THE CATHOLIC UNIVERSITY OF AMERICA

BEHAVIOR OF K-DIMENSIONAL CONVEXITY MODULI

A Dissertation
Submitted to the Faculty of the
School of Arts and Sciences
Of the Catholic University of America
In partial fulfillment of the requirements
For the Degree
Doctor of Philosophy

by

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Washington, DC
1980

a mi madre

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ACKNOWLEDGEMENTS

First and foremost, I would like to express my deep appreciation to Professor Francis Sullivan for his help and cooperation in the preparation of this thesis and for his support and friendship during the past seven years. It is due to his constant advise and encouragement that this dissertation has reached completion.

Also, I am sincerely grateful to Professor James Hagler for his encouragement throughout the years, and to Professor Parfeny Saworotnow for reading this dissertation.

INTRODUCTION

A satisfactory discussion of Banach space theory must include the analysis of well known convexity properties. An in-depth study of these concepts will reveal some of the connections between the geometry of Banach spaces and other areas of functional analysis. It is the objective of this paper to obtain new insight into this subject by studying the asymptotic behavior of finite dimensional subspaces of Banach spaces from the point of view of multidimensional moduli of convexity.

A Banach space is said to satisfy the fixed point property whenever every mapping which does not increase distances from a weakly compact subset of the space into itself has a fixed point. Among the questions of special interest we will address will be those relating to the evaluation and limiting behavior of convexity moduli, and to the existence of structures under which a Banach space has the fixed point property. We will be concerned with determining convexity conditions that imply normal structure. A Banach space has normal structure if every bounded convex subset of the space contains a point whose maximum distance to any other point of the subset is strictly less than its diameter. Since W. A. Kirk has proved this property implies the fixed point property [18], we will also examine particular examples of superreflexive spaces that satisfy the fixed point property but do not have normal structure, and show they still have some kind of structure.

A convexity condition easily seen to imply normal structure is uniform convexity. This property was first introduced by J. A. Clarkson in 1936 [2]. Independently from Kirk, F. E. Browder showed it satisfies the fixed point property [1]. Geometrically, a uniformly convex Banach space uniformly does not have arbitrarily flat arcs on the surface of the unit ball of length arbitrarily close to any given fixed positive number. Since, intuitively, length can be thought of as a measurement of the relative positions of two points with respect to each other, we will generalize uniform convexity to a property that utilizes the multiple-point concept of area as studied by E. Silverman [19] and F. Sullivan [20]. This area notion will be based on the usual definitions of areas and volumes in R^3 which can be obtained by calculating the determinant of certain matrices [3]. The property generalizing uniform convexity will be referred to as k-uniform convexity, where k can be any positive integer, and it will be equivalent to uniform convexity in case k is equal to 1 .

The first chapter of this paper deals with the behavior of moduli of convexity in Hilbert spaces. Along with some fundamental results, the definitions of k-uniform convexity and the corresponding modulus of convexity are introduced in this chapter. For a given positive integer k , k-uniform convexity of a Banach space will imply the uniformly nonexistance of arbitrarily flat k+1-dimensional convex regions on the unit sphere of the space of "area" arbitrarily close to any given fixed positive number. In a Banach space with this property the distance from the origin of the space to the centroid of any k+1-dimensional convex hull with vertices on the unit sphere and a given fixed positive "area" is at most one minus the modulus of k-uniform convexity corresponding to the given "area." Also, the modulus of k-uniform convexity is the largest positive number satisfying this condition. Given these definitions, we then evaluate the modulus of k-uniform convexity of Hilbert spaces corresponding to any given positive "area" for any given positive integer k , and show that this modulus approaches one as k goes to infinity. This will mean that the centroids of all k+1-dimensional convex hulls in the unit ball of a

Hilbert space with a given fixed positive "area" collapse into the origin of the space as k gets arbitrarily large.

In Chapter II convexity considerations play a central role. It contains the definitions of two important area-related conditions; properties A and B. The former is the stronger of the two. These properties will require a uniform behavior by all subspaces of a Banach space with some fixed finite dimension. Geometrically, if this dimension is a positive integer k, they will correspond to the definition of a round unit ball from the point of view of k-dimensionality. Following the sophisticated argument employed by R. C. James to prove that uniformly nonsquare Banach spaces are superreflexive [12], we show that property B also implies superreflexivity. In order to have a complete treatment of the notion of superreflexivity in the context of properties A and B, we include the proof of a theorem by D. van Dulst and A. J. Pach [6]. A consequence of this is that superreflexivity does not imply either A or B. Another result in this chapter shows that a Banach space has property B if its norm is close enough to an equivalent norm for which the moduli of k-uniform convexity converge to one as k goes to infinity. We use this result together with the main theorem of Chapter I to prove that the space treated in an example by L. A. Karlovitz [17] has property B even though it does not have property A. Next, a main result of this chapter shows that property A implies the space has normal structure, from which it follows that it must have the fixed point property [18]. This makes property A the most general known convexity condition for a space to have this type of structure. Finally, in the remainder of the chapter, we present a result that gives a characterization of property B in terms of certain structures related to normal structures. Since Karlovitz's space satisfies the fixed point property but does not have normal structure, this shows it still has some kind of structure, namely property B.

In Chapter III, which is mostly independent of Chapters I and II, we turn our attention to the definition of locally k-uniformly convex Banach spaces. The principal result of this chapter is that a sufficient condition for a Banach space to be reflexive is that its second dual be locally k-uniformly convex for any $k \geq 1$.

The last chapter contains a list of open questions or problems which we feel are closely related to the main subject of this paper.

The terminology used throughout this paper is standard. It should be the same as in Dunford and Schwartz [7]. The following are exceptions: S_X and B_X will denote the unit sphere and the unit ball of a Banach space X respectively; $[x_1, \ldots, x_k]$ will be the affine span of $x_1, \ldots, x_k$, i.e. if $z \in [x_1, \ldots, x_k]$ then $z = \sum_{i=1}^{k} \lambda_i x_i$, where $\sum_{i=1}^{k} \lambda_i = 1$; and $\text{dist}(x_{k+1}, [x_1, \ldots, x_k])$ will mean the distance between x_{k+1} and $[x_1, \ldots, x_k]$. The numbering of definitions, lemmas, theorems, and propositions is done in the order of appearance and there is no discrimination among them. Thus, for example, Lemma 1.10 is the tenth numbered item of Chapter I. The nine previous items may include definitions, lemmas, theorems, and propositions.

CHAPTER I

Moduli of Convexity in Hilbert Spaces

Among Banach spaces, Hilbert spaces have the most regular properties. In this chapter we confine our attention to Hilbert spaces, and prove, in Theorem 1.11, they satisfy the following regular property: By picking k arbitrarily large, the centroids of all k-dimensional convex hulls in the unit ball with a given fixed positive "area" can be made arbitrarily close to the origin of the space.

First some definitions and fundamental results are essential.

Definition 1.1: A Banach space X is said to be uniformly convex if given $\varepsilon > 0$ there exists $\delta > 0$ such that $||x-y|| < \varepsilon$ whenever $\frac{||x+y||}{2} > 1 - \delta$, and $x, y \in S_X$.

In a uniformly convex Banach space not only is it impossible to find two distinct points on the surface of the unit ball for which the segment joining them is entirely contained in the unit sphere; but, regardless of which two points are chosen, by knowing the length of the segment we are able to conclude that the norm of the midpoint of the segment does not exceed some given fixed number between zero and one.

In order to generalize this two-point convexity property to one utilizing any arbitrary finite number of points, a new concept had to be created involving the determination of some type of measurement caused by the relative positions of any given finite number of points. With this purpose in mind, the notion of area was bound to become the intuitively obvious choice.

Definition 1.2: Let X be a Banach space. If k is an integer, $k > 1$, $x_1, \ldots, x_k \in X$, then the area of the convex hull determined by $x_1, \ldots, x_k$, is the nonnegative number $\frac{1}{(k-1)!} \cdot A(x_1, \ldots, x_k)$, where

$$A(x_1, \ldots, x_k) \equiv \sup \left\{ \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_k) \\ \vdots & & \vdots \\ f_{k-1}(x_1) & \cdots & f_{k-1}(x_k) \end{vmatrix} : f_1, \ldots, f_{k-1} \in S_{X^*} \right\} .$$

Here, and throughout this paper, $|\cdot|$ will denote the determinant. We will call $A(x_1, \ldots, x_k)$ the "area" determined by $x_1, \ldots, x_k$.

The next two lemmas are the backbone of the intuition employed in defining the concept of areas. The first lemma is due to Geremia and Sullivan [10]. Its proof appears in [9].

Lemma 1.3: If X is a Banach space, n an integer, $n > 2$, $x_1, \ldots, x_n \in X$, then

$$A(x_1, \ldots, x_n) \geq A(x_1, \ldots, x_{n-1}) \cdot \operatorname{dist}(x_n, [x_1, \ldots, x_{n-1}]) .$$

In the next lemma we show that in Hilbert spaces equality always holds in the above relation [3].

Lemma 1.4: If X is a Hilbert space, n an integer, $n > 2$, $x_1, \ldots, x_n \in X$, then

$$A(x_1, \ldots, x_n) = A(x_1, \ldots, x_{n-1}) \cdot \operatorname{dist}(x_n, [x_1, \ldots, x_{n-1}]) .$$

Proof: Since the notation becomes quite involved for large n, we only present the proof for $n = 3$.

Let $A, B, C \in X$.

Without any loss of generality we may assume C is the zero element of X , and $X = R^2$.

Suppose $A = (a_1,a_2)$, $B = (b_1,b_2)$, $C = (0,0)$, $f_1 = (1,0)$, $f_2 = (0,1)$, and θ is the angle between A and B .

Letting $d = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, it follows that

$$||B-C|| \cdot \mathrm{dist}(A, [B, C]) = |A|\ |B| \sin\theta = |A \times B| = |d| .$$

Since $d = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} f_1(A) & f_1(B) \\ f_2(A) & f_2(B) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ f_1(A) & f_1(B) & f_1(C) \\ f_2(A) & f_2(B) & f_2(C) \end{vmatrix}$

it can be shown easily that

$$|d| = \sup \left\{ \begin{vmatrix} 1 & 1 & 1 \\ g_1(A) & g_1(B) & g_1(C) \\ g_2(A) & g_2(B) & g_2(C) \end{vmatrix} : g_1, g_2 \quad S_{X^*} \right\}$$

and the proof is complete.

Clearly from this lemma, if x_1, x_2, x_3 belong to a Hilbert space, then $\frac{1}{2!} A(x_1, x_2, x_3)$ is the usual area of the triangle determined by x_1, x_2, x_3 . Hence, our definition of area generalizes the well known elementary concepts of area and volume. Intuitively, from the last two lemmas we may conclude that the area of the convex hull determined by $x_1, \ldots, x_n$ is not less than $\frac{1}{n-1}$ times the product of its base, the area of the convex hull determined by $x_1, \ldots, x_{n-1}$, and its height, the distance from x_n to the affine span of $x_1, \ldots, x_{n-1}$. Therefore, $\frac{1}{(n-1)!} \cdot A(x_1, \ldots, x_n)$ makes sense as a definition of area.

The following property, introduced and examined by Sullivan [20], generalizes uniform convexity. It utilizes the concept of area defined above.

Definition 1.5: A Banach space X is said to be k-uniformly convex, (k-UR), for some positive integer k, if given $\varepsilon > 0$ there exists $\delta > 0$ such that $A(x_1, \ldots, x_{k+1}) < \varepsilon$ whenever $\dfrac{||x_1 + \ldots + x_{k+1}||}{k+1} > 1 - \delta$, and $x_1, \ldots, x_{k+1} \in S_X$.

From this definition it is easy to see that a Banach space is uniformly convex if and only if it is 1-UR.

The following simple proposition is related to the main result of this chapter.

Proposition 1.6 (Sullivan [20]): If for some positive integer k a Banach space X is k-UR, then it is $k+1$ - UR.

Proof: Let $\varepsilon > 0$ be given.

By the definition of k-UR pick $\delta > 0$ such that if $y_1, \ldots, y_{k+1} \in S_X$ then $A(y_1, \ldots, y_{k+1}) < \dfrac{\varepsilon}{k+2}$ whenever $\dfrac{||y_1 + \ldots + y_{k+1}||}{k+1} > 1 - \delta$.

We show that $A(x_1, \ldots, x_{k+2}) < \varepsilon$ if $x_1, \ldots, x_{k+2} \in S_X$, and

$$\frac{||x_1 + \ldots + x_{k+2}||}{k+2} > 1 - \left(\frac{k+1}{k+2}\right)\delta .$$

This will imply X is $k+1$ - UR.

It is clear then, by the triangle inequality, that for each i , $1 \leq i \leq k+2$,

$$\frac{||x_i||}{k+2} + \frac{||x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_{k+2}||}{k+2} > 1 - \left(\frac{k+1}{k+2}\right)\delta$$

so

$$\frac{||x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_{k+2}||}{k+2} > 1 - \left(\frac{k+1}{k+2}\right)\delta - \frac{1}{k+2} .$$

Multiplying both sides of this inequality by $\frac{k+2}{k+1}$ we get for each i , $1 \leq i \leq k+2$,

$$\frac{||x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_{k+2}||}{k+1} > \frac{k+2}{k+1} - \delta - \frac{1}{k+1} = 1 - \delta .$$

Hence, by the choice of δ we must have

$$A(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+2}) < \frac{\varepsilon}{k+2} .$$

Let $f_1, \dots, f_{k+1}$ belong to the unit sphere of X^* .

Then by the theory of determinants

$$\begin{vmatrix} 1 & \dots & 1 \\ f_1(x_1) & \dots & f_1(x_{k+2}) \\ \vdots & & \vdots \\ f_{k+1}(x_1) & \dots & f_{k+1}(x_{k+2}) \end{vmatrix} = \sum_{i=1}^{k+2} (-1)^i\, f_1\,(x_i)\, d_i \leq \sum_{i=1}^{k+2} |(-1)^i|\; ||f_1||\; |d_i|$$

$$= \sum_{i=1}^{k+2} |d_i|$$

$$\text{where} \quad d_i = \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ f_2(x_1) & \cdots & f_2(x_{i-1}) & f_2(x_{i+1}) & \cdots & f_2(x_{k+2}) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{k+1}(x_1) & \cdots & f_{k+1}(x_{i-1}) & f_{k+1}(x_{i+1}) & \cdots & f_{k+1}(x_{k+2}) \end{vmatrix} .$$

By taking the supremum over all choices of $f_1, \ldots, f_{k+1}$, it is then easy to deduce from this inequality that

$$A(x_1, \ldots, x_{k+2}) \leq \sum_{i=1}^{k+2} A(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+2})$$

$$< (k+2) \cdot \frac{\varepsilon}{k+2} = \varepsilon$$

which is the desired result. Hence X must be $k+1$ - UR .

It is clear from this proposition that if a Banach space is k-UR for some k then it is n-UR for $n > k$. Also, from its proof it might seem that as n grows, some of the centroids of n-dimensional convex hulls with vertices on the unit sphere and a given fixed positive "area" approach the surface of the unit ball. This is not necessarily the case. In Hilbert spaces the opposite occurs: They collapse into the origin of the space.

Before proving this surprising special property, a new definition and some preliminary lemmas are in order.

<u>Definition 1.7</u>: The modulus of k-uniform convexity of a Banach space X is the nonnegative function δ_k defined by

$$\delta_k(\varepsilon) = \inf \{1 - \frac{||x_1 + \ldots + x_{k+1}||}{k+1} : x_1, \ldots, x_{k+1} \in S_X \quad ,$$

$$\text{and} \quad A(x_1, \ldots, x_{k+1}) \geq \varepsilon\} \ .$$

Lemma 1.8: Let X be a Hilbert space, ε a given positive number. If $x_1, \ldots, x_k \in S_X$, $A(x_1, \ldots, x_k) = \varepsilon$, and for some $a > 0$, for all i,j, $1 \leq i,j \leq k$, $i \neq j$, $||x_i - x_j|| = a$, then $\frac{1}{2} a^2 = \frac{\varepsilon^{2/(k-1)}}{k^{1/(k-1)}}$, and

$$\frac{k-i}{k-i+1} \left|\left| x_i - \frac{x_{i+1} + \ldots + x_k}{k-1} \right|\right|^2 = \frac{1}{2} a^2 \quad \text{for all} \quad i \ , \ 1 \leq i \leq k-1 \ .$$

Proof: We first show the second assertion by induction on $k-i$.

For $k-i = 1$:

$$\frac{1}{2} ||x_{k-1} - x_k||^2 = \frac{1}{2} a^2 \ .$$

For $1 \leq k-i \leq k-2$ assume it is true for $k-i$. We show it then holds for $k-i+1$.

By the induction hypothesis then

$$\frac{k-i}{k-i+1} \left|\left| x_i - \frac{x_{i+1} + \ldots + x_k}{k-i} \right|\right|^2 = \frac{1}{2} a^2$$

and we would like to show that

$$\frac{k-i+1}{k-i+2} \left|\left| x_{i-1} - \frac{x_i + \ldots + x_k}{k-i+1} \right|\right|^2 = \frac{1}{2} a^2 \ .$$

Since $\frac{x_i + \ldots + x_k}{k-i+1} = \frac{1}{k-i+1} x_i + \frac{k-i}{k-i+1} \left(\frac{x_{i+1} + \ldots + x_k}{k-i} \right)$

then $\left|\left| \frac{x_i + \ldots + x_k}{k-i+1} - x_i \right|\right| = \frac{k-i}{k-i+1} \left|\left| \frac{x_{i+1} + \ldots + x_k}{k-i} - x_i \right|\right|$

and $$\left\| \frac{x_i + \dots + x_k}{k-i+1} - x_i \right\|^2 = \left(\frac{k-i}{k-i+1} \right)^2 \left\| \frac{x_{i+1} + \dots + x_k}{k-i} - x_i \right\|^2$$

$$= \left(\frac{k-i}{k-i+1} \right) \cdot \frac{1}{2} a^2 \ .$$

Because $\{x_1, \dots, x_k\}$ is a set of equidistant points we have by the Pythagorean theorem that

$$||x_i - x_{i-1}||^2 = \left\| x_{i-1} - \frac{x_i + \dots + x_k}{k-i+1} \right\|^2 + \left\| \frac{x_i + \dots + x_k}{k-i+1} - x_i \right\|^2$$

giving

$$\left\| x_{i-1} - \frac{x_i + \dots + x_k}{k-i+1} \right\|^2 = ||x_i - x_{i-1}||^2 - \left\| \frac{x_i + \dots + x_k}{k-i+1} - x_i \right\|^2$$

$$= a^2 - \left(\frac{k-i}{k-i+1} \right) \frac{1}{2} a^2$$

$$= \left(2 - \left(\frac{k-i}{k-i+1} \right) \right) \frac{1}{2} a^2$$

$$= \left(\frac{k-i+2}{k-i+1} \right) \frac{1}{2} a^2$$

and hence

$$\frac{k-i+1}{k-i+2} \left\| x_{i-1} - \frac{x_i + \dots + x_k}{k-i+1} \right\|^2 = \frac{1}{2} a^2$$

as required.

From this notice that

$$\left(\frac{1}{2} a^2 \right)^{k-1} = \frac{k-1}{k} \left\| x_1 - \frac{x_2 + \dots + x_k}{k-1} \right\|^2 \cdot$$

$$\frac{k-2}{k-1} \left\| x_2 - \frac{x_3 + \dots + x_k}{k-2} \right\|^2 \cdot \dots \cdot \frac{1}{2} ||x_{k-1} - x_k||^2$$

$$= \frac{1}{k}\left\|\, x_1 - \frac{x_2 + \dots + x_k}{k-1}\right\|^2 \cdot \dots$$

$$\cdot \left\|x_{k-2} - \frac{x_{k-1} + x_k}{2}\right\|^2 \cdot ||x_{k-1} - x_k||^2$$

$$= \frac{1}{k} A(x_1, \dots, x_k)^2 = \frac{1}{k} \varepsilon^2 .$$

Therefore,

$$\frac{1}{2} a^2 = \frac{\varepsilon^{2/(k-1)}}{k^{1/(k-1)}} ,$$

which proves the first assertion, and completes the proof of the lemma.

<u>Lemma 1.9</u>: Let X be an infinite dimensional Hilbert space. Given an integer k , $k \geq 2$, there exist equidistant points $x_1, \dots, x_k \in S_X$, with $A(x_1, \dots, x_k) = \left(\frac{k}{k-1}\right)^{(k-1)/2} \cdot k^{1/2}$. Thus, $\delta_k(\varepsilon)$ is well defined for each $\varepsilon > 0$, for sufficiently large k .

<u>Proof</u>: We show there exist equidistant points $x_1, \dots, x_k$ in S_X with $||x_i - x_j||^2 = 2\left(\frac{k}{k-1}\right)$ for all $i, j, 1 \leq i, j \leq k, i \neq j$. This will imply, by Lemma 1.8, that if $A(x_1, \dots, x_k) = \varepsilon$ then

$$\frac{1}{2}\left(2\left(\frac{k}{k-1}\right)\right) = \frac{\varepsilon^{2/(k-1)}}{k^{1/(k-1)}} ,$$

giving $\varepsilon = \left(\frac{k}{k-1}\right)^{(k-1)/2} \cdot k^{1/2}$ as desired.

Since X is infinite dimensional, we may choose $y_1, \dots, y_k \in S_X$, and $\langle y_i, y_j \rangle = 0$ for all $i, j, 1 \leq i, j \leq k, i \neq j$.

We prove that $x_1, \dots, x_k$, defined as follows, will be as required.

$$x_1 = y_1$$

$$x_2 = a_1 y_1 + a_2 y_2$$

$$x_i = a_1 y_1 - \sum_{j=2}^{i-1} \frac{1}{k-j} a_j y_j + a_i y_i \qquad \text{for} \qquad 3 \leq i \leq k$$

where

$$a_1 = -\frac{1}{k-1}$$

$$a_2 = (1-a_1^2)^{1/2}$$

$$a_i = (1 - \sum_{j=2}^{i-1} \frac{1}{(k-j)^2} a_j^2 - a_1^2)^{1/2} \qquad \text{for} \qquad 3 \leq i \leq k-1$$

$$a_k = 0 \quad .$$

By the definition of $y_1, \ldots, y_k, a_1, \ldots, a_k$, it is easy to see that $||x_1|| = \ldots = ||x_k|| = 1$.

Also, in order to show $||x_i - x_j||^2 = 2\left(\frac{k}{k-1}\right)$ for all $i, j, 1 \leq i, j \leq k, i \neq j$, it suffices to show $||x_i - x_{i-1}||^2 = 2\left(\frac{k}{k-1}\right)$ for $2 \leq i \leq k$.

For $i = 2$:

$$||x_2 - x_1||^2 = (a_1-1)^2 + a_2^2 = (a_1-1)^2 + (1-a_1^2) = a_1^2 - 2a_1 + 1 + 1 - a_1^2$$

$$= 2 - 2a_1 = 2\left(1 - \left(-\frac{1}{k-1}\right)\right) = 2\left(1 + \frac{1}{k-1}\right) = 2\left(\frac{k}{k-1}\right) \quad .$$

For $3 \leq i \leq k-1$:

From the definition of $a_1, \ldots, a_{k-1}$ it can be shown that for each i , $2 \leq i \leq k-1$,

$$a_i^2 = \prod_{j=1}^{i-1} \left(1 - \frac{1}{(k-j)^2}\right) .$$

Hence, for $3 \leqslant i \leqslant k-1$,

$$||x_i - x_{i-1}||^2 = (a_{i-1} + \frac{1}{k-i+1} a_{i-1})^2 + a_i^2$$

$$= \left(\frac{k-i+2}{k-i+1}\right)^2 a_{i-1}^2 + a_i^2$$

$$= \left(\frac{k-i+2}{k-i+1}\right)^2 \left(1 - \frac{1}{(k-i+2)^2}\right)\left(1 - \frac{1}{(k-i+3)^2}\right) \cdots$$

$$\left(1 - \frac{1}{(k-2)^2}\right)\left(1 - \frac{1}{(k-1)^2}\right)$$

$$+ \left(1 - \frac{1}{(k-i+1)^2}\right)\left(1 - \frac{1}{(k-i+2)^2}\right)\left(1 - \frac{1}{(k-i+3)^2}\right) \cdots$$

$$\left(1 - \frac{1}{(k-2)^2}\right)\left(1 - \frac{1}{(k-1)^2}\right)$$

$$= \left(\frac{(k-i+2)^2}{(k-i+1)^2}\right)\left(\frac{(k-i+2)^2 - 1}{(k-i+2)^2}\right)\left(\frac{(k-i+3)^2 - 1}{(k-i+3)^2}\right) \cdots$$

$$\left(\frac{(k-2)^2 - 1}{(k-2)^2}\right)\left(\frac{(k-1)^2 - 1}{(k-1)^2}\right)$$

$$+ \left(\frac{(k-i+1)^2 - 1}{(k-i+1)^2}\right)\left(\frac{(k-i+2)^2 - 1}{(k-i+2)^2}\right)\left(\frac{(k-i+3)^2 - 1}{(k-i+3)^2}\right) \cdots$$

$$\left(\frac{(k-2)^2 - 1}{(k-2)^2}\right)\left(\frac{(k-1)^2 - 1}{(k-1)^2}\right)$$

$$= \frac{(k-i+2)^2}{(k-i+1)^2} \; \frac{(k-i+3)(k-i+1)}{(k-i+2)^2} \; \frac{(k-i+4)(k-i+2)}{(k-i+3)^2} \cdots$$

$$\frac{(k-1)(k-3)}{(k-2)^2} \; \frac{(k)(k-2)}{(k-1)^2}$$

$$+ \frac{(k-i+2)(k-i)}{(k-i+1)^2} \quad \frac{(k-i+3)(k-i+1)}{(k-i+2)^2} \quad \frac{(k-i+4)(k-i+2)}{(k-i+3)^2} \quad \cdots$$

$$\frac{(k-1)(k-3)}{(k-2)^2} \quad \frac{(k)(k-2)}{(k-1)^2}$$

$$= \left(\frac{(k-i+2) + (k-i)}{(k-i+1)}\right)\left(\frac{k}{k-1}\right) = \left(\frac{(2k-2i+2)}{k-i+1}\right)\left(\frac{k}{k-1}\right)$$

$$= 2\left(\frac{k}{k-1}\right) \quad .$$

For $i=k$:

$$||x_k - x_{k-1}||^2 = (a_{k-1} + a_{k-1})^2 + a_k = (2a_{k-1})^2 + 0$$

$$= 4a^2_{k-1}$$

$$= 4\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{(k-2)^2}\right)\left(1 - \frac{1}{(k-1)^2}\right)$$

$$= 4\left(\frac{2^2 - 1}{2^2}\right)\left(\frac{3^2 - 1}{3^2}\right) \cdots \left(\frac{(k-2)^2 - 1}{(k-2)^2}\right)\left(\frac{(k-1)^2 - 1}{(k-1)^2}\right)$$

$$= \frac{4(2+1)(2-1)}{2^2} \quad \frac{(3+1)(3-1)}{3^2} \quad \cdots \quad \frac{(k-1)(k-3)}{(k-2)^2} \quad \frac{(k)(k-2)}{(k-1)^2}$$

$$= \frac{4(3)(1)}{2^2} \quad \frac{(4)(2)}{3^2} \quad \cdots \quad \frac{(k-1)(k-3)}{(k-2)^2} \quad \frac{(k)(k-1)}{(k-1)^2}$$

$$= \frac{4}{2}\left(\frac{k}{k-1}\right)$$

$$= 2\left(\frac{k}{k-1}\right) \quad .$$

This completes the proof of the lemma.

The following lemma is based on an idea communicated by J. Hagler.

Lemma 1.10: Let X be an infinite dimensional Hilbert space. Given an integer k, $k \geq 2$, and a number ε, $0 < \varepsilon \leq \left(\frac{k}{k-1}\right)^{(k-1)/2} \cdot k^{1/2}$, then there exist equidistant points $x_1, \ldots, x_k \in S_X$, with $A(x_1, \ldots, x_k) = \varepsilon$.

Proof: By Lemma 1.9 there exist equidistant points $z_1, \ldots, z_k \in S_X$, with $A(z_1, \ldots, z_k) = \left(\frac{k}{k-1}\right)^{(k-1)/2} \cdot k^{1/2}$. By Lemma 1.8 if $a = \|z_i - z_j\|$, $1 \leq i, j \leq k$, $i \neq j$, then it can be shown that $a = \left(2\left(\frac{k}{k-1}\right)\right)^{1/2}$.

Thus, again by Lemma 1.8, it suffices to show that given b, $0 < b \leq a$, then there exist $x_1, \ldots, x_k \in S_X$, with $\|x_i - x_j\| = b$ for all i, j, $1 \leq i, j \leq k$, $i \neq j$.

Since X is infinite dimensional we can find $z \in X$, such that $\|z\| = 1$, and $\langle z, z_i \rangle = 0$ for all i, $1 \leq i \leq k$.

Let $c = \left(1 - \left(\frac{b}{a}\right)^2\right)^{1/2}$, and $x_i = cz + \frac{b}{a} z_i$ for each i, $1 \leq i \leq k$.

Then it follows easily that $x_1, \ldots, x_k$ are as required, and the proof of the lemma is complete.

Theorem 1.11: Let X be an infinite dimensional Hilbert space and δ_k its modulus of k-uniform convexity. Then all of the following are true.

(i) For any given $\varepsilon > 0$, $\lim_{k\to\infty} \delta_k(\varepsilon) = 1$.

(ii) Given an integer k, $k \geq 1$, $x_1, \ldots, x_{k+1} \in S_X$, then $A(x_1, \ldots, x_{k+1}) \leq \left(\frac{k+1}{k}\right)^{k/2} (k+1)^{1/2}$.

(iii) Given ε, $0 < \varepsilon \leq \left(\frac{k+1}{k}\right)^{k/2} (k+1)^{1/2}$, k an integer, $k \geq 1$, then

$$\delta_k(\varepsilon) = 1 - \left(1 - \frac{k}{k+1}\left(\frac{\varepsilon^{2/k}}{(k+1)^{1/k}}\right)\right)^{1/2} .$$

<u>Proof</u>: The following equation, whose proof follows from the definition of the norm in terms of the inner product, holds for every integer k:

$$\left\|\frac{x}{k} + \frac{(k-1)}{k} y\right\|^2 = \frac{1}{k}\|x\|^2 + \frac{k-1}{k}\|y\|^2 - \frac{k-1}{k^2}\|x - y\|^2 .$$

In particular given $x_1, \ldots, x_k \in X$ we have by letting $x = x_1$, and $y = \frac{x_2 + \ldots + x_k}{k-1}$, that $\left\|\frac{x_1 + x_2 + \ldots + x_k}{k}\right\|^2 =$

$$\left\|\frac{x_1}{k} + \frac{(k-1)}{k}\left(\frac{x_2 + \ldots + x_k}{k-1}\right)\right\|^2 = \frac{1}{k}\|x_1\|^2$$

$$+ \frac{k-1}{k}\left\|\frac{x_2 + \ldots + x_k}{k-1}\right\|^2 - \frac{k-1}{k^2}\left\|x_1 - \frac{x_2 + \ldots + x_k}{k-1}\right\|^2 \quad (*) .$$

Now, we will show the following fact by induction:

Given $x_1, \ldots, x_k \in S_X$, then

$$\left\|\frac{x_1 + \ldots + x_k}{k}\right\|^2 = 1 - \frac{1}{k}\left(\frac{1}{2}\|x_{k-1} - x_k\|^2 + \frac{2}{3}\left\|x_{k-2} - \frac{x_{k-1} + x_k}{2}\right\|^2\right.$$

$$\left. + \frac{3}{4}\left\|x_{k-3} - \frac{x_{k-2} + \ldots + x_k}{3}\right\|^2 + \ldots + \frac{k-1}{k}\left\|x_1 - \frac{x_2 + \ldots + x_k}{k-1}\right\|^2\right).$$

For $k = 2$ $\left\|\frac{x_1 + x_2}{2}\right\|^2 = 1 - \frac{1}{4}\|x_1 - x_2\|^2 = 1 - \frac{1}{2}\left(\frac{1}{2}\|x_1 - x_2\|^2\right)$

by the parallelogram law.

Assume fact is true for k. We show it then holds for $k+1$.

By (*)

$$\left\|\frac{x_1 + x_2 + \dots + x_k + x_{k+1}}{k+1}\right\|^2 = \frac{1}{k+1}\,||x_1||^2 + \frac{k}{k+1}\left\|\frac{x_2 + \dots + x_{k+1}}{k}\right\|^2 - \frac{k}{(k+1)^2}\left\|x_1 - \frac{x_2 + \dots + x_{k+1}}{k}\right\|^2$$

and by the induction hypothesis

$$\left\|\frac{x_1 + x_2 + \dots + x_k + x_{k+1}}{k+1}\right\|^2 = \frac{1}{k+1}\,||x_1||^2$$

$$+ \frac{k}{k+1}\left(1 - \frac{1}{k}\left(\frac{1}{2}\,||x_k - x_{k+1}||^2 + \frac{2}{3}\left\|x_{k-1} - \frac{x_k + x_{k+1}}{2}\right\|^2\right.\right.$$

$$+ \frac{3}{4}\left\|x_{k-2} - \frac{x_{k-1} + x_k + x_{k+1}}{3}\right\|^2 + \dots$$

$$\left.\left.+ \frac{k-1}{k}\left\|x_2 - \frac{x_3 + \dots + x_{k+1}}{k-1}\right\|^2\right)\right) - \frac{k}{(k+1)^2}\left\|x_1 - \frac{x_2 + \dots + x_{k+1}}{k}\right\|^2$$

$$= \frac{1}{k+1} + \frac{k}{k+1} - \frac{1}{k+1}\left(\frac{1}{2}\,||x_k - x_{k+1}||^2 + \frac{2}{3}\left\|x_{k-1} - \frac{x_k + x_{k+1}}{2}\right\|^2\right.$$

$$\left.+ \frac{3}{4}\left\|x_{k-2} - \frac{x_{k-1} + x_k + x_{k+1}}{3}\right\|^2 + \dots + \frac{k-1}{k}\left\|x_2 - \frac{x_3 + \dots + x_{k+1}}{k-1}\right\|^2\right)$$

$$- \frac{k}{(k+1)^2}\left\|x_1 - \frac{x_2 + \dots + x_{k+1}}{k}\right\|^2$$

$$= 1 - \frac{1}{k+1}\left(\frac{1}{2}\,||x_k - x_{k+1}||^2 + \frac{2}{3}\left\|x_{k-1} - \frac{x_k + x_{k+1}}{2}\right\|^2 + \dots\right.$$

$$\left.+ \frac{k}{k+1}\left\|x_1 - \frac{x_2 + \dots + x_{k+1}}{k}\right\|^2\right) \quad \text{as required.}$$

As mentioned above, since $(X, \|\ \|)$ is a Hilbert space we know that given $x_1, \ldots, x_k \in X$, $A(x_1, \ldots, x_k) = \text{dist}(x_1, [x_2, \ldots, x_k]) \cdot A(x_2, \ldots, x_k)$.

Hence, inductively, $A(x_1, \ldots, x_k) = \text{dist}(x_1, [x_2, \ldots, x_k]) \cdot \text{dist}(x_2, [x_3, \ldots, x_k]) \cdot \ldots \cdot \text{dist}(x_{k-2}, [x_{k-1}, x_k]) \cdot \|x_{k-1} - x_k\|$.

Let $d_i = \text{dist}(x_i, [x_{i+1}, \ldots, x_k])$, $1 \leqslant i \leqslant k-2$,

$d_{k-1} = \|x_{k-1} - x_k\|$.

Then $A(x_1, \ldots, x_k) = d_1 \cdot d_2 \cdot \ldots \cdot d_{k-1}$.

In order to show $\lim_{k\to\infty} \delta_k(\varepsilon) = 1$ for a given $\varepsilon > 0$ let us assume $x_1, \ldots, x_k \in S_X$, and $A(x_1, \ldots, x_k) \geqslant \varepsilon$.

From the fact proved above

$$\left\|\frac{x_1 + \ldots + x_k}{k}\right\|^2 = 1 - \frac{1}{k}\left(\frac{1}{2}\|x_{k-1} - x_k\|^2 + \frac{2}{3}\left\|x_{k-2} - \frac{x_{k-1} + x_k}{2}\right\|^2\right.$$

$$\left. + \frac{3}{4}\left\|x_{k-3} - \frac{x_{k-2} + x_{k-1} + x_k}{3}\right\|^2 + \ldots + \frac{k-1}{k}\left\|x_1 - \frac{x_2 + \ldots + x_k}{k-1}\right\|^2\right)$$

$$\leqslant 1 - \frac{1}{k}\left(\frac{1}{2} d^2_{k-1} + \frac{2}{3} d^2_{k-2} + \frac{3}{4} d^2_{k-3} + \ldots + \frac{k-1}{k} d^2_1\right),$$

since $d_i = \text{dist}(x_i, [x_{i+1}, \ldots, x_k])$

$$\leqslant \left\|x_i - \frac{x_{i+1} + \ldots + x_k}{k-i}\right\| \quad \text{for} \quad 1 \leqslant i \leqslant k-2,$$

and $d_{k-1} = \|x_{k-1} - x_k\|$.

We know $d_1 \cdot d_2 \cdot \ldots \cdot d_{k-1} = A(x_1, \ldots, x_k) = \beta$ for some $\beta \geqslant \varepsilon$.

Let us now define functions f and h from R^{k-1} into R by

$h(e_1, e_2, \ldots, e_{k-1}) = \frac{1}{k}\left(\frac{1}{2} e^2_{k-1} + \frac{2}{3} e^2_{k-2} + \ldots + \frac{k-1}{k} e^2_1\right)$, and

$f = 1 - h$, where $(e_1, e_2, \ldots, e_{k-1}) \in R^{k-1}$.

We show that h attains an absolute minimum value on the closed surface $C = \{(e_1, \ldots, e_{k-1}) : e_1 \cdot e_2 \cdot \ldots \cdot e_{k-1} = \beta$, and $(e_1, \ldots, e_{k-1}) \epsilon R^{k-1}\}$.

This in turn will imply that f attains an absolute maximum value on this surface.

Notice that, since h is continuous and its range is $[0, \infty)$, $p = h|C$ is continuous and its range is contained in $[0,\infty)$.

Pick $(e_1, e_2, \ldots, e_{k-1})$ in C , and let $L = p\,(e_1, e_2, \ldots, e_{k-1})$.

Since $(e_1, e_2, \ldots, e_{k-1}) \epsilon\ p^{-1}([0,L])$ and $[0,L]$ is a closed set then $p^{-1}([0,L])$ is a nonempty closed subset of C .

We show $p^{-1}([0,L])$ is bounded. If not, we can find $(a_1, a_2, \ldots, a_{k-1})$ in $p^{-1}([0,L])$ of arbitrarily large norm. But this would mean being able to find $(a_1, a_2, \ldots, a_{k-1})$ in $p^{-1}([0,L])$ with at least one arbitrarily large coordinate. However, this is not possible since then $p\,(a_1, \ldots, a_{k-1})$ would be arbitrarily large and in particular larger than L .

Therefore, $p^{-1}([0,L])$ is bounded and thus compact.

Since $p^{-1}([0,L])$ is nonempty, p is defined at each of its elements and by compactness it must attain a minimum value at some point Z of this set.

Suppose now that $Y \epsilon C$. If $Y \epsilon p^{-1}([0,L])$ then $p(Z) \leqslant p(Y)$ by the choice of Z . If $Y \notin p^{-1}([0,L])$ then $p(Y) > L$ since p is nonnegative. Hence $p(Z) \leqslant L < p(Y)$ and thus p attains an absolute minimum value on C at Z .

Define $g : R^{k-1} \to R$ by $g(e_1, \ldots, e_{k-1}) = e_1 \cdot e_2 \cdot \ldots \cdot e_{k-1} - \beta$ where $(e_1, \ldots, e_{k-1}) \in R^{k-1}$.

Since f achieves a maximum value on the surface for which the function g is equal to zero, then we may use Lagrange's theorem to conclude that there must exist λ , a real number, such that $\nabla f(z) = \lambda \nabla g(z)$, where ∇h denotes the gradient of the function h .

By the definition of f and g it can easily be seen that

$$\left(\frac{\partial f}{\partial e_1}, \frac{\partial f}{\partial e_2}, \ldots, \frac{\partial f}{\partial e_{k-1}}\right) = \left(-\frac{2}{k}\left(\frac{k-1}{k}\right)e_1, -\frac{2}{k}\left(\frac{k-2}{k-1}\right)e_2, \ldots, -\frac{2}{k}\left(\frac{2}{3}\right)e_{k-2}, -\frac{2}{k}\left(\frac{1}{2}\right)e_{k-1}\right)$$

and

$$\left(\frac{\partial g}{\partial e_1}, \frac{\partial g}{\partial e_2}, \ldots, \frac{\partial g}{\partial e_{k-1}}\right) = \left(e_2 \cdot e_3 \cdot \ldots \cdot e_{k-1}, e_1 \cdot e_3 \cdot \ldots \cdot e_{k-1}, \ldots, e_1 \cdot \ldots \cdot e_{k-2}\right) .$$

Taking $Z = (b_1, \ldots, b_{k-1})$ we must have then

$$-\frac{2}{k}\left(\frac{k-1}{k}\right) b_1 = \lambda\, b_2 \cdot \ldots \cdot b_{k-1}$$

$$-\frac{2}{k}\left(\frac{k-2}{k-1}\right) b_2 = \lambda\, b_1 \cdot b_3 \cdot \ldots \cdot b_{k-1}$$

$$\vdots$$

$$-\frac{2}{k}\left(\frac{2}{3}\right) b_{k-2} = \lambda\, b_1 \cdot b_2 \cdot \ldots \cdot b_{k-3} \cdot b_{k-1}$$

$$-\frac{2}{k}\left(\frac{1}{2}\right) b_{k-1} = \lambda\, b_1 \cdot b_2 \cdot \ldots \cdot b_{k-2} \quad .$$

For each i , $1 \leq i \leq k-1$, multiply both sides of the ith equation above by b_i , and subtract $\lambda\beta$ from both sides to obtain the following equations:

$$-\frac{2}{k}\left(\frac{k-1}{k}\right)b_1^2 - \lambda\beta = \lambda\, b_1 \cdot b_2 \cdot \ldots \cdot b_{k-1} - \lambda\beta = 0$$

$$-\frac{2}{k}\left(\frac{k-2}{k-1}\right)b_2^2 - \lambda\beta = 0$$

$$\vdots$$

$$-\frac{2}{k}\left(\frac{2}{3}\right)b_{k-2}^2 - \lambda\beta = 0$$

$$-\frac{2}{k}\left(\frac{1}{2}\right)b_{k-1}^2 - \lambda\beta = 0 \quad .$$

Thus $\lambda = -\frac{2}{k\beta}\left(\frac{k-1}{k}\right)b_1^2 = -\frac{2}{k\beta}\left(\frac{k-2}{k-1}\right)b_2^2 = \ldots = -\frac{2}{k\beta}\left(\frac{1}{2}\right)b_{k-1}^2$

giving $\left(\frac{k-1}{k}\right)b_1^2 = \left(\frac{k-2}{k-1}\right)b_2^2 = \ldots = \left(\frac{1}{2}\right)b_{k-1}^2$.

Hence $\left(\frac{k-1}{k}\,b_1^2\right)^{k-1} = \left(\frac{k-2}{k-1}\,b_2^2\right)^{k-1} = \ldots = \left(\frac{1}{2}\,b_{k-1}^2\right)^{k-1}$

$$= \frac{k-1}{k}\,b_1^2 \cdot \frac{k-2}{k-1}\,b_2^2 \cdot \ldots \cdot \frac{1}{2}\,b_{k-1}^2$$

$$= \frac{k-1}{k} \cdot \frac{k-2}{k-1} \cdot \ldots \cdot \frac{2}{3} \cdot \frac{1}{2}\,(b_1 \cdot b_2 \cdot \ldots \cdot b_{k-1})^2 = \frac{1}{k}\,\beta^2 \quad .$$

So $\frac{k-1}{k}\,b_1^2 = \frac{k-2}{k-1}\,b_2^2 = \ldots = \frac{1}{2}\,b_{k-1}^2 = \left(\frac{1}{k}\,\beta^2\right)^{1/(k-1)} = \frac{\beta^{2/(k-1)}}{k^{1/(k-1)}}$.

Therefore $f(Z) = 1 - \frac{1}{k}\left(\frac{1}{2}\, b^2_{k-1} + \frac{2}{3}\, b^2_{k-2} + \dots + \frac{k-1}{k}\, b^2_1 \right)$

$$= 1 - \frac{1}{k}\left((k-1)\, \frac{\beta^{2/(k-1)}}{k^{1/(k-1)}} \right)$$

and as seen above

$$\left\| \frac{x_1 + \dots + x_k}{k} \right\|^2 \leqslant 1 - \frac{1}{k}\left(\frac{1}{2}\, d^2_{k-1} + \frac{2}{3}\, d^2_{k-2} + \dots + \frac{k-1}{k}\, d^2_1 \right) \leqslant f(Z)$$

$$= 1 - \frac{1}{k}\left((k-1)\, \frac{\beta^{2/(k-1)}}{k^{1/(k-1)}} \right) \leqslant 1 - \frac{1}{k}\left((k-1)\, \frac{\varepsilon^{2/(k-1)}}{k^{1/(k-1)}} \right) \text{ since } \beta \geqslant \varepsilon \text{ .}$$

Letting k go to infinity we get from this inequality that

$$\left\| \frac{x_1 + \dots + x_k}{k} \right\| \to 0 \quad .$$

Thus, we may conclude that $\lim_{k\to\infty} \delta_k(\varepsilon) = 1$, which proves (i).

Suppose now that $x_1, \dots, x_{k+1} \in S_X$, and $A(x_1, \dots, x_{k+1}) > \left(\frac{k+1}{k}\right)^{k/2} \cdot (k+1)^{1/2}$, $k \geqslant 1$.

Then, as seen above, we have

$$\left\| \frac{x_1 + \dots + x_{k+1}}{k+1} \right\|^2 \leqslant 1 - \frac{k}{k+1}\left(\frac{A(x_1, \dots, x_{k+1})^{2/k}}{(k+1)^{1/k}} \right)$$

$$< 1 - \frac{k}{k+1}\left(\frac{\left(\left(\frac{k+1}{k}\right)^{k/2} (k+1)^{1/2} \right)^{2/k}}{(k+1)^{1/k}} \right)$$

$$= 1 - 1 = 0$$

which is a contradiction. Hence $A(x_1, \dots, x_{k+1}) \leqslant \left(\frac{k+1}{k}\right)^{k/2} (k+1)^{1/2}$, and the proof of (ii) is complete.

From above it is clear that for a given ε , $k \geqslant 1$, $0 < \varepsilon \leqslant \left(\frac{k+1}{k}\right)^{k/2} \cdot (k+1)^{1/2}$, then

$$\delta_k(\varepsilon) \geqslant 1 - \left(1 - \frac{k}{k+1}\left(\frac{\varepsilon^{2/k}}{(k+1)^{1/k}}\right)\right)^{1/2} \qquad (**) .$$

By Lemma 1.10 we may choose equidistant points $x_1, \ldots, x_{k+1}$ in S_X , with $A(x_1, \ldots, x_{k+1}) = \varepsilon$.

From Lemma 1.8 if $a = ||x_i - x_j||$, $1 \leqslant i , j \leqslant k$, $i \neq j$, then

$$\frac{1}{2} a^2 = \frac{\varepsilon^{2/k}}{(k+1)^{1/k}} ,$$

and

$$\frac{k+1 - i}{(k+1) - i+1}\left\|x_i - \frac{x_{i+1} + \ldots + x_{k+1}}{k}\right\|^2 = \frac{1}{2} a^2 \quad \text{for all } i,\ 1 \leqslant i \leqslant k .$$

From above we know that

$$\left\|\frac{x_1 + \ldots + x_{k+1}}{k}\right\|^2 = 1 - \frac{1}{k+1}\left(\frac{1}{2} ||x_k - x_{k+1}||^2 + \ldots + \frac{k}{k+1}\left\|x_1 - \frac{x_2 + \ldots + x_{k+1}}{k}\right\|^2\right)$$

so

$$\left\|\frac{x_1 + \ldots + x_{k+1}}{k}\right\|^2 = 1 - \frac{1}{k+1}\left(k\left(\frac{1}{2} a^2\right)\right)$$

$$= 1 - \frac{1}{k+1}\left(k\left(\frac{\varepsilon^{2/k}}{(k+1)^{1/k}}\right)\right)$$

$$= 1 - \frac{k}{k+1}\left(\frac{\varepsilon^{2/k}}{(k+1)^{1/k}}\right)$$

which together with (**) implies

$$\delta_k(\varepsilon) = 1 - \left(1 - \frac{k}{k+1}\left(\frac{\varepsilon^{2/k}}{(k+1)^{1/k}}\right)\right)^{1/2} .$$

This proves (iii), and the proof of the theorem is complete.

In the next chapter we will exhibit an interesting consequence of (i) of this theorem.

CHAPTER II

Normal Structure and Superreflexivity

The principal purpose of this chapter is to investigate some structural properties of Banach spaces under certain area-related convexity conditions. The material we present originated from an attempt to characterize all spaces isomorphic to superreflexive Banach spaces in terms of these conditions. However, because of the nonisomorphic nature of these properties, it was to become apparent that a characterization with respect to the structure of the space would be of greater feasibility.

Thanks to some deep and fascinating arguments due to R. C. James [11, 12, 13, 14, 15] and P. Enflo [8], superreflexivity has been shown to be equivalent to several other geometrical and topological properties of Banach spaces. Because of its significance to our study, we first collect some of the material that pertains to these characterizations, including James' definition of superreflexivity.

Definition 2.1: Let X and Y be Banach spaces. Y is finitely representable in X if for each finite dimensional subspace Z of Y and each positive number ε there exists an isomorphism $T : Z \to X$ for which $(1-\varepsilon)||z|| \leqslant ||T(z)|| \leqslant (1+\varepsilon)||z||$ if $z \in Z$.

Definition 2.2: A Banach space X is superreflexive if every Banach space which is finitely representable in X is reflexive.

The next definition concerns the finite tree property.

Definition 2.3: A Banach space X is said to possess the finite tree property if there exists $\varepsilon > 0$ such that for any positive integer n

there exists a subset $\{x_i : 1 \leqslant i \leqslant 2^n - 1\}$ of B_X for which $x_i = (x_{2i} + x_{2i+1})/2$ and $||x_{2i} - x_{2i+1}|| \geqslant \varepsilon$ for each i, $1 \leqslant i \leqslant 2^{n-1} - 1$.

Definition 2.4: A Banach space X is uniformly nonsquare if there is a positive number η such that there do not exist members x and y of B_X for which

$$\left|\left|\frac{x+y}{2}\right|\right| > 1 - \eta \qquad \text{and} \qquad \left|\left|\frac{x-y}{2}\right|\right| > 1 - \eta \quad .$$

The following theorem whose proof was obtained by James [11, 12, 13, 14, 15] and Enflo [8] connects these notions and several other characterizations of superreflexivity.

Theorem 2.5: The following conditions on a Banach space X are equivalent.

(i) X is superreflexive.

(ii) X does not have the finite tree property.

(iii) For some $\varepsilon > 0$, $0 < \varepsilon < 1$, and some positive integer n, there do not exist subsets $\{x_1, \ldots, x_n\}$ of B_X and $\{f_1, \ldots, f_n\}$ of B_{X^*} for which $f_k(x_i) \geqslant \varepsilon$ if $k \leqslant i$, and $f_k(x_i) = 0$ if $k > i$.

(iv) For some $\varepsilon > 0$, $0 < \varepsilon < 1$, and some positive integer n, there does not exist a subset $\{x_1, \ldots, x_n\}$ of B_X such that, if $1 \leqslant k < n$, then $\text{dist}(\text{conv}\{x_1, \ldots, x_k\}, \text{conv}\{x_{k+1}, \ldots, x_n\}) \geqslant \varepsilon$.

(v) X is isomorphic to a uniformly convex Banach space.

(vi) X is isomorphic to a uniformly nonsquare Banach space.

As mentioned earlier we would like to determine conditions concerning the concept of area and their relationship to superreflexivity. With this purpose in mind we now introduce a list of definitions whose central theme is the notion of area.

<u>Definition 2.6</u>: A Banach space X is said to have property A if there exist ε, m, δ, with $0 < \varepsilon < 1$, m an integer, $m > 1$, $\delta > 0$, such that whenever $\{x_1, \ldots, x_m\}$ is a subset of B_X, then

$$\text{dist}(x_m, [x_{m-1}, \ldots, x_1]) \cdot \text{dist}(x_{m-1}, [x_{m-2}, \ldots, x_1]) \cdot \ldots \cdot \text{dist}(x_3, [x_2, x_1]) \cdot ||x_2 - x_1|| < \varepsilon$$

$$\text{if} \quad \left|\left|\frac{x_1 + \ldots + x_m}{m}\right|\right| > 1 - \delta \quad .$$

<u>Definition 2.7</u>: A Banach space X is said to have property A' if there exist ε, m, δ, with $0 < \varepsilon < 1$, m an integer, $m > 1$, $\delta > 0$, such that whenever $\{x_1, \ldots, x_m\}$ is a subset of B_X, then $A(x_1, x_2, \ldots, x_m) < \varepsilon$ if $\left|\left|\frac{x_1 + \ldots + x_m}{m}\right|\right| > 1 - \delta$.

<u>Definition 2.8</u>: A Banach space X is said to have property B if there exist δ, m, with $\delta > 0$, m an integer, $m > 1$, such that whenever $\{x_1, \ldots, x_m\}$ is a subset of B_X, then $\text{dist}(x_m, [x_{m-1}, \ldots, x_1]) \cdot \text{dist}(x_{m-1}, [x_{m-2}, \ldots, x_1]) \cdot \ldots \cdot \text{dist}(x_3, [x_2, x_1]) \cdot ||x_2 - x_1|| < 2^{m-1} \cdot (1 - \delta)$ if $\left|\left|\frac{x_1 + \ldots + x_m}{m}\right|\right| > 1 - \delta$.

<u>Definition 2.9</u>: A Banach space X is said to have property B' if there exist δ, m, with $\delta > 0$, m an integer, $m > 1$, such that whenever $\{x_1, \ldots, x_m\}$ is a subset of B_X, then $A(x_1, \ldots, x_m) < 2^{m-1} (1 - \delta)$ if $\left|\left|\frac{x_1 + \ldots + x_m}{m}\right|\right| > 1 - \delta$.

By Lemma 1.3, since $A(x_1, \ldots, x_m) \geqslant \text{dist}(x_m, [x_{m-1}, \ldots, x_1]) \cdot \ldots \cdot \text{dist}(x_3, [x_2, x_1]) \cdot ||x_2, x_1||$ for any $x_1, \ldots, x_m \in X$, property A'

implies property A and property B' implies property B. We will confine our attention mostly to properties A and B. Clearly, property A is a sufficient condition for a space to have property B, and as we shall see, from an example due to L. A. Karlovitz [17], property A is the stronger of the two. Since Karlovitz's example also depicts some other properties of Banach spaces which we still have not defined, we will delay its presentation until later in this chapter.

We know that if $x_1, \ldots, x_m \in X$, X a Hilbert space, then $A(x_1, \ldots, x_m) = \text{dist}(x_m, [x_{m-1}, \ldots, x_1]) \cdot \ldots \cdot \text{dist}(x_3, [x_2, x_1]) \cdot ||x_2 - x_1||$. So, intuitively, if we think of $\text{dist}(x_m, [x_{m-1}, \ldots, x_1]) \cdot \ldots \cdot \text{dist}(x_3, [x_2, x_1]) \cdot ||x_2 - x_1||$ as an alternate definition of area, $x_1, \ldots, x_m \in X$, X an arbitrary Banach space, then a Banach space with property A can be thought of as uniformly not having, for some integer m, arbitrarily flat m-dimensional convex regions on the unit sphere of area arbitrarily close to 1. On the other hand, Banach spaces having property B do not have, for some integer m, arbitrarily flat m-dimensional convex regions on the surface of the unit ball of area arbitrarily close to 2^{m-1}.

Intrinsic in the definition of property B is the fact that m-dimensional convex regions in the unit ball of area arbitrarily close to 2^{m-1} have edges of length arbitrarily close to 2. In contrast, whenever the area of a m-dimensional convex region in the unit ball is arbitrarily close to 1, it does not follow necessarily that the lengths of its edges are also arbitrarily close to 1. This last fact, then, makes property A less suitable for certain calculations.

That Banach spaces possessing property A are superreflexive is essentially due to W. Davis [4]. The proof is presented next.

<u>Theorem 2.10</u> (Davis): If a Banach space X possesses property A then X is superreflexive.

<u>Proof</u>: If X is not superreflexive, then by Theorem 2.5 given ε, $0 < \varepsilon < 1$, and a positive integer m, there exist subsets $\{x_1, \ldots, x_m\}$ of B_X and $\{f_1, \ldots, f_m\}$ of B_{X^*} for which $f_k(x_i) \geq \varepsilon$ if $k \leq i$, and $f_k(x_i) = 0$ if $k > i$. Then

$$\left\|\frac{x_1 + \ldots + x_m}{m}\right\| \geq f_1\left(\frac{x_1 + \ldots + x_m}{m}\right) = \frac{1}{m}\left(f_1(x_1) + \ldots + f_1(x_m)\right)$$

$$\frac{1}{m} \cdot m\,\varepsilon = \varepsilon .$$

Given i, $2 \leq i \leq m$, and $z \in [x_1, \ldots, x_{i-1}]$ then $z = \sum_{j=1}^{i-1} \lambda_j x_j$ with $\sum_{j=1}^{i-1} \lambda_j = 1$.

Hence $||x_i - z|| \geq f_i(x_i - z) = f_i(x_i) - f(z) \geq \varepsilon - 0 = \varepsilon$, so $\text{dist}(x_i, [x_{i-1}, \ldots, x_1]) \geq \varepsilon$ for all i, $2 \leq i \leq m$, and therefore $\text{dist}(x_m, [x_{m-1}, \ldots, x_1]) \cdot \text{dist}(x_{m-1}, [x_{m-2}, \ldots, x_1]) \cdot \ldots \cdot \text{dist}(x_3, [x_2, x_1]) \cdot ||x_2 - x_1|| > \varepsilon^{m-1}$.

Since ε is an arbitrary number between zero and one and m is any arbitrary positive integer, we get that X does not have property A, contradicting the given hypothesis. Hence X must be superreflexive.

A rather exceptional idea is the heart of R. C. James' proof that uniformly nonsquare Banach spaces are superreflexive [12]. We take advantage of this idea to improve Davis' result as follows:

Theorem 2.11: If a Banach space X is not reflexive, then given $\delta > 0$, m an integer, $m > 1$, there exist $x_1, \ldots, x_m$ in B_X such that $\mathrm{dist}(x_\ell, [x_{\ell+1}, \ldots, x_m]) > 2(1 - \delta)$ for all ℓ, $1 \leq \ell \leq m-1$, and

$$\left\|\frac{x_1 + x_2 + \ldots + x_m}{m}\right\| > 1 - \delta .$$

Hence, in particular, property B implies reflexivity.

Proof: R. C. James [11] has proved that since X is not reflexive, for some θ, $0 < \theta < 1$, there exist sequences $\{z_k\}$, $\{f_j\}$, in B_X and B_{X^*} respectively, such that $f_j(z_k) = \theta$ if $j \leq k$, $f_j(z_k) = 0$ if $j > k$.

Following James' argument, if n is a positive integer and if $p_1 < p_2 < \ldots < p_{2n}$ is a sequence of positive integers, let $S(p_1, \ldots, p_{2n}) = \{x \in X : f_j(x) = (-1)^i\theta$ if $p_{2i-1} \leq j \leq p_{2i}\}$, and $K_n = \liminf_{p_1}(\liminf_{p_2}(\ldots(\liminf_{p_{2n}}(\inf\|x\| : x \in S(p_1, \ldots, p_{2n}))) \ldots))$ for each n.

Notice that for each n, $\sum_{i=1}^{n} (z_{p_{2i}} - z_{p_{2i-1}} - 1)(-1)^i \in S(p_1, \ldots, p_{2n})$.

Since $\|\sum_{i-1}^{n} (z_{p_{2i}} - z_{p_{2i-1}} - 1)\| \leq 2n$ then $K_n \leq 2n$ for all n.

Also by the definition of K_n we have $K_{n+1} \geq K_n$. Let $\delta > 0$, $m > 1$, an integer, be given.

Since $\dfrac{K_n}{K_{n+1}} \leq 1$ and $\dfrac{K_n}{K_n} = 1$ for all n, there must exist $\varepsilon > 0$, and n a positive integer so large that $\dfrac{K_{n-1} - \varepsilon}{K_n + \varepsilon} > 1 - \delta$ and $\dfrac{K_n - \varepsilon}{K_n + \varepsilon} > 1 - \delta$.

For this n there exists a positive integer p such that if $p_1, p_2, \cdots, p_{2n}$ are positive integers with $p \leq p_1 < p_2 < \cdots < p_{2n}$ then $||z|| > K_n - \varepsilon$ whenever $z \in S(p_1, p_2, \cdots, p_{2n})$, and $||z|| > k_{n-1} - \varepsilon$ whenever $z \in S(p_1, p_2, \cdots, p_{2n-2})$.

Choose positive integers $p^1_1, p^1_2, \cdots p^1_{2n}, p^2_1, p^2_2, \cdots, p^2_{2n}, p^3_1, p^3_2, \cdots, p^3_{2n}, \cdots, p^{m-1}_1, p^{m-1}_2, \cdots, p^{m-1}_{2n}, p^m_1, p^m_2, \cdots, p^m_{2n}$, such that

$$p \leq p^1_1, < p^2_1 < p^3_1 < p^4_1 < \cdots < p^{m-1}_1 < p^m_1 < p^1_2 < p^1_3 < p^2_2 < p^2_3 < \cdots$$

$$\cdots < p^{m-1}_2 < p^{m-1}_3 < p^m_2 < p^m_3 < p^1_4 < p^1_5 < p^2_4 < p^2_5 < \cdots$$

$$\cdots < p^{m-1}_4 < p^{m-1}_5 < p^m_4 < p^m_5 < p^1_6 < p^1_7 < p^2_6 < p^2_7 < \cdots$$

$$\cdots < p^{m-1}_6 < p^{m-1}_7 < p^m_6 < p^m_7 < p^1_8 < p^1_9 < p^2_8 < p^2_9 < \cdots$$

$$\cdots$$

$$\cdots < p^m_{2n-4} < p^m_{2n-3} < p^1_{2n-2} < p^1_{2n-1} < p^2_{2n-2} < p^2_{2n-1} < \cdots$$

$$\cdots < p^{m-1}_{2n-2} < p^{m-1}_{2n-1} < p^m_{2n-2} < p^m_{2n-1} < p^1_{2n} < p^2_{2n} < p^3_{2n} < \cdots p^{m-1}_{2n} < p^m_{2n} ,$$

so large that there exist $u_1, \cdots, u_m \in X$, with

$$u_1 \in S(p^1_1, p^1_2, \cdots, p^1_{2n})$$

$$u_2 \in S(p^2_1, p^2_2, \cdots, p^2_{2n})$$

$$\vdots$$

$$u_{m-1} \in S(p^{m-1}_1, p^{m-1}_2, \cdots p^{m-1}_{2n})$$

$$u_m \in S(p^m_1, p^m_2, \cdots, p^m_{2n})$$

and $||u_1||, ||u_2||, \cdots, ||u_{m-1}||, ||u_m|| < K_n + \varepsilon$.

Let $S_1 = S(p^m_1, p^1_2, p^m_3, p^1_4, \dots, p^m_{2n-1}, p^1_{2n},)$

$S_2 = S(p^1_3, p^2_2, p^1_5, p^2_4, \dots, p^1_{2n-1}, p^2_{2n-2})$

$S_3 = S(p^2_3, p^3_2, p^2_5, p^3_4, \dots, p^2_{2n-1}, p^3_{2n-2})$

$S_4 = S(p^3_3, p^4_2, p^3_5, p^4_4, \dots, p^3_{2n-1}, p^4_{2n-2})$

$\vdots$

$S_{m-1} = S(p^{m-2}_3, p^{m-1}_2, p^{m-2}_5, p^{m-1}_4, \dots, p^{m-2}_{2n-1}, p^{m-1}_{2n-2})$

$S_m = S(p^{m-1}_3, p^m_2, p^{m-1}_5, p^m_4, \dots, p^{m-1}_{2n-1}, p^m_{2n-2})$.

By the definition of p if $z \in S_1$, then $||z|| > K_n - \varepsilon$, and if $z \in S_i$, $2 \leq i \leq m$, then $||z|| > K_{n-1} - \varepsilon$.

Letting $x_1 = \dfrac{u_1}{K_n + \varepsilon}$, $x_2 = \dfrac{u_2}{K_n + \varepsilon}$, $\dots$, $x_m = \dfrac{u_m}{K_n + \varepsilon}$, then $||x_1||$, $||x_2||, \dots, ||x_m|| < \dfrac{K_n + \varepsilon}{K_n + \varepsilon} = 1$.

We now show $\text{dist}(x_\ell, [x_{\ell+1}, \dots, x_m]) > 2\,(1 - \delta)$, $1 \leq \ell \leq m-1$.

Suppose $y \in [x_{\ell+1}, \dots, x_m]$, $1 \leq \ell \leq m-1$.

Then $y = \sum\limits_{n=\ell+1}^{m} \lambda_n x_n$ with $\sum\limits_{n=\ell+1}^{m} \lambda_n = 1$.

By the definition of $S_{\ell+1}$ it can be shown that $\dfrac{\sum\limits_{n=\ell+1}^{m} \lambda_n u_n - u_\ell}{2} \in S_{\ell+1}$,

hence $\left|\left|\dfrac{\sum\limits_{n=\ell+1}^{m} \lambda_n u_n - u_\ell}{2}\right|\right| > K_{n-1} - \varepsilon$ and therefore $\left|\left|\sum\limits_{n=\ell+1}^{m} \lambda_n x_n - x_\ell\right|\right| >$ $2\left(\dfrac{K_{n-1} - \varepsilon}{K_n + \varepsilon}\right) > 2\,(1 - \delta)$.

Since y is arbitrary in $[x_{\ell+1}, \ldots, x_m]$ then $\mathrm{dist}(x_\ell, [x_{\ell+1}, \ldots, x_m]) > 2(1 - \delta)$, $1 \leq \ell \leq m-1$.

Hence $\mathrm{dist}(x_1, [x_2, \ldots, x_m]) \cdot \mathrm{dist}(x_2, [x_3, \ldots, x_m]) \cdot \ldots \cdot \mathrm{dist}(x_{m-2}, [x_{m-1}, x_m]) \cdot ||x_{m-1} - x_m|| > 2^{m-1}(1 - \delta)^{m-1}$.

Now we show $\left|\left|\dfrac{x_1 + x_2 + \ldots + x_m}{m}\right|\right| > 1 - \delta$.

By the definition of S_1 it can be shown that $\dfrac{u_1 + u_2 + \ldots + u_m}{m} \in S_1$, hence $\left|\left|\dfrac{u_1 + \ldots + u_m}{m}\right|\right| > K_n - \varepsilon$, and therefore $\left|\left|\dfrac{x_1 + x_2 + \ldots + x_m}{m}\right|\right| > \dfrac{K_n - \varepsilon}{K_n + \varepsilon} > 1 - \delta$.

Thus $x_1, \ldots, x_m$ satisfy the desired conditions, proving the first part of the theorem.

Since $\delta > 0$, and $m > 1$, an integer, are arbitrary, we get that if X is not reflexive then it does not have property B, which is the second assertion of the theorem.

Theorem 2.12: If a Banach space X has property B then X is superreflexive.

Proof: Suppose X is not superreflexive.

Then there exists Y a Banach space finitely representable in X, and nonreflexive.

Let $\delta > 0$, and $m > 1$, an integer, be given.

If $|||\cdot|||$ is the norm on Y, then, by the previous theorem there exist $y_1, \ldots, y_m \in B_Y$ such that $\mathrm{dist}(y_\ell, [y_{\ell+1}, \ldots, y_m]) > 2(1 - \delta)$ for all

ℓ, $1 \leq \ell \leq m-1$, and $\left|\left|\left| \dfrac{y_1 + \dots + y_m}{m} \right|\right|\right| > 1 - \delta$.

Let Y_m be the linear span of $y_1, \dots, y_m$.

Since Y is finitely representable in X then there exists an isomorphism $T : Y_m \to X$ for which

$$\frac{(1-\delta)}{(1+\delta)} |||y||| \leq ||T(y)|| \leq |||y||| \quad \text{if} \quad y \in Y_m .$$

Let $x_1 = Ty_1, \dots, x_m = Ty_m$.

Then from above $||x_1||, ||x_2||, \dots, ||x_m|| \leq 1$.

Also, if $x = \sum_{n=\ell+1}^{m} \lambda_n x_n$ with $\sum_{n=\ell+1}^{m} \lambda_n = 1$, $1 \leq \ell \leq m-1$, then $||x_\ell - x|| \geq \dfrac{(1-\delta)}{(1+\delta)} |||y_\ell - y||| > \dfrac{(1-\delta)}{(1+\delta)} \cdot 2(1-\delta)$ where $y = \sum_{n=\ell+1}^{m} \lambda_n y_n$.

Hence $\text{dist}(x_\ell, [x_{\ell+1}, \dots, x_m]) > \dfrac{2(1-\delta)^2}{(1+\delta)}$ for all ℓ, $1 \leq \ell \leq m-1$, and $\text{dist}(x_1, [x_2, \dots, x_m]) \cdot \dots \cdot \text{dist}(x_{m-2}, [x_{m-1}, x_m]) ||x_{m-1} - x_m|| > 2^{m-1} \dfrac{(1-\delta)^{2m-2}}{(1+\delta)^{m-1}}$.

Similarly $\left|\left| \dfrac{x_1 + \dots + x_m}{m} \right|\right| > \dfrac{(1-\delta)}{(1+\delta)} (1-\delta)$.

Since $\delta > 0$, and $m > 1$, an integer, are arbitrary, we get that X does not have property B , contradicting the given hypothesis. Therefore X must be superreflexive.

That property A and property B are not necessary conditions for a Banach space to be superreflexive is an easy consequence of the next theorem.

Theorem 2.13 (van Dulst and Pach [6]): Let X be an infinite-dimensional Banach space with norm $||\cdot||$. Then there exist a norm $|||\cdot|||$ on X, equivalent to $||\cdot||$, and a sequence $(x_n)_{n=1}^{\infty}$ in X such that $|||x_n||| = 1$ for all n, $\frac{|||x_1 + \ldots + x_m|||}{m} = 1$ for all m, and $|||x_\ell - z||| \geq 2$ for any $\ell, k, 1 \leq \ell < k$, and $z \in [x_{\ell+1}, \ldots, x_k]$.

Proof: It can be shown that there exist $K > 0$, $(y_n)_{n=0}^{\infty}$ a sequence on the unit sphere of X, $(\overline{y}^*_n)_{n=0}^{\infty}$ a sequence in Y^*, where Y is the Banach space generated by $(y_n)_{n=0}^{\infty}$, satisfying the following properties [5]:

(i) $(y_n)_{n=0}^{\infty}$ is a basic sequence.

(ii) Whenever $\sum_{i=1}^{\infty} a_i y_i \in Y$, $||\sum_{i=1}^{n} a_i y_i|| \leq K\, ||\sum_{i=1}^{\infty} a_i y_i||$ for all n.

(iii) $\overline{y}^*_m(y_n) = 0$ if $n \neq m$, $\overline{y}^*_m(y_n) = 1$ if $n = m$.

(iv) $||\overline{y}^*_n|| \leq 2K$ for all n.

For each n let y^*_n be the Hahn-Banach extension to X of $\overline{y}^*_n$.

Also for each n, $n \geq 1$, let $x_n = y_0 - 2y_n$, $x^*_n = y^*_0 + y^*_n$.

We define $|||\cdot|||$ on X by

$$|||x||| \equiv \sup\left(\{\tfrac{1}{3}||x||\} \cup \{|x^*_n(x)| : n = 1, 2, \ldots\}\right) \text{ for } x \in X.$$

$|||\cdot|||$ is clearly a norm on X.

By the definition of x^*_n

$$||x^*_n|| \leq ||y^*_0|| + ||y^*_n|| \leq 4K \text{ for all } n,$$

so $|x^*_n(x)| \leq 4K\, ||x||$ for all n and all $x \in X$,

and $|||x||| \leq \max(\frac{1}{3}, 4K)\, ||x||$ for all $x \in X$.

Since also $|||x||| \geq \frac{1}{3} ||x||$ for all $x \in X$, it follows that $||| \cdot |||$ is equivalent to $|| \cdot ||$.

By (iii) if $m \neq n$ then

$$\begin{aligned} x^*_m(x_n) &= y^*_0(y_0 - 2y_n) + y^*_m(y_0 - 2y_n) \\ &= y^*_0(y_0) - y^*_0(2y_n) + y^*_m(y_0) - y^*_m(2y_n) \\ &= 1 - 0 + 0 - 0 \\ &= 1 \end{aligned}$$

and if $m = n$ then

$$\begin{aligned} x^*_m(x_n) &= y^*_0(y_0) - y^*_0(2y_n) + y^*_m(y_0) - y^*_m(2y_n) \\ &= 1 - 0 + 0 - 2 \\ &= -1 . \end{aligned}$$

Also $||x_n|| \leq ||y_0|| + ||2y_n|| = 3$ and $|x^*_m(x_n)| = 1$ for all n, m.

Hence $|||x_n||| = 1$ for all n.

Similarly, $\left| x^*_{m+1}\left(\frac{x_1 + \dots + x_m}{m}\right)\right| = \frac{m}{m} = 1$ for all m, so $\left|\left|\left| \frac{x_1 + \dots + x_m}{m} \right|\right|\right| = 1$ for all m.

Suppose now that $z \in [x_{\ell+1}, \dots, x_k]$, $1 \leq \ell < k$.

Then $z = \sum_{j=\ell+1}^{k} \lambda_j x_j$ where $\sum_{j=\ell+1}^{k} \lambda_j = 1$.

Again, $|x^*_\ell(x_\ell - z)| = |-1 - 1| = 2$, hence $|||x_\ell - z||| \geq 2$ which completes the proof of the theorem.

In spite of the last theorem, the following result shows that under certain circumstances, a Banach space with a given norm $|||\cdot|||$ has property B .

Theorem 2.14: Let X be a Banach space, and $||\cdot||$ its norm. If for some $\varepsilon > 0$ $\lim_{k\to\infty} \delta'_k(\varepsilon) = 1$, where $\delta'_k(\varepsilon) = \inf\{1 - \frac{||x_1 + \ldots + x_{k+1}||}{k+1} :$ $||x_1|| = \ldots = ||x_{k+1}|| = 1$, and $||x_1 - x_2|| \cdot \mathrm{dist}(x_3, [x_1, x_2]) \cdot \ldots \cdot \mathrm{dist}(x_{k+1}, [x_1, \ldots, x_k]) \geq \varepsilon\}$, then $(X, |||\cdot|||)$ has property B whenever for some $A, C > 0$ with $\frac{A}{C} > \frac{1}{2}$, $A\,|||x||| \leq ||x|| \leq C\,|||x|||$ for all $x \in X$.

Proof: For any $x \in X$ let $||x||' = \frac{A}{C}||x||$. It is easy to show that we still get $\lim_{k\to\infty} \delta'_k(\varepsilon) = 1$ if δ'_k is now defined in terms of $||\cdot||'$.

From the hypothesis we now have

$$\frac{A}{C}|||x||| \leq ||x||' \leq |||x||| \quad \text{for all } x \in X \quad (*) .$$

Assume $(X, |||\cdot|||)$ does not have property B .

Pick M , $\frac{C}{A} < M < 2$.

Then, in particular, given any integer k , $k > 1$, there exist $x_1, \ldots,$ x_k, $|||x_1|||, \ldots, |||x_k||| \leq 1$, with $\frac{|||x_1 + \ldots + x_k|||}{k} > .9$, and $|||x_1 - x_2||| \cdot \overline{\mathrm{dist}}(x_3, [x_1, x_2]) \cdot \ldots \cdot \overline{\mathrm{dist}}(x_k, [x_1, \ldots, x_{k-1}]) > M^{k-1}$, where $\overline{\mathrm{dist}}(x_i, [x_1, \ldots, x_{i-1}])$ is $\mathrm{dist}(x_i, [x_1, \ldots, x_{i-1}])$ calculated with $|||\cdot|||$.

By (*) $||x_1||', \ldots, ||x_k||' \leq 1$, and $\frac{||x_1 + \ldots + x_k||'}{k} > .9\frac{A}{C}$.

Also, since $\frac{A}{C} \cdot M > 1$, $||x_1 - x_2||' \cdot \text{dist}'(x_3, [x_1, x_2]) \cdot \ldots \cdot$ $\text{dist}'(x_k, [x_1, \ldots, x_{k-1}]) \geq \left(\frac{A}{C}\right)^{k-1} \cdot |||x_1 - x_2||| \cdot \overline{\text{dist}}(x_3, [x_1, x_2])$ $\cdot \ldots \cdot \overline{\text{dist}}(x_k, [x_1, \ldots, x_{k-1}]) > \left(\frac{A}{C}\right)^{k-1} \cdot M^{k-1} = \left(\frac{A}{C} M\right)^{k-1} > \varepsilon$ for sufficiently large k , where $\text{dist}'(x_i, [x_1, \ldots, x_{i-1}])$ is $\text{dist}(x_i,$ $[x_1, \ldots, x_{i-1}])$ calculated with $|| \cdot ||'$.

Thus, from above it is clear that $\lim_{k\to\infty} \delta'_k(\varepsilon) \neq 1$, which is a contradiction.

Hence $(X, ||| \cdot |||)$ must have property B .

<u>Corollary 2.15</u>: If $(X, || \cdot ||)$ is a Hilbert space then $(X, ||| \cdot |||)$ has property B whenever for some $A, C > 0$ with $\frac{A}{C} > \frac{1}{2}$, $A\ |||x||| \leq$ $||x|| \leq C\ |||x|||$ for all $x \in X$.

<u>Proof</u>: By Theorem 1.11 $\lim_{k\to\infty} \delta'_k(\varepsilon) = 1$ for any $\varepsilon > 0$. By Theorem 2.14 $(X, ||| \cdot |||)$ must have property B .

As we shall see, it is a consequence of the last corollary that the space given in Karlovitz's example has property B .

Because of Theorem 2.13, it is evident that properties A and B should be examined from the isometric point of view. With this in mind, we show that spaces satisfying either of the two area related conditions also possess some special geometrical structures related to fixed points.

<u>Definition 2.16</u>: A Banach space X is said to have normal structure if for each bounded convex subset K of X , consisting of more than one point, there is a point $x \in K$ such that $\sup\{||x - k|| : k \in K\} < \text{diam}(K)$ $\equiv \sup\{||h-k|| : h, k \in K\}$.

Definition 2.17: A Banach space X is said to have close to normal structure if for each bounded convex subset K of X consisting of more than one point there is a point $x \in K$ such that $||x - k|| < \operatorname{diam}(K)$ for all $k \in K$.

Throughout this chapter a subset K of a Banach space X will be called abnormal whenever K is bounded, convex, consists of more than one point, and for all $x \in K$ $\sup\{||x - k|| : k \in K\} = \operatorname{diam}(K)$.

Intuitively, in a Banach space having normal (close to normal) structure, every bounded convex subset has at least one point that acts as its "center of mass."

Banach spaces with normal structure are of importance because they satisfy the fixed point property [18], which we define next.

Definition 2.18: A Banach space X is said to have the fixed point property if for every weakly compact convex subset K of X, and every mapping $T : K \to K$ which is nonexpansive, i.e. $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in K$, there exists $z \in K$ with $Tz = z$.

We will show that Banach spaces possessing property A have normal structure so that they also must satisfy the fixed point property. First we consider some preliminary lemmas.

Lemma 2.19: If K is an abnormal subset of a Banach space with $\operatorname{diam}(K) = 1$, then, given (β_n), a decreasing sequence of positive numbers converging to zero, there exists a sequence (x_n) in K such that $||x_i - z|| \geq 1 - \beta_i$ whenever $z \in \operatorname{co}(x_1, \ldots, x_{i-1})$, $i \geq 2$.

Proof: By induction we choose (x_n) in K satisfying the following property:

$$\left\|\frac{x_1 + \dots + x_{i-1}}{i-1} - x_i\right\| > 1 - \frac{\beta_i}{(i-1)} \qquad \text{for} \quad i \geq 2 .$$

Since $K \neq \phi$, pick $x_1 \in K$.

K being abnormal implies we may choose $x_2 \in K$ with $||x_1 - x_2|| > 1 - \beta_2$ $= 1 - \frac{\beta_2}{(2-1)}$ so the property holds for $n = 2$.

Suppose it is true for $n = i$. We show it holds for $n = i+1$.

By the induction hypothesis, we may choose $x_1, \dots, x_i \in K$ such that

$$\left\|\frac{x_1 + \dots + x_{i-1}}{i-1} - x_i\right\| > 1 - \frac{\beta_i}{i-1} .$$

By the convexity of K, $\frac{x_1 + \dots + x_{i-1} + x_i}{i} \in K$, so again since K is abnormal we may choose x_{i+1} with $\left\|\frac{x_1 + \dots + x_i}{i} - x_{i+1}\right\| > 1 - \frac{\beta_{i+1}}{i}$ $= 1 - \frac{\beta_{i+1}}{i+1 - 1}$ as required. Hence, we may choose (x_n) in K satisfying the given property.

Now, suppose $z \in co(x_1, \dots, x_{i-1})$, $i \geq 2$.

We show $||z - x_i|| \geq 1 - \beta_i$.

If $i = 2$ then $z = x_1$ and $x_i = x_2$ so $||z - x_i|| = ||x_1 - x_2|| > 1 - \beta_2$.

Thus, let us assume $i > 2$ and $||z - x_i|| < 1 - \beta_i$ $(*)$.

Since $z \in co(x_1, \dots, x_{i-1})$ then $z = \sum_{j=1}^{i-1} \lambda_j x_j$, where $\sum_{j=1}^{i-1} \lambda_j = 1$, $\lambda_j \geq 0$ for $1 \leq j \leq i - 1$.

Hence, as shown above

$$\left|\left| \frac{(1-\lambda_1)\, x_1 + \dots + (1-\lambda_{i-1})\, x_{i-1}}{i-1} + \frac{\lambda_1 x_1 + \dots + \lambda_{i-1}\, x_{i-1}}{i-1} - \frac{x_i}{i-1} - \frac{(i-2)}{(i-1)}\, x_i \right|\right| = \left|\left| \frac{x_1 + \dots + x_{i-1}}{i-1} - x_i \right|\right| > 1 - \frac{\beta_i}{(i-1)}$$

so by the triangle inequality

$$\left|\left| \frac{(1-\lambda_1)\, x_1 + \dots + (1-\lambda_{i-1})\, x_{i-1}}{i-1} - \frac{i-2}{i-1}\, x_i \right|\right| + \left|\left| \frac{\lambda_1 x_1 + \dots + \lambda_{i-1}\, x_{i-1}}{i-1} - \frac{x_i}{i-1} \right|\right| > 1 - \frac{\beta_i}{(i-1)}$$

or

$$\left|\left| \frac{\sum_{j=1}^{i-1} (1-\lambda_j)\, x_j}{i-1} - \frac{i-2}{i-1}\, x_i \right|\right| > 1 - \frac{\beta_i}{i-1} - \frac{1}{i-1}\, ||z - x_i||$$

and by (*)

$$\left|\left| \frac{\sum_{j=1}^{i-1} (1-\lambda_j)\, x_j}{i-1} - \frac{i-2}{i-1}\, x_i \right|\right| > 1 - \frac{\beta_i}{i-1} - \frac{1}{i-1}\,(1 - \beta_i)$$

$$= 1 - \frac{1}{i-1} = \frac{i-2}{i-1} \quad .$$

Multiplying both sides of this inequality by $\frac{i-1}{i-2}$ we obtain

$$\left|\left| \frac{\sum_{j=1}^{i-1} (1-\lambda_j)\, x_j}{i-2} - x_i \right|\right| > 1 \qquad \text{with} \qquad \frac{\sum_{j=1}^{i-1} (1-\lambda_j)}{i-2} = \frac{i-1-1}{i-2} = 1 \, .$$

So by the convexity of K , $\frac{\sum_{j=1}^{i-1} (1-\lambda_j)\, x_j}{i-2} \in K$, but since $x_i \in K$ and $\operatorname{diam}(K) = 1$ we get a contradiction. Therefore $||z - x_i|| \geq 1 - \beta_i$

whenever $z \in co(x_1, \ldots, x_{i-1})$, $i \geq 2$, and the proof of the lemma is complete.

Lemma 2.20: If K is an abnormal subset of a Banach space X with $diam(K) = 1$, then, given (β_n), a decreasing sequence of positive numbers converging to zero, there exists a sequence (x_n) in K satisfying the following properties:

(i) $||z - x_{L+i}|| \geq 1 - \beta_L$ whenever z $co(x_{L+1}, \ldots, x_{L+i-1})$ for all positive integers $L, i \geq 2$.

(ii) $||z - x_{L+i}|| \geq 1 - N\,(i-1)\,\beta_L$ whenever $N > 0$, $z = \sum_{j=1}^{i-1} \lambda_j x_{L+j}$, $\sum_{j=1}^{i-1} \lambda_j = 1$, $B = \{-\lambda_j : \lambda_j < 0, 1 \leq j \leq i-1\} \neq \phi$, and $|\lambda_j| \leq N$, $1 \leq j \leq i-1$, for all positive integers $L, i \geq 3$.

Proof: By Lemma 2.19 there exists (x_n) a sequence in K such that $||x_i - z|| \geq 1 - \beta_i$ whenever $z \in co(x_1, \ldots, x_{i-1})$, $i \geq 2$. We show (x_n) is the desired sequence in K.

Let L be any positive integer. If $z \in co(x_{L+1}, \ldots, x_{L+i-1})$, $i \geq 2$, then from above $||x_{L+i} - z|| \geq 1 - \beta_{L+i} > 1 - \beta_L$ since (β_n) is a decreasing sequence, and $co(x_{L+1}, \ldots, x_{L+i-1}) \subseteq co(x_1, \ldots, x_{L+i-1})$.

Thus, (i) is satisfied by (x_n).

Again, let L be any positive integer, and suppose

$$||z - x_{L+i}|| < 1 - N\,(i-1)\,\beta_L \qquad (*)$$

where $z = \sum_{j=1}^{i-1} \lambda_j x_{L+j}$, $\sum_{j=1}^{i-1} \lambda_j = 1$, $B = \{-\lambda_j : \lambda_j < 0, 1 \leq j \leq i-1\} \neq \phi$, and $|\lambda_j| \leq N$, $1 \leq j \leq i-1$, N some positive number, $i \geq 3$.

Let $C = \{\lambda_j : \lambda_j \geq 0 , 1 \leq j \leq i-1\}$.

Since B and C are finite sets there must exist integers $k, h > 0$ such that $B = \{\eta_j : j = 1, \ldots, k\}$, $C = \{\gamma_j : j = 1, \ldots, h\}$.

Notice that since $\sum_{j=1}^{i-1} \lambda_j = 1$, then $\sum_{j=1}^{h} \gamma_j \neq 0$, and $\sum_{j=1}^{h} \gamma_j - \sum_{j=1}^{k} \eta_j = 1$.

Also since $B \neq \phi$, then $\sum_{j=1}^{k} \eta_j \neq 0$.

We will set $x_{\eta_j} = x_{L+n}$ if $-\eta_j = \lambda_n$, and $x_{\gamma_j} = x_{L+n}$ if $\gamma_j = \lambda_n$ for all n , $1 \leq n \leq i-1$.

Hence $z = \sum_{j=1}^{i-1} \lambda_j x_{L+j} = \sum_{j=1}^{h} \gamma_j x_{\gamma_j} - \sum_{j=1}^{k} \eta_j x_{\eta_j}$ or $z + \sum_{j=1}^{k} \eta_j x_{\eta_j} = \sum_{j=1}^{h} \gamma_j x_{\gamma_j}$.

Since $\sum_{j=1}^{h} \gamma_j \neq 0$ we may divide both sides by this amount to obtain

$$\frac{z + \sum_{j=1}^{k} \eta_j x_{\eta_j}}{\sum_{j=1}^{h} \gamma_j} = \frac{\sum_{j=1}^{h} \gamma_j x_{\gamma_j}}{\sum_{j=1}^{h} \gamma_j} .$$

Since $\dfrac{\sum_{j=1}^{h} \gamma_j}{\sum_{j=1}^{h} \gamma_j} = 1$ and $\gamma_j \geq 0$ for each j , $1 \leq j \leq h$ we get that

$$\frac{\sum_{j=1}^{h} \gamma_j x_{\gamma_j}}{\sum_{j=1}^{h} \gamma_j} \in co(x_{\gamma_1}, \ldots, x_{\gamma_h}) \subseteq co(x_{L+1}, \ldots, x_{L+i-1}) \subseteq$$

$co(x_1, \ldots, x_{L+i-1})$.

So by the choice of (x_n) we have

$$\left\| \frac{z + \sum_{j=1}^{k} \eta_j x_{\eta_j}}{\sum_{j=1}^{h} \gamma_j} - x_{L+1} \right\| = \left\| \frac{\sum_{j=1}^{h} \gamma_j x_{\gamma_j}}{\sum_{j=1}^{h} \gamma_j} - x_{L+i} \right\| > 1 - \beta_{L+i} > 1 - \beta_L$$

or

$$\left\| \frac{z}{\sum_{j=1}^{h} \gamma_j} - \frac{x_{L+i}}{\sum_{j=1}^{h} \gamma_j} + \frac{\sum_{j=1}^{k} \eta_j x_{\eta_j}}{\sum_{j=1}^{h} \gamma_j} - \frac{\sum_{j=1}^{h} \gamma_j - 1}{\sum_{j=1}^{h} \gamma_j} x_{L+i} \right\| > 1 - \beta_L$$

and by the triangle inequality

$$\frac{1}{\sum_{j=1}^{h} \gamma_j} \|z - x_{L+i}\| + \left\| \frac{\sum_{j=1}^{k} \eta_j x_{\eta_j}}{\sum_{j=1}^{h} \gamma_j} - \frac{\sum_{j=1}^{k} \eta_j}{\sum_{j=1}^{h} \gamma_j} x_{L+i} \right\| > 1 - \beta_L$$

giving

$$\left\| \frac{\sum_{j=1}^{k} \eta_j x_{\eta_j}}{\sum_{j=1}^{h} \gamma_j} - \frac{\sum_{j=1}^{k} \eta_j}{\sum_{j=1}^{h} \gamma_j} x_{L+i} \right\| > 1 - \beta_L - \frac{1}{\sum_{j=1}^{h} \gamma_j} \|z - x_{L+i}\|$$

$$> 1 - \beta_L - \frac{1}{\sum_{j=1}^{h} \gamma_j} (1 - N \cdot (i-1) \cdot \beta_L) \quad \text{by } (*)$$

$$= 1 - \beta_L - \frac{1}{\sum_{j=1}^{h} \gamma_j} + \frac{N \cdot (i-1) \cdot \beta_L}{\sum_{j=1}^{h} \gamma_j}$$

$$= 1 - \frac{1}{\sum_{j=1}^{h} \gamma_j} + \frac{N \cdot (i-1) \cdot \beta_L}{\sum_{j=1}^{h} \gamma_j} - \beta_L$$

$$= \frac{\sum_{j=1}^{h} \gamma_j - 1}{\sum_{j=1}^{h} \gamma_j} + \beta_L \left(\frac{N \cdot (i-1)}{\sum_{j=1}^{h} \gamma_j} - 1 \right)$$

$$> \frac{\sum_{j=1}^{k} \eta_j}{\sum_{j=1}^{h} \gamma_j} \quad \text{since} \quad \frac{N \cdot (i-1)}{\sum_{j=1}^{h} \gamma_j} - 1 > 0 \quad \text{for } \gamma_j \leq N \text{ for all } j\text{'s} .$$

Multiplying both sides of this inequality by $\dfrac{\sum_{j=1}^{h} \gamma_j}{\sum_{j=1}^{k} \eta_j}$ we obtain

$$\left\| \frac{\sum_{j=1}^{k} \eta_j x_{n_j}}{\sum_{j=1}^{k} \eta_j} - x_{L+i} \right\| > 1 \qquad \text{with} \qquad \frac{\sum_{j=1}^{k} \eta_j}{\sum_{j=1}^{k} \eta_j} = 1 .$$

So by the convexity of K, $\dfrac{\sum_{j=1}^{k} \eta_j x_{n_j}}{\sum_{j=1}^{k} \eta_j} \in K$, but since $x_{L+i} \in K$ and

$\operatorname{diam}(K) = 1$ we get a contradiction. Therefore $||z - x_i|| \geq 1 - N \cdot (i-1) \cdot \beta_L$, z as defined above, and (ii) is satisfied by (x_n).

This completes the proof of the lemma.

Now that the last technical lemma has become available we are ready to prove a main result of this chapter.

<u>Theorem 2.21</u>: If a Banach space X has property A then it has normal structure.

<u>Proof</u>: Suppose X does not have normal structure.

Then X contains an abnormal subset K, and without any loss of generality we may assume it is closed.

Since normality is invariant under scalar multiplication and translation, we may assume without any loss of generality that $\text{diam}(K) = 1$ and that K is bounded away from zero.

Pick ε, m, δ, $0 < \varepsilon < 1$, m an integer, $m > 1$, $\delta > 0$, as in the definition of property A.

Let (β_n) be a decreasing sequence of positive numbers converging to zero, with $\beta_n < \delta$ for all n.

Since $\text{diam}(K) = 1$, choose a sequence (x_n) in K corresponding to (β_n) as in Lemma 2.20.

Let L be any positive integer, and set $y_1^L = x_{L+1} - x_{L+m+1}$, $y_2^L = x_{L+2} - x_{L+m+1}$, ..., $y_m^L = x_{L+m} - x_{L+m+1}$.

We know from Lemma 2.20 (i) that in particular

$$\left\| \frac{x_{L+1} + x_{L+2} + \dots + x_{L+m}}{m} - x_{L+m+1} \right\| > 1 - \beta_L > 1 - \delta .$$

So $$\left\| \frac{y_1^L + y_2^L + \dots + y_m^L}{m} \right\|$$

$$= \left\| \frac{(x_{L+1} - x_{L+m+1}) + \dots + (x_{L+m} - x_{L+m+1})}{m} \right\| > 1 - \delta .$$

Also $||y_1^L||, \ldots, ||y_m^L|| \leq 1$ since $\text{diam}(K) = 1$.

By property A we must have

$$||y_2^L - y_1^L|| \cdot \text{dist}(y_3^L, [y_2^L, y_3^L]) \cdot \ldots \cdot \text{dist}(y_m^L, [y_{m-1}^L, \ldots, y_1^L]) < \varepsilon$$

which implies, by the definition of $y_1^L, \ldots, y_m^L$, that

$||x_{L+2} - x_{L+1}|| \cdot \text{dist}(x_{L+3}, [x_{L+2}, x_{L+1}]) \cdot \ldots \cdot \text{dist}(x_{L+m}, [x_{L+m-1}, \ldots, x_{L+1}]) < \varepsilon$ (*).

We show that for very large L the left side of (*) is bounded below by a number between ε and 1, which will be a contradiction.

Since by Theorem 2.10 property A implies superreflexivity and thus reflexivity we may assume without any loss of generality that there exists $x \in X$, such that $x_n \to x$ weakly.

Since K is closed and convex then K must be weakly closed so that $x \in K$. Also, since K is bounded away from zero we know $x \neq 0$, and hence, $||x|| \neq 0$.

Let $| \cdot |_i$ be the Euclidean norm on R^i.

For a given positive integer L, and $i \geq 2$, define $F_L^i : R^i \to R$ by $F_L^i((a_1, \ldots, a_i)) = ||a_1 x_{L+1} + \ldots + a_i x_{L+i}||$ for $(a_1, \ldots, a_i) \in R^i$.

It is easy to prove that F_L^i is continuous.

Let $U_i = \{(a_1, \ldots, a_i) : (a_1, \ldots, a_i) \in R^i, |(a_1, \ldots, a_i)|_i = 1, \sum_{j=1}^{i} a_j \geq 0\}$.

Since U_i is compact for each i, then for every L and i there exists $(b^i_{L+1}, \ldots, b^i_{L+i}) \in U_i$ such that $F^i_L((b^i_{L+1}, \ldots, b^i_{L+i})) = ||b^i_{L+1} x_{L+1} + \ldots + b^i_{L+i} x_{L+i}|| \leq ||a_1 x_{L+1} + \ldots + a_i x_{L+i}||$ for all $(a_1, \ldots, a_i) \in U_i$.

We show there exists L', a positive integer, and A, a positive number, such that if $L \geq L'$ then $||a_1 x_{L+1} + \ldots + a_i x_{L+i}|| \geq A$ for all $(a_1, \ldots, a_i) \in U_i$, for each i, $2 \leq i \leq m$.

Without any loss of generality, by the compactness of U_i, we may assume that for each i, $2 \leq i \leq m$ there exists $(b^i_1, \ldots, b^i_i) \in U_i$ such that $(b^i_{L+1}, \ldots, b^i_{L+i}) \to (b^i_1, \ldots, b^i_i)$.

Since $(b^i_1, \ldots, b^i_i) \in U_i$ then $|(b^i_1, \ldots, b^i_i)|_i = 1$ and $\sum_{j=1}^{i} b^i_j \geq 0$ for all i, $2 \leq i \leq m$.

For a given i, $2 \leq i \leq m$, suppose $\sum_{j=1}^{i} b^i_j > 0$.

Then since $x_n \overset{w}{\to} x$, and there exists $f \in X^*$, $||f|| = 1$, with $f(x) = ||x||$, we must have $f(b^i_1 x_{L+1} + \ldots + b^i_i x_{L+i}) \to f(\sum_{j=1}^{i} b^i_j x)$ $= \sum_{j=1}^{i} b^i_j f(x) = \sum_{j=1}^{i} b^i_j ||x|| > 0$ since $||x|| \neq 0$.

But $f(b^i_{L+1} x_{L+1} + \ldots + b^i_{L+i} x_{L+i}) \leq ||b^i_{L+1} x_{L+1} + \ldots + b^i_{L+i} x_{L+i}||$ so there must exist $A_i > 0$, and L'_i a positive integer, such that if $L \geq L'_i$ then $||a_1 x_{L+1} + \ldots + a_i x_{L+i}|| \geq A_i$ for all $(a_1, \ldots, a_i) \in U_i$.

Suppose now that $\sum_{j=1}^{i} b^i_j = 0$, and A_i, L'_i do not exist as above. Then for arbitrarily small $\beta > 0$ we can find arbitrarily large L such that

$$||b^i_1 x_{L+1} + \dots + b^i_i x_{L+i}|| < \beta .$$

Without any loss of generality assume $b^i_i \neq 0$.

Since $\sum_{j=1}^{i} b^i_j = 0$ then $b^i_1 + \dots + b^i_{i-1} = - b^i_i$ so $\frac{b^i_1}{-b^i_i} + \dots + \frac{b^i_{i-1}}{- b^i_i}$ $= 1$.

Notice $||b^i_1 x_{L+1} + \dots + b^i_{i-1} x_{L+i-1} - (- b^i_i) x_{L+i}|| =$

$$||b^i_1 x_{L+1} + \dots + b^i_{i-1} x_{L+i-1} + b^i_i x_{L+i}|| < \beta .$$

Hence $\left|\left| \frac{b^i_1}{- b^i_i} x_{L+1} + \dots + \frac{b^i_{i-1}}{- b^i_i} x_{L+i-1} - \frac{(- b^i_i)}{- b^i_i} x_{L+i} \right|\right| < \frac{\beta}{|b^i_i|}$

or $\left|\left| \frac{b^i_1}{- b^i_i} x_{L+1} + \dots + \frac{b^i_{i-1}}{- b^i_i} x_{L+i-1} - x_{L+i} \right|\right| < \frac{\beta}{|b^i_i|}$ (**) .

Letting $N = \sup \left\{ \left| \frac{b^i_1}{- b^i_i} \right| , \dots, \left| \frac{b^i_{i-1}}{- b^i_i} \right| \right\}$ we must have by (i) and (ii) of Lemma 2.20 that

$$\left|\left| \frac{b^i_1}{- b^i_i} x_{L+1} + \dots + \frac{b^i_{i-1}}{- b^i_i} x_{L+i-1} - x_{L+i} \right|\right| \geq \min(1 - \beta_L, 1 - N \cdot (i-1) \cdot$$

$\beta_L)$ for all L .

But since $\beta_L \to 0$ as $L \to \infty$ the last inequality contradicts (**) for β is very small.

So L'_i and A_i must exist as desired. Also by letting $L' = \max_{2 \leq i \leq m} \{L'_i\}$, and $A = \min_{2 \leq i \leq m} \{A_i\}$ the claim is seen to be valid.

Assume, now, $\lambda = (\lambda_1, \dots, \lambda_i) \in R^i$, $\sum_{j=1}^{i} \lambda_j = 1$, $2 \leq i \leq m$.

Then, since $\sum_{j=1}^{i} \frac{\lambda_j}{|\lambda|_i} = \frac{1}{|\lambda|_i} > 0$ and $\left|\frac{\lambda}{|\lambda|_i}\right|_i = 1$, we must have $\frac{\lambda}{|\lambda|_i} \in U_i$, and by the fact just shown above we get

$$\left\|\frac{\lambda_1}{|\lambda|_i} x_{L+1} + \dots + \frac{\lambda_i}{|\lambda|_i} x_{L+i}\right\| \geqslant A \quad \text{for} \quad L \geqslant L',$$

so $\|\lambda_1 x_{L+1} + \dots + \lambda_i x_{L+i}\| \geqslant A |\lambda|_i$ for $L \geqslant L'$.

From this inequality, then, we may easily conclude that given $P > 0$, there exists $N > 0$, such that if $z = \sum_{j=1}^{i} \lambda_j x_{L+j}$, $\sum_{j=1}^{i} \lambda_j = 1$, $2 \leqslant i \leqslant m$, and $\sup_{1 \leqslant j \leqslant i} |\lambda_j| > N$ then $\|z\| > P$, for all $L \geqslant L'$.

Let $M = \sup\{\|x\| : x \in K\}$, and pick $N > 0$ corresponding to $M + 1$ as above.

For $2 \leqslant i \leqslant m$, and $L \geqslant L'$, we show $\mathrm{dist}(x_{L+i}, [x_{L+1}, \dots, x_{L+i-1}]) \geqslant \min(1 - \beta_L, 1 - N \cdot (i-1) \cdot \beta_L)$ (***).

If $z = \sum_{j=1}^{i-1} \lambda_j x_{L+j}$, $\sum_{j=1}^{i-1} \lambda_j = 1$, then by (i) and (ii) of Lemma 2.20 we get for all L $\|x_{L+i} - z\| \geqslant \min(1 - \beta_L, 1 - N \cdot (i-1) \cdot \beta_L)$ whenever $|\lambda_j| \leqslant N$ for all j, $1 \leqslant j \leqslant i-1$.

On the other hand, for $L \geqslant L'$, if $\sup_{1 \leqslant j \leqslant i-1} |\lambda_j| > N$, we know by the choice of N that $\|z\| > M+1$.

So $\|x_{L+i} - z\| \geqslant \|z\| - \|x_{L+i}\| > M+1-M = 1 > \min(1 - \beta_L, 1 - N \cdot (i-1) \cdot \beta_L)$.

Thus, $\|x_{L+i} - z\| \geqslant \min(1 - \beta_L, 1 - N \cdot (i-1) \cdot \beta_L)$ for $L \geqslant L'$, for all $z \in [x_{L+1}, \dots, x_{L+i-1}]$, and (***) must hold.

Since N is fixed, $2 \leqslant i \leqslant m$, and $\beta_L \to 0$, then from (***), by picking L

very large, we can make $\text{dist}(x_{L+i}, [x_{L+i-1}, \ldots, x_{L+1}])$ bounded below by a number arbitrarily close to 1, for all i, $2 \leq i \leq m$. This, in turn, will imply that for very large L we can make $||x_{L+2} - x_{L+1}|| \cdot \text{dist}(x_{L+3}, [x_{L+2}, x_{L+1}]) \cdot \ldots \cdot \text{dist}(x_{L+m}, [x_{L+m-1}, \ldots, x_{L+1}])$ bounded below by a number between ε and 1, which is the desired contradiction to $(*)$.

Therefore, X must have normal structure.

Corollary 2.22: If a Banach space X has property A then it satisfies the fixed point property.

Proof: By Theorem 2.21 property A implies X has normal structure, and this, in turn, implies the fixed point property [18].

At this point we will examine Karlovitz's example [17]. The purpose of this example was to expose a superreflexive space which does not have normal structure but in which the fixed point property is satisfied. We will show that it has property B even though it does not possess property A.

Example 2.23: Let X be the space ℓ_2 renormed as follows: $||x|| = \max\{||x||_\infty, ||x||_2/\sqrt{2}\}$, for all $x \in X$, where $||\cdot||_\infty$ is the ℓ_∞ norm and $||\cdot||_2$ the ℓ_2 norm.

Let $K = \{x : x = x(i), x(i) \in X, x(i) \geq 0, ||x||_2 \leq 1\}$. It can easily be proved that K is abnormal in X.

Thus, by Theorem 2.21, X does not have property A.

It is also easy to see by the definition of $||\cdot||$ that

$$||x|| \leq ||x||_2 \leq \sqrt{2}\,||x||$$

and since ℓ_2 is a Hilbert space, $\frac{1}{\sqrt{2}} > \frac{1}{2}$, we must have by Corollary 2.15 that X satisfies property B.

In the rest of this chapter we will concentrate on proving the following theorem in which property B is characterized in terms of a property related to normal sets. Here, a convex bounded subset of a Banach space consisting of more than one point will be normal (close to normal) if it has a center of mass as in the definition of normal (close to normal) structure.

Theorem 2.24: The following conditions on a Banach space X are equivalent.

(i) If Y is a Banach space finitely representable in X, then every subset of S_Y with diameter equal to 2 is normal.

(ii) If Y is a Banach space finitely representable in X, then every subset of S_Y with diameter equal to 2 is close to normal.

(iii) There exist δ, m, $\delta > 0$, m an integer, $m > 1$, such that if $x_1, \ldots, x_m \in B_X$, then $\text{dist}(x_m, \text{co}(x_{m-1}, \ldots, x_1)) \cdot \ldots \cdot \text{dist}(x_3, \text{co}(x_2, x_1)) \cdot ||x_2 - x_1|| < 2^{m-1}(1 - \delta)$ whenever $\left\| \frac{x_1 + \ldots + x_m}{m} \right\| > 1 - \delta$.

(iv) X has property B.

Proof: (i) → (ii): This follows easily since normal sets are also close to normal.

(ii) → (iii): Assume (iii) is not true.

Let (ε_n) be an increasing sequence of real numbers between zero and one converging to one.

Since X fails to satisfy (iii), given an integer m, $m > 1$, there exist

$x_1^m, \ldots, x_m^m \in B_X$ with $\left\| \dfrac{x_1^m + \ldots + x_m^m}{m} \right\| > \varepsilon_m$, and $\text{dist}(x_m^m, \text{co}(x_{m-1}^m, \ldots, x_1^m)) \cdot \ldots \cdot \text{dist}(x_3^m, \text{co}(x_2^m, x_1^m)) \cdot \|x_2^m - x_1^m\| > 2^{m-1} \varepsilon_m$.

Since $\text{dist}(x_\ell^m, \text{co}(x_{\ell-1}^m, \ldots, x_1^m)) \leq 2$ for all ℓ, $2 \leq \ell \leq m$, then $\text{dist}(x_\ell^m, \text{co}(x_{\ell-1}^m, \ldots, x_1^m)) > 2\,\varepsilon_m$ for all ℓ, $2 \leq \ell \leq m$, so $\text{dist}(x_\ell^m, \text{co}(x_{\ell-1}^m, \ldots, x_1^m)) \to 2$ as $m \to \infty$ for all ℓ, $2 \leq \ell \leq m$.

Let Y be the vector space consisting of all finite linear combinations of the infinite sequence of symbols $\{\xi_i\}$.

Use a diagonal process to obtain a sequence of integers $\{k_m\}$ for which the following limit exists for all finite sets $\{a_1, \ldots, a_r\}$ of rational numbers:

$$\lim_{m\to\infty} \left\| \sum_{n=1}^{r} a_n \, x_n^{k_m} \right\| .$$

A norm $|||\cdot|||$ can be defined for all finite linear combinations of members of $\{\xi_i\}$ with rational coefficients, by letting

$$\left|\left|\left| \sum_{n=1}^{r} a_n \, \xi_n \right|\right|\right| = \lim_{m\to\infty} \left\| \sum_{n=1}^{r} a_n \, x_n^{k_m} \right\| .$$

This norm can be extended to all of Y and without any loss of generality we may assume Y is complete so that it is a Banach space.

By the definition of $|||\cdot|||$, Y is finitely representable in X.

Let $K = \text{co}(\{\xi_i\})$.

We show $\operatorname{diam}(K) = 2$, K is not close to normal, and is contained in the unit sphere of Y, which will be a contradiction to (ii).

By the definition of $|||\cdot|||$, given i, $|||\xi_i||| = \lim_{m\to\infty} ||x_i^{k_m}|| \leq 1$, so k is contained in the unit ball of Y and therefore, $\operatorname{diam}(K) \leq 2$.

Also, given ℓ, $\left|\left|\left|\frac{\xi_1 + \dots + \xi_\ell}{\ell}\right|\right|\right| = \lim_{m\to\infty} \left|\left|\frac{x_1^{k_m} + \dots + x_\ell^{k_m}}{\ell}\right|\right| \geq \lim_{m\to\infty} (\varepsilon_m) = 1$, so $\left|\left|\left|\frac{\xi_1 + \dots + \xi_\ell}{\ell}\right|\right|\right| = 1$ for all ℓ, which implies $K = co(\{\xi_i\})$ lies on the surface of the unit ball since we already know that $|||\xi_i||| \leq 1$ for all i.

On the other hand, given ℓ, $z \in co(\xi_1, \dots, \xi_{\ell-1})$,

then $z = \sum_{j=1}^{\ell-1} \lambda_j \xi_j$, $\sum_{j=1}^{\ell-1} \lambda_j = 1$, $\lambda_j \geq 0$, $1 \leq j \leq \ell-1$,

and $\left|\left|\left|\xi_\ell - \sum_{j=1}^{\ell-1} \lambda_j \xi_j\right|\right|\right| = \lim_{m\to\infty} \left|\left|x_\ell^{k_m} - \sum_{j=1}^{\ell-1} \lambda_j x_j^{k_m}\right|\right| = 2$ as mentioned above.

Thus, $\operatorname{diam}(K) = 2$ and K is not close to normal, which completes the proof of (ii) → (iii).

(iii) → (iv): This follows easily by the definition of property B.

(iv) → (i): Suppose Y is a Banach space finitely representable in X containing a set K, $\operatorname{diam}(K) = 2$, K not normal, and $||y|| = 1$ if $y \in K$.

As in the proof of Theorem 2.21, for a given m, we can find $y_1, \dots, y_m \in K$, such that for each ℓ, $2 \leq \ell \leq m$, $\operatorname{dist}(y_\ell, [y_{\ell-1}, \dots, y_1])$ is bounded below by a number arbitrarily close to 2.

Let $|||\cdot|||$ be the norm on Y.

We know K lies on the surface of unit sphere,

so $|||y_1||| = |||y_2||| = \dots = |||y_m||| = 1$ and $\left|\left|\left| \frac{y_1 + \dots + y_m}{m} \right|\right|\right| = 1.$

Thus, since Y is finitely representable in X , by an argument similar to that of Theorem 2.12, we may utilize $y_1, \dots, y_m$, to obtain $x_1, \dots, x_m$ in B_X contradicting the definition of property B . This contradiction, then, gives (iv) $\to$ (i).

The proof of the theorem is now complete.

CHAPTER III

Locally K-uniformly Convex Spaces and Reflexivity

In this chapter we find, in the context of areas, a condition under which a Banach space is reflexive. First, we introduce some important definitions.

Definition 3.1: A Banach space X is said to be locally uniformly convex if given $x \in S_X$, $\varepsilon > 0$, there exists $\delta > 0$ such that $||x - y|| < \varepsilon$ whenever $\frac{||x+y||}{2} > 1 - \delta$, and $y \in S_X$.

This definition resembles that given for uniformly convex spaces. However, here the choice of δ not only depends on ε, but it is also determined by the given x. Therefore, locally uniform convexity is a weaker condition than uniform convexity.

In the following definition we describe a property which is a generalization of k-uniform convexity.

Definition 3.2: A Banach space X is said to be locally k-uniformly convex, $(Lk - UR)$, for some positive integer k, if given $x \in S_X$, $\varepsilon > 0$, there exists $\delta > 0$ such that $A(x, x_1, \ldots, x_k) < \varepsilon$ whenever

$$\frac{||x + x_1 + \ldots + x_k||}{k+1} > 1 - \delta \text{ , and } x_1, \ldots, x_k \in S_X .$$

From this definition it is easy to see that a Banach space is locally uniformly convex if and only if it is $L1 - UR$.

Sullivan has shown that if X^{**} is L2 - UR , then X is reflexive [20]. Here, in Theorem 3.6, we prove that actually this is true for any $k \geq 1$.

In what follows, for a sequence (x_n) in a Banach space X , $V_k\,(x_n)$ will mean the set of k-tuples of the (x_n) , i.e. the set $\{(w_1, \ldots, w_k) : w_i \in (x_n) , 1 \leq i \leq k\}$. Since $V_k\,(x_n)$ is countable we may say $V_k\,(x_n) = \{v_m : m = 1, 2, \ldots\}$.

Given a positive integer m , if $v_m = (w_1, \ldots, w_k)$, by $[v_m]$ we will mean the affine span of the points $w_1, \ldots, w_k$.

The next several lemmas are required for the proof of the main theorem.

<u>Lemma 3.3</u>: If (x_n) is a sequence in a Banach space X , $||x_n|| = 1$ for all n , k a given positive integer, then one of the following is true:

(a) There exists a subsequence (x_{n_i}) of (x_n) , a positive integer L , and $\delta^k > 0$, such that $\text{dist}(x_{n_i}, [v_\ell]) > \delta^k$ for $L \leq \ell \leq i$, $v_\ell \in V_k\,(x_n)$.

(b) Given $\varepsilon > 0$ there exists a subsequence (x_{n_i}) of (x_n) and $v_\ell \in V_k(x_n)$ such that $\text{dist}(x_{n_i}, [v_\ell]) < \varepsilon$ for all i .

<u>Proof</u>: Pick (ε_m) , a sequence of decreasing positive numbers converging to zero.

We construct sequences of real numbers (δ^k_m), subsets of the integers I_m , for each positive integer m , satisfying the following:

(i) I_m is an infinite subset of the integers for each m .

(ii) If $m > \ell$ then $I_m \subseteq I_\ell$.

(iii) $0 \leq \delta^k_m \leq 2$ for all m.

(iv) $\frac{1}{2} \delta^k_m \leq \mathrm{dist}(x_j, [v_m]) < \delta^k_m + \varepsilon_m$ for all $j \in I_m$, for each m.

The construction is done by induction on m.

For $m = 1$: Let $\delta^k_1 = \liminf_{j \in I} \mathrm{dist}(x_j, [v_1])$ where I is the set of the integers.

Clearly $0 \leq \delta^k_1 \leq 2$, and we can easily get I_1, an infinite subset of the integers, such that $\frac{1}{2} \delta^k_1 \leq \mathrm{dist}(x_j, [v_1]) < \delta^k_1 + \varepsilon_1$ for all $j \in I_1$.

This takes care of the case for $m = 1$.

Suppose the construction has been accomplished for m.

We show it holds for $m + 1$.

By the induction hypothesis I_m exists as required. Let $\delta^k_{m+1} = \liminf_{j \in I_m} (x_j, [v_{m+1}])$.

Again, it is clear that $0 \leq \delta^k_{m+1} \leq 2$, and we can easily get $I_{m+1} \subseteq I_m$, an infinite subset of the integers, such that $\frac{1}{2} \delta^k_{m+1} \leq \mathrm{dist}(x_j, [v_{m+1}]) \leq \delta^k_{m+1} + \varepsilon_{m+1}$ for all $j \in I_{m+1}$, which completes the construction for $m + 1$.

Therefore, we get (δ^k_m) and a subset of the integers I_m, for each m, as required.

Let $\beta^k = \liminf_m \frac{1}{2} \delta^k_m$ $(*)$, and $\delta^k = \frac{1}{2} \beta^k$.

Then $0 \leq \delta^k \leq \frac{1}{2} < \infty$ by (iii).

Suppose $\delta^k > 0$.

By (*) choose an integer L such that if $\ell \geqslant L$ then $\frac{1}{2}\delta^k_\ell > \delta^k$.

By (i) and (ii) choose an increasing sequence of integers (n_i) with $n_i \in I_i$ for each i .

By (ii) and (iv) $\text{dist}(x_{n_i}, [v_\ell]) \geqslant \frac{1}{2}\delta^k_\ell > \delta^k$ if $i \geqslant \ell \geqslant L$, which implies (a).

Suppose $\delta^k = 0$.

Then $\liminf_m \frac{1}{2}\delta^k_m = 0$.

So for a given $\varepsilon > 0$ there must exist ℓ so large that $\delta^k_\ell + \varepsilon_\ell < \varepsilon$ since $\varepsilon_n \to 0$.

Thus, from (iv) $\text{dist}(x_j, [v_\ell]) < \delta^k_m + \varepsilon_m < \varepsilon$ for $j \in I_\ell$, which by (i) clearly implies (b).

Lemma 3.4: If (x_n) is a sequence in a Banach space X , $||x_n|| = 1$ for all n , k a given positive integer, then one of the following is true:

(a) There exists a subsequence (x_{n_i}) of (x_n), a positive integer L, and $\delta^k > 0$, such that $\text{dist}(x_{n_i}, [v_\ell]) > \delta^k$ for $L \leqslant \ell \leqslant i$, $v_\ell \in V_k(x_n)$.

(b) (x_n) has a strongly converging subsequence.

Proof: Suppose (a) is not true for (x_n) .

We construct sequences $(y^m_n) \subseteq X$, and subsets of the integers I_m , for each positive integer m , satisfying the following:

(i) I_m is an infinite subset of the integers for each m.

(ii) If $m > \ell$, $I_m \subseteq I_\ell$.

(iii) $||x_j - y_j^m|| < \frac{1}{3m}$ for $j \in I_m$, for each m.

(iv) $||y_j^m - y_i^m|| < \frac{1}{3m}$ for $i, j \in I_m$, for each m.

The construction is done by induction on m.

For $m = 1$: Since (a) is not true by Lemma 3.3 we may assume by passing to a subsequence that $\operatorname{dist}(x_j, [v\ell]) < \frac{1}{3}$ for some $v_\ell \in V_k(x_n)$ for all j's.

So there exists $(y_n^1) \subseteq [v_\ell]$ such that $||y_j^1 - x_j|| < \frac{1}{3}$ for all j's.

Notice $||y_j^1|| \leqslant ||y_j^1 - x_j|| + ||x_j|| < 1 + \frac{1}{3}$ for all j's.

This implies (y_n^1) is bounded.

Since $[v_\ell]$ is contained in a k-dimensional subspace of X then there exists I_1, an infinite subset of the integers, such that $\{y_j^1 : j \in I_1\}$ is a converging subsequence of (y_n^1).

By the definition of (y_n^1) it is clear that in particular $||y_j^1 - x_j|| < \frac{1}{3} = \frac{1}{3(1)}$ for $j \in I_1$, and since $\{y_j^1 : j \in I_1\}$ must be Cauchy, we may assume without any loss of generality that $||y_j^1 - y_i^1|| < \frac{1}{3} = \frac{1}{3(1)}$ for all $i, j \in I_1$.

This takes care of the case for $m = 1$.

Suppose the construction has been accomplished for m.

We show it holds for $m + 1$.

Since $\{x_j : j \in I_m\}$ is a subsequence of (x_n) then (a) can not be true for $\{x_j : j \in I_m\}$, either.

So by Lemma 3.3 we may assume by passing to a subsequence that

$$\mathrm{dist}(x_j, [v_\ell]) < \frac{1}{3(m+1)} \quad \text{for all } j \in I_m \text{, for some } v_\ell \in V_k(x_n).$$

So there exists $(y_n^{m+1}) \subseteq [v_\ell]$ such that $\|y_j^{m+1} - x_j\| < \frac{1}{3(m+1)}$ for all $j \in I_m$.

Again, as above, there exists I_{m+1}, an infinite subset of the integers with $I_{m+1} \subseteq I_m$, such that $\{y_j^{m+1} : j \in I_{m+1}\}$ is a converging subsequence of $\{y_j^{m+1} : j \in I_m\}$.

By the definition of (y_n^{m+1}) it is clear that in particular $\|y_j^{m+1} - x_j\| < \frac{1}{3(m+1)}$ for all $j \in I_{m+1}$, and since $\{y_j^{m+1} : j \in I_{m+1}\}$ must be Cauchy, we may assume without any loss of generality that $\|y_j^{m+1} - y_i^{m+1}\| < \frac{1}{3(m+1)}$ for all $i, j \in I_{m+1}$.

Thus the construction is complete for $m + 1$, and we get (y_n^m), I_m, for each m, as required.

Now, choose an increasing sequence of integers (n_i) with $n_i \in I_i$ for each i.

We show (x_{n_i}) is a Cauchy subsequence of (x_n), and hence it must converge.

By (i) and (ii) (x_{n_i}) must be a subsequence of (x_n) .

Let $\varepsilon > 0$ be given.

Pick an integer N so that $\frac{1}{N} < \varepsilon$.

Suppose $m, \ell \geq N$.

Then by (ii) $I_m \subseteq I_N$, and $I_\ell \subseteq I_N$, so $n_m, n_\ell \in I_N$, and by (iii) and (iv)

$$||x_{n_m} - x_{n_\ell}|| \leq ||x_{n_m} - y^N_{n_m}|| + || x_{n_\ell} - y^N_{n_\ell}|| + ||y^N_{n_m} - y^N_{n_\ell}||$$

$$\leq \frac{1}{3_N} + \frac{1}{3_N} + \frac{1}{3_N} = \frac{1}{N} < \varepsilon .$$

Thus, (x_{n_i}) is a Cauchy sequence, and this clearly implies (b) must be true.

Lemma 3.5: If (x_n) is a sequence in a Banach space X , $||x_n|| = 1$ for all n , k a given positive integer, then one of the following is true:

(a) There exists a subsequence (x_{n_i}) of (x_n), a positive integer L, and numbers $\delta^j > 0$ for $j = 2, \ldots, k-1$, such that given $v_\ell \in V_j(x_{n_i})$, $\ell \geq L$, there exists r for which $\text{dist}(x_{n_p}, [v_\ell]) > \delta^j$ whenever $p \geq r$, for $2 \leq j \leq k-1$.

(b) (x_n) has a strongly convergent subsequence.

Proof: Suppose (b) is not true.

For each i , let $x^1_{n_i} = x_i$, so that $(x^1_{n_i}) = (x_i)$.

We construct sequences $(x^j_{n_i})$, positive numbers δ^j, $j = 2, \ldots, k-1$, satisfying the following:

(i) For each j, $2 \leq j \leq k-1$, $(x^j_{n_i})$ is a subsequence of (x_n).

(ii) If $j < \ell$, $2 \leq j, \ell \leq k-1$, then $(x^\ell_{n_i})$ is a subsequence of $(x^j_{n_i})$.

(iii) For some positive integer L_j, if $L_j \leq \ell \leq i$ then $\mathrm{dist}(x^j_{n_i}, [v_\ell]) > \delta^j$ whenever $v_\ell \in V_j(x^{j-1}_{n_i})$, for $j = 2, \ldots, k-1$.

The construction is done by induction on j.

For $j = 2$: By Lemma 3.4 since (b) is not true there exists $(x^2_{n_i})$ a subsequence of (x_n), a positive integer L_2, and $\delta^2 > 0$, such that $\mathrm{dist}(x^2_{n_i}, [v_\ell]) > \delta^2$ for $L_2 \leq \ell \leq i$, $v_\ell \in V_2(x_n) = V_2(x^1_{n_i})$.

This takes care of the case for $j = 2$.

Suppose the construction has been accomplished for j, $2 \leq j \leq k-2$. We show it can also be done for $j + 1$.

Since $(x^j_{n_i})$ is a subsequence of $\{x_n\}$ then (b) can not be true for $(x^j_{n_i})$, either.

So by Lemma 3.4 we can obtain $(x^{j+1}_{n_i})$, a subsequence of $(x^j_{n_i})$, L_{j+1} a positive integer, and $\delta^{j+1} > 0$ such that $\mathrm{dist}(x^{j+1}_{n_i}, [v_\ell]) > \delta^{j+1}$ for $L_{j+1} \leq \ell \leq i$, $v_\ell \in V_{j+1}(x^j_{n_i})$.

Thus, the construction is complete, and we obtain $(x^j_{n_i})$, δ^j, as required.

By (ii) $V_j(x^{k-1}_{n_i}) \subseteq V_j(x^{j-1}_{n_i})$ for each j, $2 \leq j \leq k-1$.

So given $v_\ell \in V_j(x^{k-1}_{n_i})$, $v_\ell = v_h \in V_j(x^{j-1}_{n_i})$, where h is a positive integer.

Choose L so that if $\ell \geq L$ and $v_\ell \in V_j(x^{k-1}_{n_i})$ then $h \geq L_j$ if $v_\ell = v_h \in V_j(x^{j-1}_{n_i})$, for any j, $2 \leq j \leq k-1$.

We show that $L, \delta^2, \ldots, \delta^{k-1}$, $(x^{k-1}_{n_i})$ are respectively the numbers and subsequence of (x_n) satisfying (a).

By (i) $(x^{k-1}_{n_i})$ is a subsequence of (x_n).

For a given j, $2 \leq j \leq k-1$, suppose $v_\ell \in V_j(x^{k-1}_{n_i})$, $\ell \geq L$.

Then $v_\ell = v_h \in V_j(x^{j-1}_{n_i})$ and $h \geq L_j$ by the definition of L.

Thus, by (iii) $\text{dist}(x^j_{n_i}, [v_h]) > \delta^j$ if $i \geq h$ $(*)$.

By (ii) $(x^{k-1}_{n_i})$ is a subsequence of $(x^j_{n_i})$, so for some q, $q \geq h$, $x^j_{n_q} \in (x^{k-1}_{n_i})$, and hence, $x^j_{n_q} = x^{k-1}_{n_r}$ for some r.

From this, if $p \geq r$ and $x^j_{n_i} = x^{k-1}_{n_p}$ then $i \geq q \geq h$, which by $(*)$ gives $\text{dist}(x^j_{n_i}, [v_h]) > \delta^j$, or $\text{dist}(x^{k-1}_{n_p}, [v_\ell]) > \delta^j$, implying (a) must be true.

Now we are in a position to exhibit the connection between (Lk - UR) and reflexivity.

Theorem 3.6: Let X be a Banach space. If X^{**} is (Lk-UR) , $k \geq 1$, then X is reflexive.

Proof: Suppose X is not reflexive. By James' theorem [16] there exists $f \in X^*$, $||f|| = 1$, such that $f(x) < 1$ for all $x \in X$, $||x|| = 1$.

By the Hahn-Banach theorem there exists $x^{**} \in X^{**}$ with $x^{**}(f) = 1$, $||x^{**}|| = 1$. Obviously, $x^{**} \notin X$.

Since $||f|| = 1$ we may choose (x_n) , a sequence in X , $||x_n|| = 1$ for all n , such that $f(x_n) \to 1$.

No subsequence of (x_n) converges strongly in X .

Otherwise, if $x_{n_i} \to x$, (x_{n_i}) a subsequence of (x_n) , $x \in X$, then $||x|| = 1$ and $f(x_{n_i}) \to f(x)$.

But by the definition of (x_n) , $f(x_{n_i}) \to 1$.

So $f(x) = 1$ which contradicts the definition of f . Thus, (a) of Lemma 3.5 must be true for (x_n) , so that (x_{n_i}) , a subsequence of (x_n) , and numbers L , δ^2, δ^3, ... δ^{k-1} , must exist as described in the lemma.

Since $x^{**} \notin X$ then $\text{dist}(x^{**}, X) > \eta$ where $\eta > 0$.

Let $\varepsilon > 0$ be given, and let δ correspond to $\varepsilon \cdot \delta^1 \cdot \delta^2 \cdot \ldots \cdot \delta^{k-1} \cdot \eta$ as in the definition of Lk-UR .

Since $||x^{**} + x_{n_{i(k)}} + x_{n_{i(k-1)}} + \ldots + x_{n_{i(1)}}|| \geq x^{**}(f) + f(x_{n_{i(k)}}) + \ldots + f(x_{n_{i(1)}})$, and $x^{**}(f) = 1$, $f(x_{n_i}) \to 1$, then the right side of the above inequality converges to $k+1$ as $i(k), i(k-1), \ldots, i(1)$ approach infinity.

So there must exist a positive integer N such that

$$\left\| \frac{x^{**} + x_{n_{i(k)}} + \dots + x_{n_{i(1)}}}{k+1} \right\| > 1 - \delta$$

whenever $i(k), i(k-1), \dots, i(1) \geq N$.

Thus, by the definition of δ we get $A(x^{**}, x_{n_{i(k)}}, \dots, x_{n_{i(1)}}) < \varepsilon \cdot \delta^2 \cdot \delta^3 \cdot \dots \cdot \delta^{k-1} \cdot \eta$ if $i(k), i(k-1), \dots, i(1) \geq N$.

We show (x_{n_i}) is a Cauchy sequence.

With L given as above, we may assume without any loss of generality that N is large enough so that given any j, $2 \leq j \leq k-1$, whenever $i(j), i(j-1), \dots, i(1) \geq N$, and $[x_{n_{i(j)}}, x_{n_{i(j-1)}}, \dots, x_{n_{i(1)}}] = v_\ell \in V_j (x_{n_i})$, then $\ell \geq L$.

So suppose $i(1), i(2) \geq N$.

We show by induction that we can pick $i(3), \dots, i(k)$, positive integers, such that $i(3), \dots, i(k) \geq N$, and $\text{dist}(x_{n_{i(j)}}, [x_{n_{i(j-1)}}, \dots, x_{n_{i(1)}}]) > \delta^{j-1}$ for $j = 3, \dots, k$.

For $j = 3$: If $[x_{n_{i(2)}}, x_{n_{i(1)}}] = v_\ell \in V_2 (x_{n_i})$, then by the choice of N we have that $\ell \geq L$. Hence, by the definition of δ^2 , there exists r so that if $p \geq r$ then $\text{dist}(x_{n_p}, [x_{n_{i(2)}}, x_{n_{i(1)}}]) > \delta^2$.

By picking $i(3) \geq \max (r, N)$, $x_{n_{i(3)}}$ is as desired.

Suppose we have chosen $i(3), \dots, i(h)$ satisfying the given properties, $3 \leq h \leq k-1$. We choose $i(h+1)$ as follows.

Again, if $[x_{n_{i(h)}}, \ldots, x_{n_{i(1)}}] = v_\ell \in V_h(x_{n_i})$ then $\ell \geq L$, and by the definition of δ^h there exists r so that if $p \geq r$ then $\text{dist}(x_{n_p}, [x_{n_{i(h)}}, \ldots, x_{n_{i(1)}}]) > \delta^h$.

By picking $i(h+1) \geq \max(r, N)$, $x_{n_{i(h+1)}}$ is as desired.

Thus, we may choose $i(3), \ldots, i(k)$ as required.

Since $i(k), \ldots, i(2), i(1) \geq N$ then $A(x^{**}, x_{n_{i(k)}}, \ldots, x_{n_{i(1)}}) < \varepsilon\, \delta^2 \cdot \ldots \cdot \delta^k \cdot \eta$.

By Lemma 1.3 we know

$$A(x^{**}, x_{n_{i(k)}}, \ldots, x_{n_{i(1)}}) \geq \text{dist}(x^{**}, [x_{n_{i(k)}}, \ldots, x_{n_{i(1)}}]) \cdot$$

$$\text{dist}(x_{n_{i(k)}}, [x_{n_{i(k-1)}}, \ldots, x_{n_{i(1)}}]) \cdot \ldots \cdot$$

$$\text{dist}(x_{n_{i(3)}}, [x_{n_{i(2)}}, x_{n_{i(1)}}]) \, ||x_{n_{i(2)}} - x_{n_{i(1)}}|| .$$

Therefore $||x_{n_{i(2)}} - x_{n_{i(1)}}|| < \dfrac{\varepsilon \cdot \delta^2 \cdot \ldots \cdot \delta^{k-1} \cdot \eta}{\delta^2 \cdot \ldots \cdot \delta^{k-1} \cdot \eta} = \varepsilon$ for $i(1)$, $i(2) \geq N$. So (x_{n_i}) must be Cauchy and, thus, it must converge.

But this is a contradiction, hence, X must be reflexive.

CHAPTER IV

Open Questions and Problems

It is not known whether in every uniformly convex Banach space the modulus of k-uniform convexity converges to one for every positive number ε as k gets arbitrarily large. Theorem 1.9 says that this is the case in Hilbert spaces. Does this property characterize Hilbert spaces? If not, what type of spaces does it characterize?

In Example 2.23 we showed that the space given in Karlovitz's example [17] has property B. The purpose of Karlovitz's example was to exhibit a superreflexive Banach space without normal structure but satisfying the fixed point property. However, by Theorem 2.24, property B is equivalent to certain structures related to normal structure. Thus, Karlovitz's space still has some kind of structure. The obvious question, then, is as follows: Does property B imply the fixed point property? This is a very interesting question since by Corollary 2.22 property A implies the fixed point property. The following questions are also of related interest: Does the fixed point property have an equivalent formulation in terms of area? Does the fixed point property imply property B? Does every superreflexive space have the fixed point property? Can superreflexivity be characterized in terms of area?

A property more general than property B can be given as follows: A Banach space X is said to have property C if there exist δ, m, with $\delta > 0$, m an integer, $m > 1$, such that whenever $\{x_1, \ldots, x_m\}$ is a subset of B_X, then $A(x_1, \ldots, x_m) < m^{m/2}(1 - \delta)$ if

$$\left\| \frac{x_1 + \ldots + x_m}{m} \right\| > 1 - \delta .$$

Here we have chosen the number $m^{m/2}$ since it is the maximum value the determinant of a m by m matrix can attain [3]. We may ask the next questions about this property: Does it imply the fixed point property? What spaces does it characterize? How does it relate to normal structure and superreflexivity? What spaces have property C while failing to have property B ?

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