: None of q_1, q_2, q_3 is artificial. The sign can then be determined through direct computations of v, $p - q_3$, and $v \cdot (p - q_3)$.

Case 2: Exactly one of q_1, q_2, q_3 is artificial. Without any loss of generality we asssume the one point is q_1 so that q_2 and q_3 are not artificial. Let k be an integer, $1 \leq k \leq 8$, so that q_1 equals $p_{k\mu}$.
Define numbers d_0, d_1, as follows:

$$d_0 \equiv ((\bar{p} - q_3) \times (q_2 - q_3)) \cdot (p - q_3).$$

$$d_1 \equiv (e_k \times (q_2 - q_3)) \cdot (p - q_3).$$

If d_1 is non-zero then the sign is that of d_1.
Else, if d_1 is zero then it is that of d_0.

Case 3: Exactly two of q_1, q_2, q_3 are artificial. Without any loss of generality we asssume the two points are q_1 and q_2 so that q_3 is not artificial. Let k and l be integers, $1 \leq k,\ l \leq 8$, so that q_1 equals $p_{k\mu}$ and q_2 equals $p_{l\mu}$.
Define numbers d_1, d_2, as follows:

$$\begin{aligned} d_1 &\equiv ((\bar{p} - q_3) \times (e_l - e_k)) \cdot (p - q_3). \\ d_2 &\equiv (e_k \times e_l) \cdot (p - q_3). \end{aligned}$$

If d_2 is non-zero then the sign is that of d_2.
Else, if d_2 is zero then it is that of d_1.

Case 4: q_1, q_2, q_3 are all artificial. The sign is positive.

8 Flipping Determination

In this section we present direct computations and lexicographical manipulations used in REGTET for solving the *flipping determination problem*, i. e.

the problem of determining the sign of $\pi_p(z(t)) - w(t)$. We do this by cases, each case depending on the number of artificial vertices of t. This solution is justified in [3].

Case 1: None of q_1, q_2, q_3, q_4 is artificial. The sign can then be determined through direct computations of $z(t)$, $w(t)$, $\pi_p(z(t))$, and $\pi_p(z(t)) - w(t)$.

Case 2: Exactly one of q_1 q_2, q_3, q_4 is artificial. Without any loss of generality assume q_1 is artificial.
Assume $((q_2 - q_4) \times (q_3 - q_4)) \cdot (q_1 - q_4) < 0$.
Compute $d \equiv ((q_2 - q_4) \times (q_3 - q_4)) \cdot (p - q_4)$.
If d is non-zero then the sign is that of d.
Else, if d is zero then let f be the facet of t whose vertices are q_2, q_3, and q_4, and let H be the plane in $\mathcal{R}^3$ that contains f. Compute $\bar{z}$, the orthogonal center of f in the plane H, and $\bar{w}$, the power distance of $\bar{z}$ from any of the vertices of f.
Compute $\pi_p(\bar{z})$ and $\pi_p(\bar{z}) - \bar{w}$.
The sign is that of $\pi_p(\bar{z}) - \bar{w}$.

Case 3: Exactly two of q_1 q_2, q_3, q_4 are artificial. Without any loss of generality assume q_1 and q_2 are artificial, and let k, l be integers, $1 \leq k, l \leq 8$, so that q_1 equals $p_{k\mu}$ and q_2 equals $p_{l\mu}$.
Assume $((q_2 - q_1) \times (q_3 - q_4)) \cdot (q_1 - q_4) < 0$.
Compute $d \equiv ((e_l - e_k) \times (q_3 - q_4)) \cdot (p - q_4)$.
If d is non-zero then the sign is that of d.
Else, if d is zero then let $\tilde{H}$ be the plane in $\mathcal{R}^3$ that is the *chordale* of q_3 and q_4, i. e. the plane of points x in $\mathcal{R}^3$ for which $\pi_{q_3}(x) = \pi_{q_4}(x)$. Let $\bar{H}$ be the plane in $\mathcal{R}^3$ that is the chordale of $p_{k\mu}$ and $p_{l\mu}$ for all positive values of μ, and let H be the plane in $\mathcal{R}^3$ that contains q_3 and q_4, and is perpendicular to $\tilde{H} \cap \bar{H}$. Compute $\bar{z}$, the one point in $\tilde{H} \cap \bar{H} \cap H$, and $\bar{w}$, the power distance of $\bar{z}$ from either q_3 or q_4.
Compute $\pi_p(\bar{z})$ and $\pi_p(\bar{z}) - \bar{w}$.
The sign is that of $\pi_p(\bar{z}) - \bar{w}$.

Case 4: Exactly three of q_1, q_2, q_3, q_4 are artificial. Without any loss of generality assume q_1, q_2 and q_3 are artificial, and let k, l, m be integers, $1 \leq k, l, m \leq 8$, so that q_1 equals $p_{k\mu}$, q_2 equals $p_{l\mu}$, and q_3 equals $p_{m\mu}$.

Assume $((q_2 - q_1) \times (q_3 - q_1)) \cdot (q_1 - q_4) < 0$.
Compute $d \equiv ((e_l - e_k) \times (e_m - e_k)) \cdot (p - q_4)$.
If d is non-zero then the sign is that of d.
Else, if d is zero then let $\tilde{H}$ and $\bar{H}$ be the planes in $\mathcal{R}^3$ that are the chordales, respectively, of $p_{k\mu}$ and $p_{l\mu}$, and $p_{k\mu}$ and $p_{m\mu}$, for all positive values of μ. Let H be the plane in $\mathcal{R}^3$ that contains q_4 and is perpendicular to $\tilde{H} \cap \bar{H}$. Compute $\bar{z}$, the one point in $\tilde{H} \cap \bar{H} \cap H$, and $\bar{w}$, the power distance of $\bar{z}$ from q_4.
Compute $\pi_p(\bar{z})$ and $\pi_p(\bar{z}) - \bar{w}$.
The sign is that of $\pi_p(\bar{z}) - \bar{w}$.

9 Flipping History

At all times during its execution, REGTET maintains a list of all tetrahedra in the current and previous tetrahedralizations. This list is in the form of a directed acyclic graph that represents the history of the flips REGTET has performed [6], and it is used by REGTET for identifying a tetrahedron in the current tetrahedralization that contains a new point. Identifying a tetrahedron that contains a point this way is a generalization of a technique used in [7] for 2−dimensional triangulations. Essentially, given a tetrahedron t in this list, links exist from t to at most four other tetrahedra in the list. If t is in the current tetrahedralization then the tetrahedra to which t is linked are those in the tetrahedralization that share a facet with t. Otherwise, if t was in a previous tetrahedralization then at some point during the execution of REGTET, t was part of a tetrahedralization for a set of five points on which a flip was applied. Accordingly, the tetrahedra to which t is linked are those in the tetrahedralization for the set of five points that resulted from that flip. Since a tetrahedron that is eliminated through a flip stays eliminated throughout the execution of REGTET then it follows that the directed graph defined by the list of tetrahedra is acyclic.

For some positive integer n, let p_j, $j = 1, \ldots, n$, be the points in S, excluding the artificial points, in the order in which they are added by REGTET. At the start of the execution of REGTET, so that at all times the set of previously added points is not empty, REGTET first computes a regular tetrahedralization for the set of artificial points together with p_1. Essentially, REGTET does this by dividing the cube R_μ (Section 4) in the obvious way into twelve tetrahedra, two per facet of R_μ, with p_1 as a vertex of all twelve tetrahedra. That the resulting tetrahedralization is regular for very large μ is not hard to show. Clearly these twelve tetrahedra are the first to be placed in the list of tetrahedra that REGTET maintains.

Assume inductively that for some integer j, $1 < j \leq n$, the points p_i, $i = 1, \ldots, j-1$, have been added by REGTET into the tetrahedralization. REGTET then proceeds to identify a tetrahedron in the current tetrahedralization that contains the point p_j through an iterative procedure as follows. Using the solution to the point location determination problem (Section 7) REGTET initially identifies a tetrahedron t that contains p_j among the first twelve in the list of tetrahedra. Let m be an integer variable whose initial value equals one. For l, $l = 1, \ldots, m+1$, if the value of l equals that of m plus one the procedure terminates. Otherwise REGTET determines whether t is in the current tetrahedralization. If it is then t is the desired tetrahedron and REGTET proceeds to the next value of l (since the next value of l is that of m plus one the procedure terminates). Otherwise, if it is not then from the list of tetrahedra, REGTET identifies the tetrahedra to which t is linked. Since t is contained in their union it follows that at least one of them contains p_j. Again, using the solution to the point location determination problem (Section 7) REGTET identifies one that does, t becomes this tetrahedron, the value of m is increased by one, and REGTET proceeds to the next value of l. Clearly when the procedure terminates it then follows that t is in the current tetrahedralization and contains p_j. In addition, the value of m equals the number of tetrahedra that contain p_j and that were identified by REGTET for the purpose of identifying the final t.

10 Execution time

REGTET has the capability of adding the points in S in a random sequence. For some positive integer n, let n be number of points in S. Using an analysis similar to the one in [7] for 2−dimensional Delaunay triangulations, Edelsbrunner and Shah [6] show that if the points in S are added in a random sequence then the expected running time of their algorithm for computing a regular tetrahedralization for S is $O(n \log n + n^2)$. As pointed out in [6], the actual expected time could be much less, i. e. the second term (n^2) in the above expectation could be much less, depending on the distribution of the points in S. Accordingly this should be the case for sets of uniformly distributed points in a cube or a sphere. As proven for a cube in [2] and for a sphere in [5], the complexity of the Voronoi diagram, and therefore of the Delaunay tetrahedralization, for such sets is expected linear. Indeed we have obtained good execution times when computing with REGTET regular tetrahedralizations for sets of uniformly distributed points in cubes: on a SGI ONYX2 (300 MHz R12000 CPU)[1] the running time is about 25 CPU minutes for a set of 512,000 points with random weights. A similar time was obtained for the same set without weights. Finally, REGTET has also been executed successfully and efficiently to compute Delaunay tetrahedralizations for non-uniformly distributed point sets representing sea floors and cave walls.

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[1]The identification of any commercial product or trade name does not imply endorsement or recommendation by the National Institute of Standards and Technology.

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