Fast Dynamic Programming for Elastic Registration of Curves

Javier Bernal, Gunay Dogan, Robert C. Hagwood, NIST, Theiss Research

National Institute of Standards and Technology
U.S. Department of Commerce

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Goal: Fast registration of curves using dynamic programming (DP) for the efficient computation of elastic shape distances, alignment of functions, etc.

Our contribution: Development and implementation of linear DP algorithm for computing optimal diffeomorphisms for elastic registration of curves.

Elastic registration formulation:

For $F:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$, minimize energy

$$E(\gamma)=\int_0^1 F(t,\gamma(t),\dot\gamma(t))dt$$

with respect to γ, γ a diffeomorphism of $[0,1]$ onto itself $(\gamma(0)=0,\gamma(1)=1,\dot\gamma>0)$.

A typical function F (from elastic shape analysis):

$$F(t,\gamma(t),\dot\gamma(t))=\|R(\theta)q_1(t+t_0)-\sqrt{\dot\gamma(t)}q_2(\gamma(t))\|^2.$$

$t_0\equiv$ seed or starting point.

$R(\theta)\equiv 2\times 2$ rotation matrix defined by angle θ.

$q_i(t)\equiv\dot\beta_i(t)/\|\dot\beta_i(t)\|^{1/2},\ i=1,2.$

$\beta_i:[0,1]\to\mathbb{R}^2$, C^2 closed curve of unit length.

Discretized formulation:

$$E(\vec\gamma)=\frac{1}{2}\sum_{l=1}^{N-1}(t_{l+1}-t_l)(F(t_{l+1},\gamma_{l+1},\dot\gamma_{l+1})+F(t_l,\gamma_l,\dot\gamma_l)).$$

$\{t_l\}_{l=1}^N\equiv$ a partition of $[0,1]$.

$\vec\gamma\equiv\{\gamma_l\}_{l=1}^N\equiv\{\gamma(t_l)\}_{l=1}^N,\ \{\dot\gamma_l\}_{l=1}^N\equiv\{\dot\gamma(t_l)\}_{l=1}^N.$

Approximating γ **with a grid:**

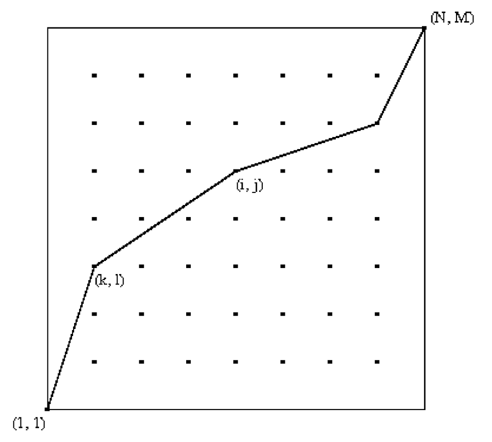
$\{t_i\}_{i=1}^N\equiv$ a partition of $[0,1]$ (x axis).

$\{z_j\}_{j=1}^M\equiv$ a partition of $[0,1]$ (y axis).

Point (t_i,z_j) coincides with grid point (i,j).

γ approximated by increasing piecewise linear γ^*.

Vertices of graph of γ^* are grid points.



Energy over line segment $\overline{(k,l)(i,j)}$:

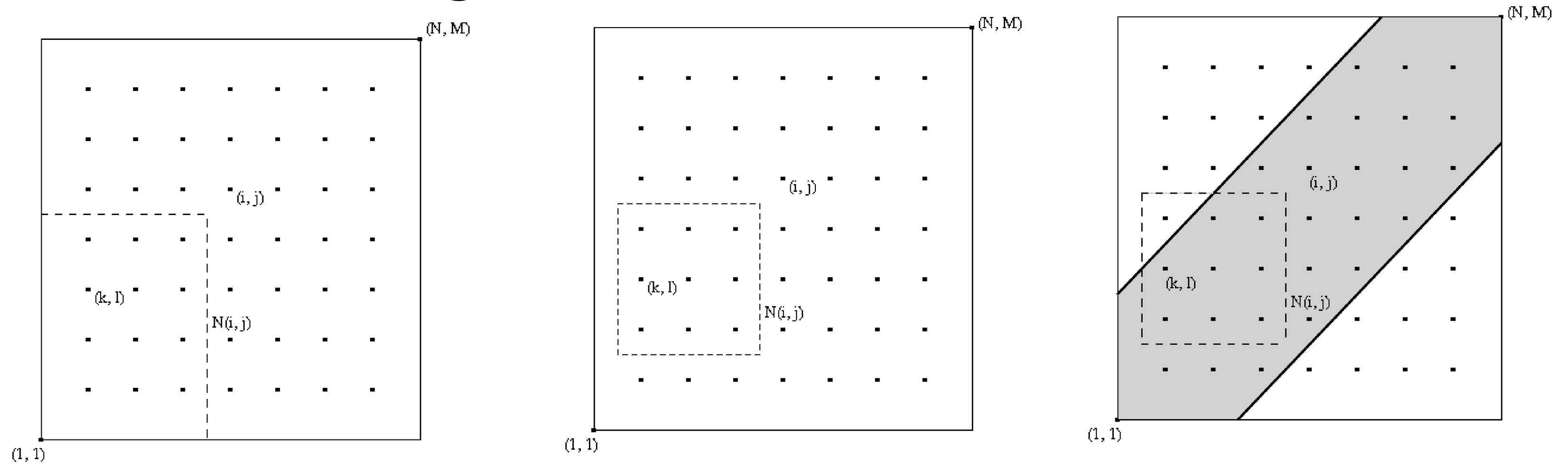
$$E_{(k,l)}^{(i,j)}=\frac{1}{2}\sum_{m=k}^{i-1}(t_{m+1}-t_m)(F_{m+1}+F_m).$$

$F_m\equiv F(t_m,\alpha(t_m),L),\ m=k,\dots,i.$

$\alpha\equiv$ linear function with $\alpha(t_k)=z_l,\ \alpha(t_i)=z_j.$

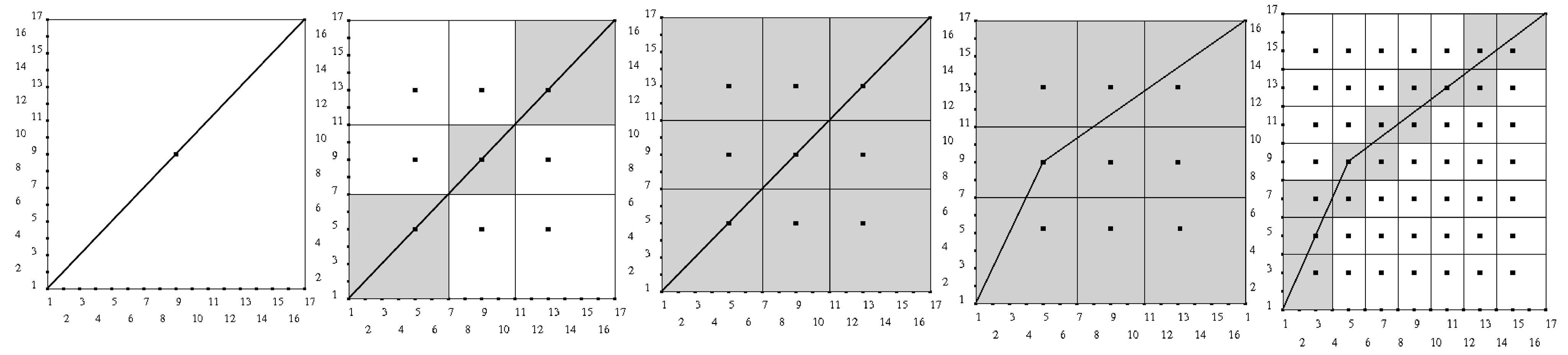
$L\equiv$ slope of line segment $\overline{(k,l)(i,j)}$.

Known DP algorithms:

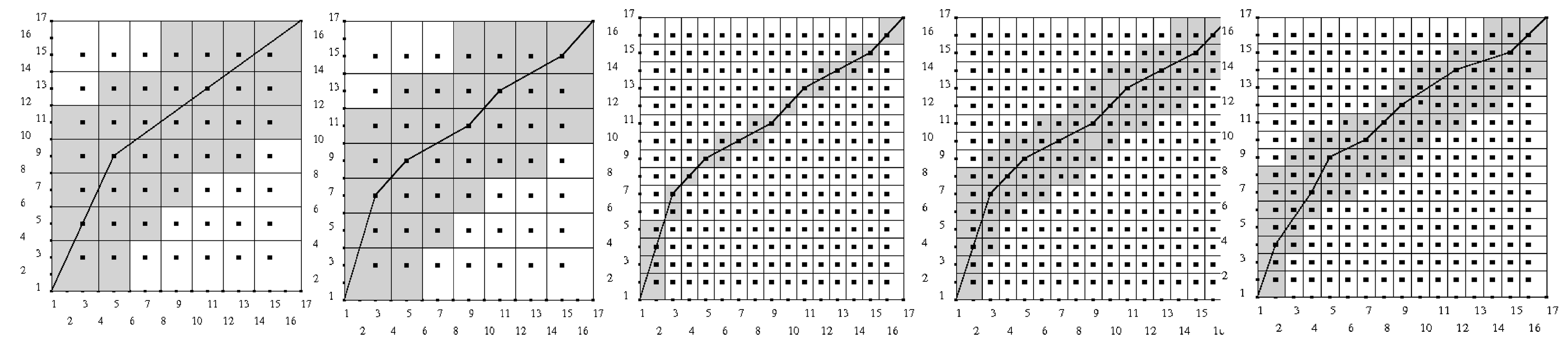


Left: the $O(N^4)$ DP algorithm guaranteed to compute optimal solution.
Center: the $O(N^2)$ original DP algorithm. Right: the $O(N^2)$ fast DP algorithm.
In all cases, $N(i,j)$ is a trailing set of (i,j) and the optimal energy at (i,j) relative to $N(i,j)$ is $E(i,j)=\min_{(k,l)\in N(i,j)}(E(k,l)+E_{(k,l)}^{(i,j)})$.

New DP algorithm: O(N) adapt DP algorithm using strips of width O(1):



From 3 x 3 subgrid to 5 x 5 subgrid to 9 x 9 subgrid



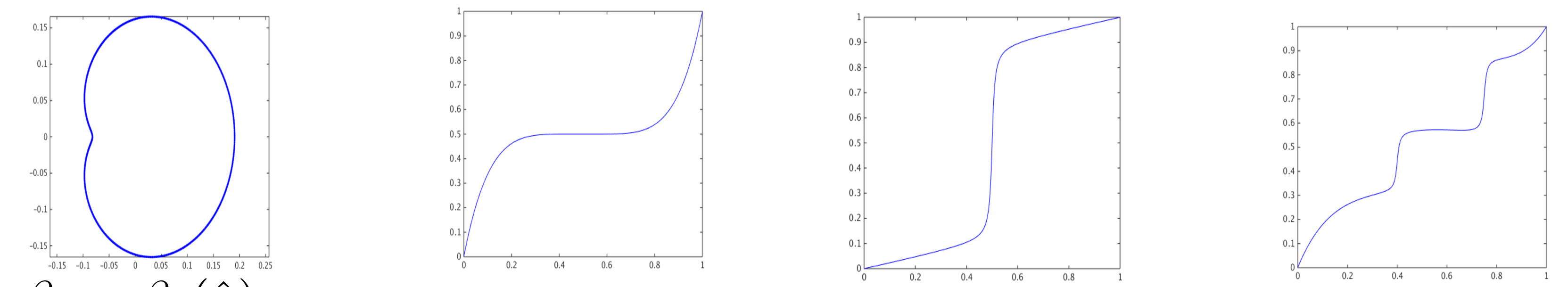
to 17 x 17 subgrid

Complexity = O(N + N/2 + N/4 …) = O(N)

Numerical results:

Diffeomorphism recovery:

$\beta_2\equiv$ a limacon | Flat $\hat\gamma$ | Steep $\hat\gamma$ | Bumpy $\hat\gamma$



$\beta_1\equiv\beta_2(\hat\gamma).$

$$F(t,\gamma(t),\dot\gamma(t))=\|q_1(t)-\sqrt{\dot\gamma(t)}q_2(\gamma(t))\|^2,\quad q_i(t)=\dot\beta_i(t)/\|\dot\beta_i(t)\|^{1/2},\ i=1,2.$$

	$N=$	64	128	256	512	1024	2048
	$\omega=$	64	32	16	12	12	12
$original-DP$	all $\hat\gamma$	0.49	1.26	1.00	2.07	8.48	34.3
$fast-DP$	all $\hat\gamma$	0.24	0.66	0.51	1.04	4.20	17.0
$adapt-dP$	Flat $\hat\gamma$	0.65	0.67	0.27	0.29	0.57	1.20
	Steep $\hat\gamma$	0.65	0.56	0.28	0.31	0.62	1.32
	Bumpy $\hat\gamma$	0.81	1.04	0.37	0.36	0.70	1.46

Times in seconds. $\omega\equiv$ number of grid points per side of $N(i,j)$.
In all cases, L^2 error between true solution $\hat\gamma$ and computed γ^* $\left(\frac{1}{N-1}\left(\sum_{l=1}^{N-1}(\hat\gamma(t_l)-\gamma^*(t_l))^2\right)^{\frac{1}{2}}\right)$ was less than 10^{-3}.

Function alignment by warping:

$$F(t,\gamma(t),\dot\gamma(t))=(q_1(t)-\sqrt{\dot\gamma(t)}q_2(\gamma(t)))^2.$$

$q_i(t)\equiv\text{sign}(\dot f_i(t))\sqrt{|\dot f_i(t)|}$, absolutely continuous $f_i:[0,1]\to\mathbb{R},\quad i=1,2.$

Function 1 in blue has 19,693 points, function 2 in red has 19,763 points.

Function 1 was aligned to function 2 in **172 seconds.**

Warping function γ on the right inside last strip (boundary in green).