Inserting Line Segments into Triangulations and Tetrahedralizations

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Abstract. In this paper, we further develop an algorithm by Bernal, De Floriani, and Puppo, for inserting a line segment into a Constrained Delaunay triangulation. The new version of the algorithm inserts the line segment in exactly the same manner in which the old version does but has the additional capability that it does not delete the triangles intersected by the line segment but transforms them through edge-swapping. Since the concept of edge-swapping generalizes to 3-dimensional space, a 3-dimensional version of the algorithm without the optimization step for the Delaunay property is also presented for attempting to insert a line segment into a tetrahedralization. It is shown that for certain cases the failure of the 3-dimensional algorithm to insert a line segment is an indication that it can not be done. Finally, 3-dimensional problems that can be approached algorithmically as 2-dimensional problems are identified.

Key words. computational geometry, constrained Delaunay triangulation, edge-swapping, empty circle criterion, locally equiangular, segment insertion, Voronoi diagram

AMS(MOS) subject classifications. 68U05
1. Introduction

A triangulation for a finite set of points \( S \) in the plane is a finite collection of triangles in the plane having pair-wise disjoint interiors, each of which intersects \( S \) exactly at its vertices, and the union of which is the convex hull of \( S \). Given a triangulation \( T \) for \( S \), we say that \( T \) is a Delaunay triangulation for \( S \) if for each triangle in \( T \) there does not exist a point of \( S \) inside the circumcircle of the triangle [10]. A (Delaunay) tetrahedralization for a finite set of points in 3-dimensional space is similarly defined with tetrahedra and spheres taking the place of triangles and circles.

A triangulation more general than the Delaunay triangulation can be defined. Let \( S \) be a finite set of points in the plane, and let \( E \) be a finite collection, possibly empty, of line segments that intersect only at points in \( S \) and whose endpoints belong to \( S \). We say that a triangulation \( T \) for \( S \) is a triangulation for \( S \) constrained by \( E \) if for each \( e \) in \( E \) and each \( t \) in \( T \), \( e \) does not intersect the interior of \( t \). Given \( T \), a triangulation for \( S \) constrained by \( E \), we say that \( T \) is a Delaunay triangulation for \( S \) constrained by \( E \) if for each \( t \) in \( T \) there does not exist a point \( P \) of \( S \) inside the circumcircle of \( t \) such that no \( e \) in \( E \) intersects the interior of the convex hull of \( t \cup \{ P \} \).

Let \( S \) and \( E \) be as above. Given \( T \), a triangulation for \( S \) constrained by \( E \), we say that \( T \) is locally equiangular constrained by \( E \) if given any two triangles in the triangulation that share a common edge not contained in any edge belonging to \( E \) and whose union is a strictly convex quadrilateral, then replacement of the common edge by the alternative diagonal of the quadrilateral does not increase the minimum of the six angles in the two triangles making up the quadrilateral. That a triangulation is constrained Delaunay if and only if it is constrained locally equiangular has been proven in [8]. On the other hand, given \( S \) and \( E \) as above, \( T \) a triangulation for \( S \) constrained by \( E \), we say that \( T \) constrained by \( E \) satisfies the empty circle criterion on a local basis if given any two triangles \( t, t' \) in \( T \) that share a common edge not contained in any edge belonging to \( E \), then the vertex of \( t' \) not in \( t \) is not inside the circumcircle of \( t \). That a triangulation is constrained locally equiangular if and only if it is constrained satisfying the empty circle criterion on a local basis has been proven in [4], [8].

Algorithms for the computation of a Delaunay triangulation for the vertices of a polygon constrained by the boundary of the polygon have been presented in [4], [8], [9], the algorithm in [8] having complexity \( O(n \log n) \), where \( n \) is the number of vertices of the polygon. As for the general problem of computing a Delaunay triangulation for a set of \( n \) points constrained by a set of line segments, an \( O(n^2) \) algorithm has been presented in [8], \( O(n \log n) \) divide-and-conquer algorithms have been presented in [3], [12], and an \( O(n \log n) \) plane-sweep algorithm has been presented in [11]. Each one of these algorithms has the disadvantage that the set
of line segments must be known before the execution of the algorithm.

In [5] a method has been presented for the incremental computation of a constrained Delaunay triangulation by stepwise insertion of points and line segments. Accordingly, algorithms are presented in [5] for point insertion and line segment insertion into a constrained Delaunay triangulation. Independently, the algorithm for line segment insertion was also presented in [1]. In the following section, we describe a new version of the segment insertion algorithm that works in the same manner in which the old one does, but that has the additional capability of not deleting the triangles intersected by the line segment, transforming them instead through edge-swapping (Lawson's transformation [7]). In Section 3, we take advantage of the fact that edge-swapping generalizes to 3-dimensional space and present what would be considered the generalization to 3-dimensional space of the new line insertion algorithm without the optimization step for the Delaunay property. It is shown there that for certain cases the failure of this algorithm to insert a line segment into a tetrahedralization is an indication that it cannot be done. Finally, in the same section, 3-dimensional problems are identified that can be approached algorithmically as if they are 2-dimensional.

2. Segment insertion by edge-swapping

Let $T$ be a triangulation in the plane, not necessarily Delaunay, let $P_1, P_2, P_1 \neq P_2$, be vertices in $T$, and let $T^*$ be the collection of triangles in $T$ whose interiors are intersected by the line segment with endpoints $P_1, P_2$. We say that the line segment with endpoints $P_1, P_2$ has been inserted into $T$ producing $\hat{T}$ if $\hat{T}$ is a triangulation for the vertices of $T$ such that the line segment is the union of one or more edges in $\hat{T}$ and each triangle in $T \setminus T^*$ is also in $\hat{T}$. In what follows, and assuming that $T$ is constrained Delaunay, we present procedure INSERT_SEGMENT which inserts the line segment with endpoints $P_1, P_2$ into the triangulation $T$ by edge-swapping (Lawson's transformation [7]), producing a constrained Delaunay triangulation with the line segment as an additional constraint. Without any loss of generality, we assume that the line segment is not an edge in $T$ and that its relative interior does not contain any vertices in $T$.

In [1] and [5] this algorithm was presented but without edge-swapping. This older version consists essentially of two steps. In the first step, the triangles whose interiors are intersected by the line segment are detected and deleted so that a non-triangulated region inside the convex hull of the original triangulation results. In the second step, this region is divided into two polygons separated by the line segment, and a Delaunay triangulation is then computed for each polygon. Each polygon satisfies the property that each point in the polygon is visible through the polygon from the line segment, i. e. given a point $P$ in the
polygon but not in the line segment, there exists a point $P'$ in the relative interior of the line segment such that the relative interior of the line segment with endpoints $P$, $P'$ is contained in the interior of the polygon. Because of this property, each polygon can be triangulated through an incremental insertion of triangles in the polygon, and optimized for the Delaunay property with procedures based on the empty circle criterion. Outlines of this older version, justifications, optimization procedures, and related results can be found in [1], [2], [5], [6].

The new version of the algorithm presented here works essentially in the same manner in which the old one does, thus producing exactly the same triangles, but has the capability through edge-swapping of maintaining at all times a complete triangulation for the vertices in the original triangulation. This will be illustrated below with an example.

In the following, we list and describe, in the order of their first appearance in procedure INSERT_SEGMENT, procedures used there as primitives.

**INTERSECTED_TRIANGLES**($T$, $T^*$, $P_1$, $P_2$, $Q$, $t_F$): Assuming that $P_1$, $P_2$, $P_1 \neq P_2$, are vertices in triangulation $T$, this procedure identifies $T^*$, $T^* \subseteq T$, which is made up of those triangles in $T$ with interiors intersected by the line segment with endpoints $P_1$, $P_2$. It also locates $t_F$ and $Q$, where $t_F$ is the triangle in $T^*$ with $P_1$ as one of its vertices and $Q$ is any one of the vertices of $t_F$ different from $P_1$.

**NEXT_TRIANGLE**($T$, $P_1$, $P_2$, $t_P$, $t_C$): Assuming that $P_1$, $P_2$, $P_1 \neq P_2$, are vertices in triangulation $T$, that the line segment with endpoints $P_1$, $P_2$ intersects the interior of $t_P$, a triangle in $T$, and that $P_2$ is not a vertex of $t_P$, this procedure locates triangle $t_C$ in $T$ which shares a facet with $t_P$ intersected by the line segment, and which is closer to $P_2$ than $t_P$ in the direction of the line segment.

**NEXT_VERTEX**($t_P$, $t_C$, $P$): Assuming that $t_P$ and $t_C$ are adjacent triangles in some triangulation, this procedure locates vertex $P$ of $t_C$ not in $t_P$.

**PREVIOUS_VERTEX**($t_C$, $P_1$, $P_2$, $P$, $Q$): Assuming that $P$ is a vertex of triangle $t_C$ and that the line segment with endpoints $P_1$, $P_2$ intersects exactly one of the edges of $t_C$ with $P$ as an endpoint, this procedure locates the vertex $Q$ of $t_C$ for which the line segment does not intersect the edge with endpoints $P$, $Q$.

**STRIC_T_CONVEXITY**($t_C$, $t_P$, $flag2$): Assuming that $t_C$ and $t_P$ are adjacent triangles in some triangulation, and that $flag2$ equasl 1, this procedure sets $flag2$ to zero whenever the
union of $t_C$ and $t_P$ is not a strictly convex quadrilateral.

**EDGE_SWAP($t_C, t_P, Q, T, T^*$):** Assuming that $t_C$, $t_P$ are adjacent triangles in $T^*$, $T^* \subseteq T$, whose union is a strictly convex quadrilateral, and that $Q$ is one of the vertices that $t_C$ and $t_P$ have in common, this procedure transforms $t_C$, $t_P$, and therefore in the same manner $T^*$ and $T$, through the replacement of the common edge by the alternative diagonal of the quadrilateral in such a way that $Q$ is the vertex of the transformed $t_P$ not in the transformed $t_C$.

**OPTIMIZE($T, T^*, t_P, P, Q, R$):** Assuming that $P$, $Q$, $R$ are the vertices of triangle $t_P$ in $T^*$, $T^* \subseteq T$, this procedure is essentially the same as procedure UPDTRI in [1], which transforms $T^*$, and therefore in the same manner $T$, through edge-swapping in such a way that if after the execution of this procedure $t$ is a triangle in $T^*$ that either equals $t_P$ or that was not in $T^*$ before the execution of the procedure, then $t$ satisfies the following properties:
1. $P$ is a vertex of $t$.
2. The two edges of $t$ with $P$ as an endpoint intersect the line segment with endpoints $Q$, $R$.
3. The circumcircle of $t$ does not contain in its interior any vertex $P'$ in $T^*$ for which the relative interior of the line segment with endpoints $P$, $P'$ lies entirely in the interior of the union of the triangles in $T^*$ and intersects the edge of $t$ that does not have $P$ as an endpoint.

**PREVIOUS_TRIANGLE($T, P_1, P_2, t_C, t_P$):** Assuming that $P_1$, $P_2$, $P_1 \neq P_2$, are vertices in triangulation $T$, that the line segment with endpoints $P_1$, $P_2$ intersects the interior of $t_C$, a triangle in $T$, and that $P_i$ is not a vertex of $t_C$, this procedure locates triangle $t_P$ in $T$ which shares a facet with $t_C$ intersected by the line segment, and which is closer to $P_1$ than $t_C$ in the direction of the line segment.

**THIRD_VERTEX($t_C, R, P, Q$):** Assuming that $R$, $P$, $R \neq P$, are vertices of triangle $t_C$, this procedures identifies $Q$, a vertex of $t_C$ with $Q \neq R$ and $Q \neq P$.

The outline of INSERT_SEGMENT follows. Throughout the procedure and for the purpose of keeping track of where in $T$ the optimization procedure can be applied, a collection of triangles $T^*$ is maintained, $T^* \subseteq T$, which is made up of those triangles or their transformations through edge-swapping (with procedures EDGE_SWAP and OPTIMIZE) that initially are intersected by the relative interior of the line segment with endpoints $P_1$, $P_2$. Also, throughout the procedure, a function $F$ is defined on certain elements of $\{1, 2\} \times T^*$. Essentially, given $t$ in $T^*$ with interior intersected by the line segment, it is assumed that
the line segment intersects exactly two edges of \( t \) at different points. \( F(1,t), F(2,t) \) are then defined to represent the endpoints of the edge of \( t \) not intersected by the line segment, in such a way that \( F(2,t) \) also represents an endpoint of the one of the two edges of \( t \) intersected by the line segment that is closer to \( P_2 \) in the direction of the line segment. Finally, we notice that if the optimization steps are eliminated in INSERT_SEGMENT (steps 20 and 32) then the procedure simply becomes one for inserting a line segment into a triangulation.

```plaintext
procedure INSERT_SEGMENT(T, P_1, P_2)
begin
  1. INTERSECTED_TRIANGLES(T, T*, P_1, P_2, Q, t_F);
  2. F(1, t_F) := P_1; F(2, t_F) := Q; flag1 := 1;
  3. while (flag1 = 1) do
     begin
       4. t_P := t_F;
       5. NEXT_TRIANGLE(T, P_1, P_2, t_P, t_C);
       6. NEXT_VERTEX(t_P, t_C, P);
       7. if (P \neq P_2) then
          begin
            8. PREVIOUS_VERTEX(t_C, P_1, P_2, P, Q);
            9. t_F := t_C
          end
       else
          begin
            10. Q := F(2, t_P); flag1 := 0
          end
       11. if (F(1, t_P) = P_1) then F(2, t_P) := Q;
       12. F(1, t_C) := Q; F(2, t_C) := P; flag2 := 1;
       13. while (flag2 = 1) do
          begin
            14. STRICT_CONVEXITY(t_C, t_P, flag2);
            15. if (flag2 = 1) then
               begin
                 16. R := F(1, t_P); t_L := t_C;
                 17. EDGE_SWAP(t_C, t_P, Q, T, T*);
                 18. if (t_F = t_L) then t_F := t_C;
                 19. if (F(1, t_C) = F(2, t_P)) then
                    begin
```
20. \[ \text{OPTIMIZE} (T, T^*, t_P, P, Q, R); \]
21. \[ F(1, t_C) := R; Q := R \]
   \hspace{1em} \text{end} \\
   \text{else} \\
   \hspace{1em} \text{begin} \\
   \hspace{2em} F(1, t_C) := R; F(2, t_C) := F(2, t_P); \\
   \hspace{2em} F(1, t_P) := Q; F(2, t_P) := P; t_C := t_P \\
   \hspace{1em} \text{end} \\
24. \text{if } (R \neq P_1) \text{ then} \\
   \hspace{1em} \text{begin} \\
   \hspace{2em} \text{PREVIOUS\_TRIANGLE} (T, P_1, P_2, t_C, t_P); \\
   \hspace{2em} \text{if } (F(1, t_P) = P_1) \text{ then } F(2, t_P) := Q; \\
   \hspace{2em} \text{if } (P = P_2) \text{ then} \\
   \hspace{3em} \text{begin} \\
   \hspace{4em} Q := F(2, t_P); F(1, t_C) := Q \\
   \hspace{3em} \text{end} \\
   \hspace{1em} \text{end} \\
   \text{else} \\
   \hspace{1em} \text{begin} \\
29. \hspace{2em} \text{flag2 := 0;} \\
30. \hspace{2em} \text{if } (P = P_2) \text{ then} \\
   \hspace{3em} \text{begin} \\
31. \hspace{4em} \text{THIRD\_VERTEX} (t_C, R, P, Q); \\
32. \hspace{4em} \text{OPTIMIZE} (T, T^*, t_C, P, Q, R) \\
   \hspace{3em} \text{end} \\
   \hspace{1em} \text{else} \\
   \hspace{1em} \text{begin} \\
33. \hspace{2em} \text{NEXT\_TRIANGLE} (T, P_1, P_2, t_C, t_N); \\
34. \hspace{2em} F(2, t_C) := F(1, t_N) \\
   \hspace{2em} \text{end} \\
   \hspace{1em} \text{end} \\
   \hspace{1em} \text{end} \\
   \hspace{1em} \text{end} \\
   \hspace{1em} \text{end} \\
\text{end} \\

Algorithm \text{INSERT\_SEGMENT} \text{ has been mostly justified in [1], [6]. In what follows, we}
illustrate with an example the way in which the algorithm works and the new aspects of its justification. Since the insertion of the line segment is the significant aspect of the algorithm we do not assume that the initial triangulation is Delaunay and ignore the optimization steps throughout. Starting with triangulation (i) in Figure 1, we enumerate and describe in the order of their executions the crucial steps of INSERT_SEGMENT that are executed in order to obtain the desired triangulation, triangulation (viii) in Figure 1, and all other intermediate triangulations, triangulations (ii)-(vii) also in Figure 1. Here, given points $X$, $Y$, $Z$, whenever we refer to the triangle $XYZ$ it is implied that a triangle exists in the plane with vertices $X$, $Y$, $Z$, and that this is the order in which they appear in the boundary of the triangle in a counterclockwise direction around the interior of the triangle. Given points $W$, $X$, $Y$, $Z$, a similar assumption goes along with any reference to the quadrilateral $WXYZ$.

1. **Step 1 (triangulation (i))**: $T^*$ is obtained and it is made up of the triangles intersected by line segment $P_1P_2$: $P_1AF$, $FAE$, $EAB$, $EBD$, $DBC$, $DCP_2$. The first triangle is $t_P = P_1AF$.

2. **Step 5 (triangulation (i))**: The next triangle is $t_C = FAE$.

3. **Step 6 (triangulation (i))**: The next vertex is $P = E$.

4. **Step 14 (triangulation (i))**: The quadrilateral $P_1AEF$ is not strictly convex ($flag2$ is set to zero, no edge-swapping is possible, and a new $P$ must be obtained).

5. **Step 5 (triangulation (i))**: The next triangle is $t_C = EAB$. 

6. **Step 6 (triangulation (i))**: The next vertex is $P = B$.

7. **Step 14 (triangulation (i))**: The quadrilateral $FAEB$ is strictly convex.

8. **Step 17 (triangulation (ii))**: Triangles $FAE$, $EAB$ are transformed into triangles $FAB$, $FBE$ through edge-swapping; $t_C$ becomes $FAB$.

9. **Step 14 (triangulation (ii))**: The quadrilateral $P_1ABF$ is strictly convex.

10. **Step 17 (triangulation (iii))**: Triangles $P_1AF$, $FAB$ are transformed into triangles $P_1AB$, $P_1BF$ through edge-swapping; $t_C$ becomes $P_1BF$.

11. **Step 29 (triangulation (iii))**: Because $P_1$ is a vertex of $t_C$, $flag2$ is set to zero (a new $P$ must be obtained).

12. **Step 5 (triangulation (iii))**: The next triangle is $t_C = EBD$. 

13. **Step 6 (triangulation (iii))**: The next vertex is $P = D$.

14. **Step 14 (triangulation (iii))**: The quadrilateral $FADE$ is strictly convex.
Figure 1: A triangulation and the insertion into it of line segment with endpoints $P_1, P_2$. 
13. Step 6 (triangulation (iii)): The next vertex is \( P = D \).

14. Step 14 (triangulation (iii)): The quadrilateral \( FBDE \) is strictly convex.

15. Step 17 (triangulation (iv)): Triangles \( FBE, EBD \) are transformed into triangles \( FBD, FDE \) through edge-swapping; \( t_C \) becomes \( FBD \).

16. Step 14 (triangulation (iv)): The quadrilateral \( P_1 BDF \) is not strictly convex.

17. Step 5 (triangulation (iv)): The next triangle is \( t_C = BCD \).

18. Step 6 (triangulation (iv)): The next vertex is \( P = C \).

19. Step 14 (triangulation (iv)): The quadrilateral \( FBCD \) is strictly convex.

20. Step 17 (triangulation (v)): Triangles \( FBD, BCD \) are transformed into triangles \( FBC, FCD \) through edge-swapping; \( t_C \) becomes \( FBC \).

21. Step 14 (triangulation (v)): The quadrilateral \( P_1 BCF \) is not strictly convex.

22. Step 5 (triangulation (v)): The next triangle is \( t_C = DCP_2 \).

23. Step 6 (triangulation (v)): The next vertex is \( P = P_2 \).

24. Step 10 (triangulation (v)): \( flag1 \) is set to zero (this is the last \( P \)).

25. Step 14 (triangulation (v)): The quadrilateral \( FCP_2 D \) is strictly convex.

26. Step 17 (triangulation (vi)): Triangles \( FCD, DCP_2 \) are transformed into triangles \( FCP_2, FP_2 D \) through edge-swapping; \( t_C \) becomes \( FCP_2 \).

27. Step 14 (triangulation (vi)): The quadrilateral \( FBCP_2 \) is strictly convex.

28. Step 17 (triangulation (vii)): Triangles \( FBC, FCP_2 \) are transformed into triangles \( FBP_2, BCP_2 \) through edge-swapping; \( t_C \) becomes \( FBP_2 \).

29. Step 14 (triangulation (vii)): The quadrilateral \( P_1 BP_2 F \) is strictly convex.

30. Step 17 (triangulation (viii)): Triangles \( P_1 BF, FBP_2 \) are transformed into triangles \( P_1 BP_2, P_1 P_2 F \) through edge-swapping (line segment \( P_1 P_2 \) has now been inserted).

31. Step 29 (triangulation (viii)): \( flag2 \) is set to zero (combined with \( flag1 \) equal to zero, it signifies that the execution of the algorithm must come to an end).
We can use the same example to illustrate the significant new aspects of the justification of the algorithm. A rigorous proof would involve showing that the triangles produced by the old version of the algorithm are also produced by the new one in exactly the same manner. Here we avoid a rigorous proof and concentrate our efforts on simply pointing out the reasons why triangulation (i) in Figure 1 is transformed into triangulation (iii), thus producing triangle \( P_1 AB \) which is a triangle that the old version would also produce. Equivalently, we point out the reasons why the quadrilaterals \( FABE \) and \( P_1 ABF \) are strictly convex under the assumptions that the quadrilateral \( P_1 AEF \) is not strictly convex and that the triangle \( P_1 AB \) is also produced by the old version of the algorithm. The latter assumption is equivalent to the fact that the internal angle of triangle \( P_1 AB \) at \( A \) measures less than \( \pi \) radians. In order to show that the quadrilateral \( FABE \) is strictly convex we must show that each of its internal angles measures less than \( \pi \) radians. The internal angles of the quadrilateral at \( F \) and \( B \) satisfy this property because they are also internal angles, respectively, of triangles \( FAE \) and \( EAB \). Since the internal angle of the quadrilateral at \( A \) is contained in the internal angle of triangle \( P_1 AB \) at \( A \) it follows that it must also satisfy the property. Finally, because of the way in which line segment \( P_1 P_2 \) intersects the quadrilateral \( P_1 AEF \) and the fact that the quadrilateral is not strictly convex it follows that the ray with origin \( E \) through \( F \) must intersect the line segment. This, combined with the fact that the ray with origin \( E \) through \( B \) also intersects the line segment \( (E \text{ is above it, } B \text{ is below it}) \), implies that the internal angle of the quadrilateral at \( E \) measures less than \( \pi \) radians. That the quadrilateral \( P_1 ABF \) is also strictly convex follows through essentially similar arguments.

3. The 3-dimensional version of the algorithm

Let \( T \) be a tetrahedralization, not necessarily Delaunay, let \( P_1, P_2, P_1 \neq P_2 \), be vertices in \( T \), and let \( T^* \) be the collection of tetrahedra in \( T \) each of which is intersected by the line segment with endpoints \( P_1, P_2 \) at either its interior or the relative interior of one of its facets. We say that the line segment with endpoints \( P_1, P_2 \) can be inserted into \( T \) if a tetrahedralization \( \hat{T} \) for the vertices of \( T \) exists such that the line segment is the union of one or more edges in \( \hat{T} \) and each tetrahedron in \( T \setminus T^* \) is also in \( \hat{T} \). In what follows, we take advantage of the fact that edge-swapping generalizes to 3-dimensional space and present procedure 3D_INSERT_ATTEMPT which attempts to insert the line segment with endpoints \( P_1, P_2 \) into \( T \), and which can be considered as the generalization to 3-dimensional space of procedure INSERT_SEGMENT of the previous section without the optimization step for the Delaunay property. We notice that only the case for which the relative interior of the line segment with endpoints \( P_1, P_2 \) does not intersect any edges in the tetrahedralization is addressed in what follows.
In the following, we list and describe, in the order of their first appearance in procedure 3D_INSERT_ATTEMPT, procedures used there as primitives.

FIRST_TETRAHEDRON\((T, P_1, P_2, Q, t_p)\): Assuming that \( P_1, P_2, P_1 \neq P_2 \), are vertices in tetrahedralization \( T \), this procedure identifies \( t_p \), where \( t_p \) is the tetrahedron in \( T \) with \( P_1 \) as one of its vertices and interior intersected by the line segment with endpoints \( P_1, P_2 \). It also identifies \( Q \), where \( Q \) is any one of the vertices of \( t_p \) different from \( P_1 \).

NEXT_TETRAHEDRON\((T, P_1, P_2, t_p, t_C)\): Assuming that \( P_1, P_2, P_1 \neq P_2 \), are vertices in tetrahedralization \( T \), that the line segment with endpoints \( P_1, P_2 \) intersects the interior of \( t_p \), a tetrahedron in \( T \), and that \( P_2 \) is not a vertex of \( t_p \), this procedure locates tetrahedron \( t_C \) in \( T \) which shares a facet with \( t_p \) intersected by the line segment, and which is closer to \( P_2 \) than \( t_p \) in the direction of the line segment.

NEXT_VERTEX\((t_p, t_C, P)\): Assuming that \( t_p \) and \( t_C \) are adjacent tetrahedra in some tetrahedralization, this procedure locates vertex \( P \) of \( t_C \) not in \( t_p \).

PREVIOUS_VERTEX\((t_C, P_1, P_2, P, Q)\): Assuming that \( P \) is a vertex of tetrahedron \( t_C \) and that the line segment with endpoints \( P_1, P_2 \) intersects exactly one of the facets of \( t_C \) with \( P \) as a vertex, this procedure locates the vertex \( Q \) of \( t_C \) for which the line segment does not intersect the facets of \( t_C \) that have in common the edge with endpoints \( P, Q \).

PAIR\((t_C, t_p, flag)\): Assuming that \( t_C \) and \( t_p \) are adjacent tetrahedra in some tetrahedralization, and that \( flag \) equals 1, this procedure sets \( flag \) to zero whenever the union of \( t_C \) and \( t_p \) is not a strictly convex hexahedron.

COMMON_VERTEX\((t_C, t_p, Q, S, U)\): Assuming that \( t_C \) and \( t_p \) are adjacent tetrahedra in some tetrahedralization, and that \( Q \) and \( S \) are vertices, not necessarily different, common to \( t_C \) and \( t_p \), this procedure locates \( U \), a vertex common to \( t_C \) and \( t_p \) such that \( U \neq Q \) and \( U \neq S \).

TWO_THREE\((T, t_C, t_p, P, R, Q, U)\): Assuming that \( t_C \), \( t_p \) are adjacent tetrahedra in \( T \) whose union is a strictly convex hexahedron, that \( P \) is the vertex of \( t_C \) not in \( t_p \), that \( R \) is the vertex of \( t_p \) not in \( t_C \), and that \( Q, U, Q \neq U \), are vertices that \( t_C \) and \( t_p \) have in common, this procedure transforms \( T \) by transforming \( t_C \) and \( t_p \) into the three tetrahedra that have in common the edge with endpoints \( P, R \), in such a way that \( t_C \) becomes the one
of the three tetrahedra that does not have $Q$ as a vertex, and $t_F$ the one that has $Q$ and $U$

as vertices.

**FACET.INTERSECT($P, R, U, P_1, P_2, \text{flag}2$):** Assuming that $P, R, U$ are the vertices of a
facet of some tetrahedron, and that $\text{flag}2$ equals 1, this procedure sets $\text{flag}2$ to zero whenever
the line segment with endpoints $P_1, P_2$ does not intersect the relative interior of the
triangle with vertices $P, R, U$.

**PREVIOUS.TETRAHEDRON($T, P_1, P_2, t_C, t_F$):** Assuming that $P_1, P_2, P_1 \neq P_2$, are
vertices in tetrahedralization $T$, that the line segment with endpoints $P_1, P_2$ intersects the
interior of $t_C$, a tetrahedron in $T$, and that $P_1$ is not a vertex of $t_C$, this procedure locates
tetrahedron $t_F$ in $T$ which shares a facet with $t_C$ intersected by the line segment, and which
is closer to $P_1$ than $t_C$ in the direction of the line segment.

The outline of 3D_INSERT_ATTEMPT follows. Throughout the procedure, and serving
the same purpose as in procedure INSERT_SEGMENT of the previous section, a function $F$
is defined on certain elements of $\{1, 2\} \times T$. Essentially, given $t$ in $T$ with interior intersected
by the line segment with endpoints $P_1, P_2$, it is assumed that the line segment intersects
exactly two facets of $t$ at different points. $F(1,t)$, $F(2,t)$ are then defined to represent the
endpoints of the edge of $t$ that is shared by the two facets of $t$ that the line segment does not intersect, in such a way that $F(2,t)$ also represents a vertex of the one of the two facets of $t$
intersected by the line segment that is closer to $P_2$ in the direction of the line segment. Also,
throughout the procedure, a variable $\text{flag}$ is defined which at the end of the execution of the
procedure equals 1 if the line segment has been inserted, zero otherwise. We point out here
that in the procedure, tetrahedra can only be transformed at steps 21 and 27 with procedure
TWO_THREE which, when possible, transforms two adjacent tetrahedra into three in the
obvious fashion.

**procedure** 3D_INSERT_ATTEMPT($T, P_1, P_2, \text{flag}$)

```plaintext
begin
1. $\text{flag} := 0$;
2. FIRST_TETRAHEDRON($T, P_1, P_2, Q, t_F$);
3. $F(1, t_F) := P_1; F(2, t_F) := Q; \text{flag}1 := 1$;
4. while ($\text{flag}1 = 1$) do
   begin
   5. $t_F := t_F$;
   6. NEXT_TETRAHEDRON($T, P_1, P_2, t_F, t_C$);
end
```
7. \text{NEXT\_VERTEX}(t_P, t_C, P);
8. \text{if} (P \neq P_2) \text{ then}
   \begin{align*}
   &9. \quad \text{PREVIOUS\_VERTEX}(t_C, P_1, P_2, P, Q); \\
   &10. \quad t_F := t_C \\
   &\text{end}
   \end{align*}
\text{else}
   \begin{align*}
   &11. \quad Q := F(2, t_P); \ flag1 := 0 \\
   &\text{end}
   \end{align*}
12. \text{if} (F(1, t_P) = P_1) \text{ then } F(2, t_P) := Q;
13. \quad F(1, t_C) := Q; F(2, t_C) := P; flag2 := 1;
14. \text{while} (flag2 = 1) \text{ do}
   \begin{align*}
   &15. \quad \text{STRICT\_CONVEXITY}(t_C, t_P, flag2); \\
   &16. \quad \text{if} (flag2 = 1) \text{ then}
      \begin{align*}
      &17. \quad R := F(1, t_P); S := F(2, t_P); \\
      &18. \quad \text{COMMON\_VERTEX}(t_C, t_P, Q, S, U); \\
      &19. \quad \text{if} (F(1, t_C) = F(2, t_P)) \text{ then}
          \begin{align*}
          &20. \quad t_L := t_C; \\
          &21. \quad \text{TWO\_THREE}(T, t_C, t_P, P, R, Q, U); \\
          &22. \quad \text{if} (t_F = t_L) \text{ then } t_F := t_C; \\
          &23. \quad F(1, t_C) := R; Q := R \\
          &\text{end}
          \end{align*}
      &\text{else}
          \begin{align*}
          &24. \quad \text{FACET\_INTERSECT}(P, R, U, P_1, P_2, flag2); \\
          &25. \quad \text{if} (flag2 = 1) \text{ then}
              \begin{align*}
              &26. \quad t_L := t_C; \\
              &27. \quad \text{TWO\_THREE}(T, t_C, t_P, P, R, Q, U); \\
              &28. \quad \text{if} (t_F = t_L) \text{ then } t_F := t_C; \\
              &29. \quad F(1, t_C) := R; F(2, t_C) := F(2, t_P); \\
              &30. \quad F(1, t_P) := Q; F(2, t_P) := P; t_C := t_P
          \end{align*}
          \end{align*}
      \end{align*}
31. if \((flag2 = 1)\) then 
   begin
32. if \((R \neq P_1)\) then 
       begin
33. \(\text{PREVIOUS}_TETRAHEDRON}(T, P_1, P_2, t_C, t_P);\)
34. if \((F(1, t_P) = P_1)\) then \(F(2, t_P) := Q;\)
35. if \((P = P_2)\) then 
       begin 
6. \(Q := F(2, t_P); F(1, t_C) := Q\)
       end 
36. else 
       begin 
37. \(flag2 := 0;\)
38. if \((P = P_2)\) then \(flag := 1\)
   else 
       begin 
39. \(\text{NEXT}_TETRAHEDRON}(T, P_1, P_2, t_C, t_N);\)
40. \(F(2, t_C) := F(1, t_N)\)
       end 
   end 
end 
end 
end 
end 
end 
end 

Experiments show that procedure 3D_INSERT_ATTEMPT seldom succeeds in inserting a line segment. However, this may just be an indication that it is seldom possible to insert a line segment into a tetrahedralization. Let \(T, P_1, P_2, T^*\) be as above. In what follows, we show that for a certain kind of \(T^*\) the failure of the procedure simply signifies that the line segment can not be inserted into \(T\). Namely, we prove the following proposition.

**Proposition 1.** If points \(Q_1, Q_2\) exist, \(Q_1 \neq Q_2\), that are vertices of every tetrahedron in \(T^*\), then at the end of the execution of 3D_INSERT_ATTEMPT, variable \(flag\) equals 1 if
and only if the line segment can be inserted into $T$.

**Proof.** That $\text{flag}$ equal to 1 implies that the line segment can be inserted into $T$ follows trivially. Thus, it remains to be shown that if $\text{flag}$ equals zero then the line segment cannot be inserted into $T$.

For some positive integer $n$, let $t_i$, $i = 1, \ldots, n$, be the tetrahedra in $T^*$ in the order in which they are intersected by the line segment from $P_1$ to $P_2$.

At the end of the execution of the procedure let $T^{**}$ be the collection of tetrahedra in $T$ that are intersected by the relative interior of the line segment, and for some positive integer $m$, let $t'_i$, $i = 1, \ldots, m$, be the tetrahedra in $T^{**}$ in the order in which they are intersected by the line segment from $P_1$ to $P_2$.

Clearly, $n \geq m$, and since $\text{flag}$ equals zero it follows that $m \geq 3$.

Let $R_0 = P_1$, and, inductively, for each $i$, $i = 1, \ldots, n$, let $R_i$ be the vertex of $t_i$ different from $R_{i-1}$, $Q_1$, and $Q_2$. Similarly, points $R'_i$, $i = 0, \ldots, m$ are defined with respect to $t'_i$, $i = 1, \ldots, m$. Figure 2 illustrates an example of the facets of the tetrahedra in $T^*$ that do not have $Q_2$ as a vertex, and of $R_i$, $i = 0, \ldots, n$, $R'_i$, $i = 0, \ldots, m$, where $n$ equals 14 and $m$ equals 5.

We define a function $f$ from $\{0, \ldots, m\}$ into $\{0, \ldots, n\}$ in such a way that for each $i$, $i = 0, \ldots, m$, $R'_i$ equals $R_{f(i)}$. Based on this definition, for each $i$, $i = 1, \ldots, m$, we then define sets $W_i \subset \{R_0, \ldots, R_n\}$, by

$$W_i = \{R_{f(i-1)} = R'_{i-1}, R_{f(i-1)+1}, \ldots, R_{f(i)} = R'_i\}.$$ 

From the definition of $T^{**}$ it follows that given $i$, $2 \leq i \leq m$, the union of $t'_{i-1}$ and $t'_i$ is not a strictly convex hexahedron (step 15 of procedure). Thus, it is not possible to insert the line segment at the same time to have a new tetrahedron in $T$ with vertices $Q_1$, $R'_{i-2}$, $R'_{i-1}$, $R'_i$. The same is true for a tetrahedron with vertices $Q_2$, $R'_{i-2}$, $R'_{i-1}$, $R'_i$. From this and the fact that it is always true that $F(1, t_O)$ equals $F(2, t_P)$ in step 19 of the procedure, it follows that for each $i$, $i = 2, \ldots, m$, it is not possible to insert the line segment at the same time to have a new tetrahedron with one vertex equal to either $Q_1$ or $Q_2$, two vertices in $W_{i-1}$, and one vertex in $W_i \setminus \{R'_{i-1}\}$.

In what follows, we assume that the line segment can be inserted into $T$. Thus, we must assume that $T^*$ has been transformed in such a way that the line segment is one of its edges. Clearly, in the transformed $T^*$, which we denote by $T^*$, only one tetrahedron can have both $Q_1$ and $Q_2$ as vertices, namely the tetrahedron with vertices $Q_1$, $Q_2$, $P_1$, and $P_2$. All other tetrahedra with either $Q_1$ or $Q_2$ as a vertex have in addition three vertices of the form $R_j$, $R_k$, $R_l$, $0 \leq j < k < l \leq n$. 

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Figure 2: Facets of tetrahedra in $T^*$ that do not have $Q_2$ as a vertex, and points $R_i$, $i = 0, \ldots, n$, $R'_i$, $i = 0, \ldots, m$. 
For some integer $n', 1 \leq n' < n$, we define integers $h_i, l_i, i = 0, \ldots, n'$, as follows. We let $h_0$ and $l_0$ equal 0 and $n$, respectively. Inductively, given $i, i > 0$, we assume integers $h_{i-1}, l_{i-1}, 0 \leq h_{i-1} < l_{i-1} \leq n$, have been defined such that for integers $j, k, 1 \leq j < k \leq m, R_{h_{i-1}} \in W_j, R_{h_i} \in W_k, R_{h_{i-1}} \neq R_j, R_{l_{i-1}} \neq R_k$, and the triangle with vertices $Q_1, R_{h_{i-1}}, R_{l_{i-1}}$ is a facet of a tetrahedron in $T^*$. Then from the geometry of $T^*$ and the last fact about the triangle with vertices $Q_1, R_{h_{i-1}}, R_{l_{i-1}}$, it follows that integers $h_i, l_i$ exist, $h_{i-1} < h_i < l_i \leq l_{i-1}$, for which $R_{h_i} \in W_j, R_{l_i} \notin W_j$, and the tetrahedron with vertices $Q_1, R_{h_{i-1}}, R_{h_i}, R_{l_i}$ belongs to $T^*$. If $R_{l_i}$ belongs to $W_{j+1}$ then we let $n'$ equal $i$. That for some $i, 1 \leq i < n$, and some $j, 1 \leq j < m$, $R_{l_i}$ belongs to $W_{j+1}$, while $R_{h_{i-1}}, R_{h_i}$ belong to $W_j$, follows from the fact that $\{h_i\}$ is an increasing sequence of integers bounded above by $\{l_i\}$ which is itself a non-increasing sequence of integers. Thus, $n'$ is well defined. However, this is a contradiction, for it implies for some $j, 1 \leq j < m$, the existence of a tetrahedron in $T^*$ with one vertex equal to $Q_1$, two vertices in $W_j$, namely $R_{h_{i-1}}$ and $R_{h_i}$, and one vertex in $W_{j+1} \setminus \{R_j\}$, namely $R_{l_{i-1}}$. This completes the proof of the proposition.

Finally, in what follows we shed more light on the fundamental differences between the 2-dimensional and the 3-dimensional line insertion problems by identifying those 3-dimensional problems that can be approached algorithmically as 2-dimensional problems. In particular, we look at the following problem: Let $T$ be a triangulation in the $x - y$ plane of 3-dimensional space, let $P_1, P_2, P_1 \neq P_2$, be vertices in $T$ such that the line segment with endpoints $P_1, P_2$ is not an edge in $T$ and its relative interior does not contain any vertices in $T$, and let $T^*$ be the collection of triangles in $T$ that are intersected by the relative interior of the line segment. Let $\tilde{T}$ be a collection of contiguous tetrahedra in 3-dimensional space that have a vertex $\tilde{Q}$ in common and let $\tilde{T}$ be the collection of 2-dimensional triangles in 3-dimensional space that are the facets of the tetrahedra in $\tilde{T}$ that do not have $\tilde{Q}$ as a vertex. Assume that the perpendicular projection of $\tilde{T}$ onto the $x - y$ plane equals $T^*$ and that each triangle in $T^*$ is the perpendicular projection of only one triangle in $\tilde{T}$. Assume, in addition, that $P'_1, P'_2$ are the vertices in $\tilde{T}$ whose perpendicular projections onto the $x - y$ plane are $P_1, P_2$, respectively, and, without any loss of generality, for the purpose of executing 3D_INSERT_ATTEMPT with $\tilde{T}, P'_1, P'_2$ as input, that $\tilde{T}$ is a complete tetrahedralization for its vertices. What conditions must the tetrahedra in $\tilde{T}$ satisfy so that the relative interior of the line segment with endpoints $P'_1, P'_2$ lies in the interior of the union of the tetrahedra in $\tilde{T}$ and the line segment can then be inserted into $\tilde{T}$ with procedure 3D_INSERT_ATTEMPT in a manner that mimics exactly what the 2-dimensional algorithm (without the optimization steps) does when inserting into $T$ the line segment with endpoints $P_1, P_2$?

In order to answer the above question we further formalize the notation. For a positive
integer $n$, let $P_i, i = 1, \ldots, n$, be distinct points in the $x - y$ plane of 3-dimensional space, and for each $i, i = 1, \ldots, n$, let $x_i, y_i$ be the $x$- and $y$-coordinates, respectively, of $P_i$. Given a triangulation $T$ for the set of points $P_i, i = 1, \ldots, n$, we say that a collection $T'$ of distinct 2-dimensional triangles in 3-dimensional space is a triangulation that generalizes $T$ (a generalized triangulation for short) if numbers $z_i, i = 1, \ldots, n$, exist such that if points $P_i', i = 1, \ldots, n$, in 3-dimensional space are defined by setting $P_i'$ equal to $(x_i, y_i, z_i)$ for each $i, i = 1, \ldots, n$, then the set of vertices of $T'$ equals the set of points $P_i', i = 1, \ldots, n$, and the perpendicular projection onto the $x - y$ plane of $T'$ is $T$.

Let $P_i, P_i', x_i, y_i, z_i, i = 1, \ldots, n$, $T, T'$ be as above. Assume that the line segment with endpoints $P_1, P_2$ is not an edge in $T$ and that its relative interior does not contain any vertices in $T$. Let $T^*$ be the collection of triangles in $T$ that are intersected by the relative interior of this line segment, and let $\hat{T}$ be the collection of triangles in $T'$ whose perpendicular projection onto the $x - y$ plane is $T^*$. For arbitrarily large positive $\epsilon$ we let $\hat{Q}$ represent the point $(0, 0, \epsilon)$, and $\hat{T}$ the collection of tetrahedra obtained by computing the convex hulls of $\hat{Q}$ together with each of the triangles in $\hat{T}$. In what follows, we say that the line segment with endpoints $P_1', P_2'$ can be inserted into $\hat{T}$ if a collection of tetrahedra $\hat{T}$ exists such that the tetrahedra in $\hat{T}$ have pair-wise disjoint interiors, the relative interior of the line segment is contained in the interior of the union of the tetrahedra in $\hat{T}$, the line segment is an edge in $\hat{T}, \hat{T}$ and $\hat{T}$ have the same set of vertices, and the union of the tetrahedra in $\hat{T}$ equals the union of the tetrahedra in $\hat{T}$. Accordingly, we say that the line segment with endpoints $P_1', P_2'$ can be inserted into the positive side of $T'$ if it can be inserted into $\hat{T}$. Based on these definitions, we notice that if the line segment with endpoints $P_1', P_2'$ satisfies the prerequisite for insertion into $\hat{T}$, i.e., its relative interior lies entirely in $\hat{T}$ and does not intersect any edges of tetrahedra in $\hat{T}$, then an attempt can be made to insert it into $\hat{T}$ with procedure 3D_INSERT_ATTEMPT even though $\hat{T}$ is not necessarily a complete tetrahedralization for its vertices.

We assume that the line segment with endpoints $P_1', P_2'$ satisfies the prerequisite for insertion into $\hat{T}$, that procedure INSERT_SEGMENT (without the optimization steps) has been executed for inserting into $T$ the line segment with endpoints $P_1, P_2$, and that procedure EDGE_SWAP (step 17 of INSERT_SEGMENT) has been executed $m$ times during the insertion. Similarly, we assume that procedure 3D_INSERT_ATTEMPT has been executed for attempting to insert into $\hat{T}$ the line segment with endpoints $P_1', P_2'$ and that procedure TWO_THREE (steps 21 and 27 of 3D_INSERT_ATTEMPT) has been executed $m'$ times during the attempt.

We define functions $a, e$ from $\{1, \ldots, m\}$ into $\{(i, j) : 1 \leq i < j \leq n\}$ as follows: Given $l, 1 \leq l \leq m$, we set $a(l)$ and $e(l)$ equal to $(h, k)$ and $(q, r)$, respectively, where $h, k, q,$ $r, 1 \leq h < k \leq n, 1 \leq q < r \leq n,$ are those integers for which after the $l^{th}$ execution of
EDGE_SWAP in INSERT_SEGMENT the edge with endpoints $P_h$, $P_k$ is the new edge in the triangulation and the edge with endpoints $P_q$, $P_r$ is the edge that has been eliminated. Correspondingly, assuming $m' > 0$, we also define functions $a'$, $e'$ from $\{1, \ldots, m'\}$ into $\{(i, j) : 1 \leq i < j \leq n\}$ as follows: Given $l$, $1 \leq l \leq m'$, we set $a'(l)$ and $e'(l)$ equal to $(h, k)$ and $(q, r)$, respectively, where $h, k, q, r$, $1 \leq h < k \leq n$, $1 \leq q < r \leq n$, are those integers for which after the $l$th execution of TWO_THREE in 3D_INSERT_ATTEMPT the edge with endpoints $P'_h$, $P'_k$ is the edge that the three new tetrahedra have in common and the edge with endpoints $P'_q$, $P'_r$ is the edge that the two eliminated tetrahedra had in common and that does not have $Q$ as an endpoint. Clearly, $a(m)$ equals $(1, 2)$, and if 3D_INSERT_ATTEMPT is successful then $m' > 0$ and $a'(m')$ also equals $(1, 2)$.

Finally, in what follows, given integers $h, k, q, r$, $1 \leq h < k \leq n$, $1 \leq q < r \leq n$, we say that $(h, k)$ crosses $(q, r)$ if the relative interiors of the line segment with endpoints $P_h$, $P_k$ and the line segment with endpoints $P_q$, $P_r$, have one and only one point in common. Assuming $(h, k)$ crosses $(q, r)$, let $\hat{x}$, $\hat{y}$ be the $x -$ and $y -$coordinates, respectively, of the point at which the line segment with endpoints $P_h$, $P_k$ intersects the line segment with endpoints $P_q$, $P_r$, and let $\hat{z}_{hk}$, $\hat{z}_{qr}$ be the numbers for which the points defined by $(\hat{x}, \hat{y}, \hat{z}_{hk})$, $(\hat{x}, \hat{y}, \hat{z}_{qr})$ belong, respectively, to the line segment with endpoints $P'_h$, $P'_k$, and the line segment with endpoints $P'_q$, $P'_r$. Based on these definitions, we say then that $(h, k)$ is below $(q, r)$ if $\hat{z}_{hk} < \hat{z}_{qr}$. We are now ready to answer the question formulated above.

**Proposition 2.** The line segment with endpoints $P'_1$, $P'_2$ satisfies the prerequisite for insertion into $\hat{T}$, $m$ equals $m'$, and for each integer $l$, $l = 1, \ldots, m$, $a(l)$ equals $a'(l)$, and $e(l)$ equals $e'(l)$ so that the line segment can be inserted into $\hat{T}$ if and only if for each integer $l$, $l = 1, \ldots, m$, $e(l)$ is below $a(l)$.

**Proof.** The ‘only if’ part follows easily. In order to prove the ‘if’ part it suffices to prove that for each integer $l$, $l = 1, \ldots, m$, $e(l)$, which obviously crosses $(1, 2)$, is below $(1, 2)$. This will imply that the line segment satisfies the prerequisite for insertion in $\hat{T}$, and that $\text{flag2}$ always equals 1 in step 25 of 3D_INSERT_ATTEMPT (after the execution of procedure FACET_INTERSECT in step 24).

Let $T^*$ be as defined above, and let $T_0$ equal $T^*$. Inductively, for each $l$, $l = 1, \ldots, m$, let $T_l$ be the collection of triangles in the $x$ -- $y$ plane of 3-dimensional space which is the transformation of $T_{l-1}$ after the $l$th edge swap.

Let $\hat{T}$ be as defined above. For each $l$, $l = 0, \ldots, m$, let $\hat{T}_l$ be the collection of distinct 2-dimensional triangles in 3-dimensional space whose perpendicular projection onto the $x$ -- $y$ plane equals $T_l^*$, and whose set of vertices equals that of $\hat{T}$. 

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For each $l$, $l = 0, \ldots, m$, we define a real-valued function $f_l$ with domain the union of the triangles in $T^*$ as follows. Given a point $P$ in a triangle in $T^*$ we let $\hat{x}$, $\hat{y}$ be the $x-$ and $y-$coordinates, respectively, of $P$, and let $f_l(P)$ be the unique number for which the point defined by $(\hat{x}, \hat{y}, f_l(P))$ belongs to a triangle in $\hat{T}_l$.

Given an integer $l$, $1 \leq l \leq m$, let $h$, $k$, $q$, $r$, $1 \leq h < k \leq n$, $1 \leq q < r \leq n$, be those integers for which $a(l)$ equals $(h, k)$ and $e(l)$ equals $(q, r)$. By definition $T^*_l$ is the transformation of $T^*_{l-1}$ obtained by replacing the edge with endpoints $P_q, P_r$ by the edge with endpoints $P_h, P_k$. Clearly, the replaced edge is shared by two triangles in $T^*_{l-1}$ whose union is a strictly convex quadrilateral and the new edge is the alternative diagonal of this quadrilateral. These observations and the fact that $e(l)$ is below $a(l)$ imply that $f_{l-1}$ equals $f_l$ everywhere except in the relative interior of the aforementioned quadrilateral in which $f_{l-1}$ is strictly less than $f_l$. In particular, given a point $P$ in the relative interior of the replaced edge, it then follows that $f_{l-1}(P) < f_l(P)$. Thus, since the edge with endpoints $P_1, P_2$ belongs to $T^*_m$, given an integer $l$, $1 \leq l \leq m$, and a point $P$ which is the intersection of the edge with endpoints $P_1, P_2$ and the edge replaced in $T^*_{l-1}$ during the $l$th edge swap, it must follow that $f_{l-1}(P) < f_l(P) \leq f_m(P)$. Hence, $e(l)$ is below $(1, 2)$ and the proof of the proposition is complete.

We notice that Proposition 2 provides conditions for identifying 3-dimensional problems that can be approached algorithmically as if they are 2-dimensional. However, it does not provide a method or procedure for selecting the vertices of $\hat{T}$ so that these conditions are satisfied. In what follows, we describe one such method that is based on the order in which the 2-dimensional swapping of edges occurs. As will be pointed out below, the implementation of this method requires that procedures INSERT_SEGMENT and 3D_INSERT_ATTEMPT be somewhat modified.

Let $P$ and $Q$ be variables as they appear in procedure INSERT_SEGMENT. We define variable $W$ as the pair $(P, Q)$. During the execution of INSERT_SEGMENT, $W$ takes on different values, each value being taken on by $W$ only once. Let $W'$ be one such value and assume that while $W$ equals $W'$ procedure EDGE_SWAP in step 17 of INSERT_SEGMENT is executed at least once. Accordingly, we notice that $W$ ceases to equal $W'$ in one of two ways: when an execution of procedure STRICT_CONVEXITY returns a value of zero for flag2, and when $F(1, t_C)$ equals $F(2, t_P)$ in step 19 of INSERT_SEGMENT. When the first possibility occurs it signifies that while $W$ equaled $W'$ the swapping of edges (through the execution of EDGE_SWAP) did not lead to the creation of a triangle that does not intersect the line segment with endpoints $P_1, P_2$ (since the second possibility never occurred). Thus, we can think of the swapping of edges as being unnecessary while $W$ equals a value for which the first
possibility occurs, and point out that procedure INSERT SEGMENT can be modified so that such a value can be identified as soon as $W$ is set to it (before any swapping of edges occurs) and $W$ can then be set to its next value. Correspondingly, we can define variable $W$ in the same manner with respect to variables $P$ and $Q$ in procedure 3D INSERT ATTEMPT. Here, again assuming that $W'$ is one of the values that $W$ takes on, we consider the possibility that $W$ ceases to equal $W'$ because an execution of procedure STRICT CONVEXITY in step 15 of 3D INSERT ATTEMPT returns a value of zero for flag2. In a manner similar to what we did for the 2-dimensional algorithm, we point out that procedure 3D INSERT ATTEMPT can be modified so that such a value can be identified as soon as $W$ is set to it (before any executions of procedure TWO THREE in step 27 of 3D INSERT ATTEMPT take place while $W$ is set to the value) and $W$ can then be set to its next value.

In what follows, we assume that INSERT SEGMENT and 3D INSERT ATTEMPT have been modified as just described. We also assume that $T$, $T^*$, $n$, $P_i$, $P'_i$, $i = 1, \ldots, n$, $\hat{T}$, $m$, $a$, $e$ are defined as above and that the modified version of INSERT SEGMENT (without the optimization steps) has been executed for inserting into $T$ the line segment with endpoints $P_1, P_2$. We let $S^*$ be the set of vertices of $T^*$, and without any loss of generality we let $P_3$ be the point in $S^*$ which is the last value that $Q$ takes on during the execution of INSERT SEGMENT (obtained through the execution of THIRD VERTEX in step 31). We let $n^*$ be the number of points in $S^*$ and notice that $1 \leq n^* - 3 \leq m$ since for each point in $S^*$ different from $P_1, P_2, P_3$, one execution of EDGE SWAP with $Q$ equal to the point takes place that produces a triangle $T_P$ that does not intersect the line segment with endpoints $P_1, P_2$. We define a one-to-one function $g$ from $\{1, \ldots, n^*\}$ onto $S^*$ by setting $g(n^*)$, $g(n^* - 1)$, $g(n^* - 2)$ equal to $P_1, P_2, P_3$, respectively, and by setting for each $i$, $i = 1, \ldots, n^* - 3$, $g(i)$ equal to the point in $S^*$ which is the value of $Q$ the $i^{th}$ time during the execution of INSERT SEGMENT that the execution of EDGE SWAP produces a triangle $T_P$ that does not intersect the line segment with endpoints $P_1, P_2$. In addition, we define functions $\text{init}$, $\text{fin}$ from $\{1, \ldots, n^* - 3\}$ into $\{1, \ldots, m\}$, by setting for each $i$, $i = 1, \ldots, n^* - 3$, $\text{init}(i)$ equal to the positive integer $l$ for which the $l^{th}$ time EDGE SWAP is executed it is also the first time it is executed with $Q$ equal to $g(i)$, and $\text{fin}(i)$ equal to the positive integer $k$ for which the $k^{th}$ time EDGE SWAP is executed it is also the last time it is executed with $Q$ equal to $g(i)$.

Assume that procedure INSERT SEGMENT has been further modified so that it computes and returns variables that correspond to $m$, $a$, $e$, $n^*$, $g$, $\text{init}$, $\text{fin}$. Based on this information, we now describe a procedure for computing for each integer $i$, $i = 1, \ldots, n^*$, a number $z_i$ that can be used as the $z$-coordinate for the vertex of $\hat{T}$ that corresponds to $g(i)$ so that for each integer $l$, $l = 1, \ldots, m$, $e(l)$ is below $a(l)$. Essentially, the procedure consists of two steps. In the first step, the numbers $z_{n^*-2}$, $z_{n^*-1}$, $z_{n^*}$ are selected arbitrarily from the set of real numbers. In the second step, inductively, for each $i$, $i = 1, \ldots, n^*-3$, assuming
that the numbers $z_{i+1}, \ldots, z_{n^*}$ have been computed, $z_i$ is selected arbitrarily from the set of possible values of $z_i$ for which $e(l)$ is below $a(l)$ for each integer $l$, $l = \text{ini}(i), \ldots, \text{fin}(i)$. In order to see that this set can be computed under the given assumption in a manner consistent with the required conditions, it suffices to show that its computation depends solely on $z_{i+1}, \ldots, z_{n^*}$. For this purpose, let $l, h, k, q, r$, $\text{ini}(i) \leq l \leq \text{fin}(i)$, $1 \leq h < k \leq n$, $1 \leq q < r \leq n$, be integers such that $a(l)$ equals $(h, k)$ and $e(l)$ equals $(q, r)$. Then from the definitions of $g$ and $e$, $g(i)$ must equal either $P_q$ or $P_r$, and assuming without any loss of generality that $g(i)$ equals $P_q$, then from the definitions of $g$, $a$, and $e$, integers $i_1, i_2, i_3$ must exist, $i + 1 \leq i_1, i_2, i_3 \leq n^*$, such that $P_h$ equals $g(i_1)$, $P_k$ equals $g(i_2)$, and $P_r$ equals $g(i_3)$. Thus, the computation of the set of possible values of $z_i$ for which $e(l)$ is below $a(l)$ depends solely on $z_{i+1}, \ldots, z_{n^*}$, and the assertion follows.

Figure 3 can be used to illustrate the way in which information must be gathered in order to compute the numbers $z_i, i = 1, \ldots, n^*$, as just described. Triangulation (i) in Figure 3 is the initial triangulation into which the line segment with endpoints $P_1, P_2$ is to be inserted, triangulation (vii) is the desired triangulation obtained from the execution of the modified version of INSERT_SEGMENT, and triangulations (ii)-(vi) are intermediate triangulations in the order in which they are computed during the execution of INSERT_SEGMENT. Each triangulation, except triangulation (i), corresponds to one instance of the swapping of edges, the dotted edge in each triangulation being the one that is eliminated. It is easy to see that for this example $m$ equals 6, $n^*$ equals 8, the values for $e(i)$, $i = 1, \ldots, 6$, are $(5, 6), (4, 8), (4, 7), (5, 7), (3, 8), (3, 7)$, respectively, and for $a(i)$, $i = 1, \ldots, 6$ are $(1, 7), (3, 7), (3, 5), (1, 3), (2, 7), (1, 2)$, respectively. In addition, from triangulations (ii), (iv)-(vii), respectively, the values for $g(i)$, $i = 1, \ldots, 5$, are obtained (the edge swapping depicted in triangulation (iii) does not produce a triangle that is not intersected by the line segment) and they are, respectively, the points $P_6, P_4, P_5, P_8, P_7$. Finally, we notice that for this example the values for $\text{ini}(i)$, $i = 1, \ldots, 5$, are, respectively, 1, 2, 4, 5, 6, and for $\text{fin}(i)$, $i = 1, \ldots, 5$, are, respectively, 1, 3, 4, 5, 6 ($P_4$, the value for $g(2)$, is the only point to which two instances of the swapping of edges are assigned: those depicted in triangulations (iii) and (iv)). Based on this information, the numbers $z_i, i = 1, \ldots, 8$, are then computed as described above.
Figure 3: A triangulation and the swapping of edges during the insertion into it of line segment with endpoints $P_1, P_2$. 
References


