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BANACH SPACES THAT HAVE NORMAL STRUCTURE AND ARE ISOMORPHIC TO A HILBERT SPACE

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ABSTRACT. We prove that given a Hilbert space $(E, \|\cdot\|)$, and $|\cdot|$ a norm on E such that for all $x \in E$, $1/\beta|x| \leqslant \|x\| \leqslant |x|$ for some β, if $1 \leqslant \beta < \sqrt{2}$, then $(E, |\cdot|)$ satisfies a convexity property from which normal structure follows.

1. Introduction. A Banach space E is said to have normal structure if for each bounded, closed and convex subset C of E, consisting of more than one point, there is an $x \in C$ such that

$$\sup\{\|x - y\| : y \in C\} < \operatorname{diam}(C) \equiv \sup\{\|y_1 - y_2\| : y_1, y_2 \in C\}.$$

In [**4**] it was proved that if E has normal structure, $C \subseteq E$ is a nonempty weakly compact convex set, and $T: C \to C$ is a mapping such that for all $x, y \in C$, $\|Tx - Ty\| \leqslant \|x - y\|$, then T has a fixed point in C.

For $r \geqslant 1$ let E_r be the space l_2 renormed by

$$|x|_r = \max\{\|x\|_2, r\|x\|_\infty\},$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the l_2 and l_∞ norms, respectively. It is known from [**1**] that E_r has normal structure when $r < \sqrt{2}$.

We use the idea of multidimensional volumes and their related convexity moduli to prove E_r satisfies a convexity property that implies this result. The notion of volumes in Banach spaces was introduced by Silverman and its use in defining moduli of convexity was introduced in [**5**]. Roughly speaking, the modulus of k-rotundity, $\delta_k(\varepsilon)$, measures the depth below the surface of the unit sphere of the centroid of a simplex of $k + 1$ norm-1 vectors enclosing a k-dimensional volume larger than ε. In symbols,

$$A(x_1, \ldots, x_{k+1}) \geqslant \varepsilon$$

implies that

$$\|(x_1 + \cdots + x_{k+1})/(k+1)\| \leqslant 1 - \delta_k(\varepsilon).$$

Here $A(x_1, \ldots, x_{k+1})$ denotes the enclosed volume. In case $k = 1$, $A(x_1, x_2) = \|x_1 - x_2\|$ and $\delta_1(\varepsilon)$ is the usual modulus of convexity. In all cases

$$D(\|\cdot\|, x_1, \ldots, x_{k+1}) \leqslant A(x_1, \ldots, x_{k+1}).$$

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Here

$$D(\|\cdot\|, x_1, \ldots, x_{k+1}) \equiv \|x_k - x_{k+1}\| \cdot \operatorname{dist}(x_{k-1}, [x_k, x_{k+1}]) \cdot \cdots \cdot \operatorname{dist}(x_1, [x_2, \ldots, x_{k+1}])$$

where $[x_{i+1}, \ldots, x_{k+1}]$ is the affine span of the vectors $x_{i+1}, \ldots, x_{k+1}$ and

$$\operatorname{dist}(x_i, [x_{i+1}, \ldots, x_{k+1}]) = \inf\{\|x_i - x\| : x \in [x_{i+1}, \ldots, x_{k+1}]\}.$$

For a Hilbert space the inequality is always equality.

A connection between these moduli and normal structure of a Banach space E was given in [3], namely

LEMMA. *Suppose that for some* $\delta > 0$ *and some* $0 < \varepsilon < 1$ *there is an integer* m *such that for all norm*-1 $x_1, \ldots, x_m \in E$, *if* $\|(x_1 + x_2 + \cdots + x_m)/m\| > 1 - \delta$ *then* $D(\|\cdot\|, x_1, \ldots, x_m) < \varepsilon$.

Then E is super-reflexive and has normal structure.

2. The result.

THEOREM. *Let* $(E, \|\cdot\|)$ *be a Hilbert space, and let* $|\cdot|$ *be a norm on E such that for all* $x \in E$, $1/\beta|x| \leqslant \|x\| \leqslant |x|$ *for some* β, $1 \leqslant \beta < \sqrt{2}$. *Given* $\varepsilon > 0$, *there exists* $\delta > 0$ *and M, a positive integer, such that for* $m \geqslant M$, *if* $x_1, \ldots, x_m \in E$, $|x_1|, \ldots, |x_m| \leqslant 1$, *and* $|(x_1 + \cdots + x_m)/m|^2 > 1 - \delta$, *then* $D(|\cdot|, x_1, \ldots, x_m) < \varepsilon$.

The proof requires some preliminary results.

LEMMA 1. *Given k, a positive integer and* $\beta > 0$, *let f, g be the functions from* R^k *into R defined by*

$$f(x_1, \ldots, x_k) = \frac{1}{k+1} \sum_{i=1}^{k} \frac{i}{i+1} (x_{k+1-i})^2,$$

$$g(x_1, \ldots, x_k) = \beta - \prod_{i=1}^{k} x_i, \qquad (x_1, \ldots, x_k) \in R^k.$$

Then $f(x) \geqslant (k/(k+1))(\beta^{2/k}/(k+1)^{1/k})$ *whenever* $x \in R^k$ *and* $g(x) = 0$.

PROOF. Let $w \in \Omega \equiv \{x \in R^k : g(x) = 0\}$ and $y = f(w) > 0$. Then $\hat{\Omega} \equiv f^{-1}([0, y]) \cap \Omega$ is nonempty and compact and, thus, there exists $x^* \in \hat{\Omega}$, a global minimum point of f over $\hat{\Omega}$. Also, if $z \in \Omega \setminus f^{-1}([0, y])$ then $f(z) > y$ so that x^* is a global minimum point of f over Ω.

With $x^* = (b_1, \ldots, b_k)$ then $\prod_{i=1}^{k} b_i = \beta > 0$. Thus $\nabla g(x^*) \neq 0$, where ∇g is the gradient of g. It now follows, by Lagrange's theorem, that for some $\lambda \in R$,

$$\nabla f(x^*) = \lambda \nabla g(x^*).$$

So, for each i, $1 \leqslant i \leqslant k$,

$$\frac{2}{k+1} \frac{k+1-i}{k+2-i} b_i = \lambda \prod_{\substack{j=1 \\ j \neq i}}^{k} b_j$$

or

$$\frac{2}{k+1}\frac{k+1-i}{k+2-i}b_i^2 - \lambda\beta = 0.$$

Thus, for each i, $1 \leqslant i \leqslant k$, $((k+1)/2)\lambda\beta = ((k+1-i)/(k+2-i))b_i^2$, and

$$\left[\frac{k+1-i}{k+2-i}b_i^2\right]^k = \prod_{j=1}^{k}\frac{k+1-j}{k+2-j}b_j^2 = \frac{1}{k+1}\beta^2.$$

Therefore, $((k+1-i)/(k+2-i))b_i^2 = \beta^{2/k}/(k+1)^{1/k}$ for each i, $1 \leqslant i \leqslant k$. So, given $x \in \Omega$,

$$f(x) \geqslant f(x^*) = \frac{1}{k+1}\sum_{i=1}^{k}\frac{i}{i+1}b_{k+1-i}^2$$

$$= \frac{1}{k+1}\sum_{i=1}^{k}\frac{k+1-i}{k+2-i}b_i^2 = \frac{k}{k+1}\frac{\beta^{2/k}}{(k+1)^{1/k}}.$$

LEMMA 2. *Let* $(E, \|\cdot\|)$ *be a Hilbert space and* k *any positive integer. Given* $x_1,\ldots,x_{k+1} \in E$, $\|x_1\|,\ldots,\|x_{k+1}\| \leqslant 1$, *then*

$$\left\|\frac{x_1+\cdots+x_{k+1}}{k+1}\right\|^2 \leqslant 1 - \frac{1}{k+1}\sum_{i=1}^{k}\frac{i}{i+1}\left\|x_{k+1-i} - \frac{\sum_{j=k+2-i}^{k+1}x_j}{i}\right\|^2.$$

PROOF. Since $(E, \|\cdot\|)$ is a Hilbert space, given $x, y \in E$, then

$$\left\|\frac{1}{k+1}x + \frac{k}{k+1}y\right\|^2 = \frac{1}{k+1}\|x\|^2 + \frac{k}{k+1}\|y\|^2 - \frac{k}{(k+1)^2}\|x-y\|^2.$$

In particular, given $x_1,\ldots,x_{k+1} \in E$,

$$\left\|\frac{x_1+\cdots+x_{k+1}}{k+1}\right\|^2 = \frac{1}{k+1}\|x_1\|^2 + \frac{k}{k+1}\left\|\frac{x_2+\cdots+x_{k+1}}{k}\right\|^2 - \frac{k}{(k+1)^2}\left\|x_1 - \frac{x_2+\cdots+x_{k+1}}{k}\right\|^2.$$

The proof of the lemma now follows by induction on k.

LEMMA 3. *Let* $(E, \|\cdot\|)$ *be a Hilbert space and* k *a positive integer. If* $x_1,\ldots,x_{k+1} \in E$, $\|x_1\|,\ldots,\|x_{k+1}\| \leqslant 1$, $D(\|\cdot\|, x_1,\ldots,x_{k+1}) \geqslant \varepsilon > 0$, *then*

$$\|(x_1+\cdots+x_{k+1})/(k+1)\|^2 \leqslant 1 - (k/(k+1))\left(\varepsilon^{2/k}/(k+1)^{1/k}\right).$$

PROOF. Since $D(\|\cdot\|, x_1 \cdots x_{k+1}) \geqslant \varepsilon$ then $D(\|\cdot\|, x_1,\ldots,x_{k+1}) = \beta$, where $\beta \geqslant \varepsilon$. Let $d_i = \operatorname{dist}(x_i, [x_{i+1},\ldots,x_{k+1}])$, for each i, $1 \leqslant i \leqslant k$. By Lemma 2, with f as defined in Lemma 1, it follows that

$$\left\|\frac{x_1+\cdots+x_{k+1}}{k+1}\right\|^2 \leqslant 1 - \frac{1}{k+1}\sum_{i=1}^{k}\frac{i}{i+1}\left\|x_{k+1-i} - \frac{\sum_{j=k+2-i}^{k+1}x_j}{i}\right\|^2$$

$$\leqslant 1 - \frac{1}{k+1}\sum_{i=1}^{k}\frac{i}{i+1}d_{k+1-i}^2 = 1 - f(d_1,\ldots,d_k).$$

However, $\prod_{i=1}^{k} d_i = D(\|\cdot\|, x_1,\ldots,x_{k+1}) = \beta$. So, by Lemma 1,

$$f(d_1,\ldots,d_k) \geqslant (k/(k+1))\left(\beta^{2/k}/(k+1)^{1/k}\right),$$

and, therefore,

$$\|(x_1 + \cdots + x_{k+1})/(k+1)\|^2 \leqslant 1 - (k/(k+1))\left(\varepsilon^{2/k}/(k+1)^{1/k}\right).$$

REMARK. Extending these ideas [2] gives the exact value of the modulus of k-rotundity of a Hilbert space, e.g.

$$\delta_k(\varepsilon) = 1 - \left[1 - \frac{k}{k+1}\frac{\varepsilon^{2/k}}{(k+1)^{1/k}}\right]^{1/2}.$$

PROOF OF THE THEOREM. Choose $\eta > 0$ so that $\beta^2 + \eta < 2$. Given any $\varepsilon > 0$, k a positive integer, let $\Delta_k(\varepsilon) = (k/(k+1))(\varepsilon^{2/k}/(k+1)^{1/k})$. Since $\lim_{k\to\infty}\Delta_k(\varepsilon) = 1$, select $M > 1$ so large that $\Delta_{m-1}(\varepsilon) > 1 - \eta$ whenever $m \geqslant M$.

Now, let $\delta = 2 - \beta^2 - \eta$, and suppose $m \geqslant M$, $|(x_1 + \cdots + x_m)/m|^2 > 1 - \delta$, $|x_1|,\ldots,|x_m| \leqslant 1$, while $D(|\cdot|, x_1,\ldots,x_m) \geqslant \varepsilon$. Then $\|x_1\|,\ldots,\|x_m\| \leqslant 1$ and

$$D(\|\cdot\|, x_1,\ldots,x_m) \geqslant (1/\beta)^{m-1}\cdot D(|\cdot|, x_1,\ldots,x_m) \geqslant (1/\beta)^{m-1}\varepsilon.$$

It follows from Lemma 3 that

$$\left\|\frac{x_1 + \cdots + x_m}{m}\right\|^2 \leqslant 1 - \Delta_{m-1}\left(\left(\frac{1}{\beta}\right)^{m-1}\varepsilon\right).$$

However,

$$\begin{aligned} 1 - \delta < \left|\frac{x_1 + \cdots + x_m}{m}\right|^2 &\leqslant \beta^2\left\|\frac{x_1 + \cdots + x_m}{m}\right\|^2 \\ &\leqslant \beta^2\left(1 - \Delta_{m-1}\left(\left(\frac{1}{\beta}\right)^{m-1}\varepsilon\right)\right) = \beta^2\left(1 - \left(\frac{1}{\beta}\right)^2\Delta_{m-1}(\varepsilon)\right) \\ &= \beta^2 - \Delta_{m-1}(\varepsilon) < \beta^2 + \eta - 1. \end{aligned}$$

This contradicts the definition of δ. Therefore, $D(|\cdot|, x_1,\ldots,x_m) < \varepsilon$. Q.E.D.

The result proven in [1] now follows from this theorem and the lemma mentioned in the introduction.

COROLLARY. *Let* $(E, \|\cdot\|)$ *be a Hilbert space and let* $|\cdot|$ *be a norm on* E *such that for all* $x \in E$, $1/\beta|x| \leqslant \|x\| \leqslant |x|$ *for some* β, $1 \leqslant \beta < \sqrt{2}$. *Then* $(E, |\cdot|)$ *has normal structure.*

Note that the Theorem is sharp because Baillon and Schöneberg proved that E_r fails to have normal structure for $r \geqslant \sqrt{2}$.

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