Special values for continuous q-Jacobi polynomials and applications

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NIST DLMF: compendium of special function properties

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Index
Notations

Search
Help?
Citing
Customize

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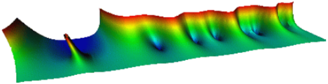


NIST Digital Library of Mathematical Functions

Project News

2021-06-15 *DLMF Update; Version 1.1.2*
2021-03-15 *DLMF Update; Version 1.1.1*
2020-12-15 *DLMF Update; Version 1.1.0*
2020-09-15 *DLMF Update; Version 1.0.28*
More news

NIST Digital Library of Mathematical Functions (DLMF)

- Since 2010
- Online compendium of important mathematical properties of the special functions and orthogonal polynomials of applied mathematics. https://dlmf.nist.gov/
- Update of **Abramowitz** and **Stegun**: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (1964).
- Contains 36 Chapters (each written in LaTeX) written by 29 authors, each eminent mathematicians in their contributed field (**Olver**, **Askey**, **Temme**, **Andrews**, **Roy**, **Koornwinder**, ...).
- Editorial team: **Adri Olde Daalhuis**, **Dan Lozier**, **Ron Boisvert**, **Barry Schneider**, **Charles Clark**, **Bruce Miller**, **Bonita Saunders**, and **Howard Cohl**.
- **Frank W. J. Olver** served as Editor-in-Chief and Mathematics Editor for the DLMF from its beginning until his death on April 23, 2013.

Introduction: q–calculus

- It has limits, derivatives, integrals, series, products, continuity, etc.
- q–calculus is not a differential calculus, it's a difference calculus.
- Scaling in the difference operators are parametrized by q.

$$(D_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}.$$

- Limit $q \to 1$: q-derivative becomes the ordinary derivative.

$$\lim_{q\to 1}(D_q f)(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

- Typically we take $|q| < 1$, but that's not necessary
- q-analogues: $q \to 1$ may produce ordinary results, but not always
- We can take q to be everywhere in the complex plane (with the exception of non-roots-of-unity, remove those points)

History: q–calculus

- Leonhard Euler in 1748 (1707-1783, Swiss mathematician, physicist, astronomer, geographer, and engineer) considered the product

$$(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} (1-q^{k+1})^{-1}$$

as a generating function for the number of partitions of a positive integer n into positive integers.

- Gauss in 1812 (1777-1855, German mathematician, physicist) presented to the Royal Society of Sciences at Göttinger the famous paper where he considered the hypergeometric series

$$1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3+\dots$$

work on basic hypergeometric functions, but most of Gauss' work was unpublished until after his death.

History: q–calculus

- Thirty-tree years after Gauss's paper, Heinrich Eduard Heine [1846, 1847, 1878] (1821-1881, German mathematician) introduced the basic hypergeometric series

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\dots$$

- Starting in 1904 Frank Hilton Jackson (1870-1960, English Clergyman) "*... embarked on a lifelong program of developing the theory of basic hypergeometric series [and q–calculus] in a systematic manner.*" [Gasper & Rahman (2004)] He introduced several q–analogs such as the q–gamma function, q–derivative, the q–integral, Jackson q–Bessel functions, etc.
- Ever since then the theory of basic hypergeometric series has been developing further and further.

Adopted set notation

- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$;
- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the sets of integers, real numbers and complex numbers respectively
- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^\dagger := \{z \in \mathbb{C}^* : |z| < 1\}$;
- We also use the product conventions:

$$\pm a := \{a, -a\},$$

$$\mathrm{e}^{\pm i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\},$$

$$z^{\pm} := \{z, z^{-1}\}.$$

Generalized hypergeometric function

- **Generalized hypergeometric function**: ${}_rF_s : \mathbb{C}^r \times (\mathbb{C} \setminus -\mathbb{N}_0)^s \times \mathbb{C} \setminus [1,\infty) \to \mathbb{C}$, defined as

$$ {}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r)_n}{(b_1,\ldots,b_s)_n} \frac{z^n}{n!}, $$

- **Pochhammer symbol (shifted factorial)**, $(\cdot)_n : \mathbb{C} \to \mathbb{C}$, defined as

$$ (z)_0 := 1, \quad (z)_n := (z)(z+1)\cdots(z+n-1),\ n \geq 1, $$

$$ (z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad z \in \mathbb{C} \setminus -\mathbb{N}_0, $$

$$ (1)_n = n!, $$

$$ (-n)_k = 0, \quad k \geq n+1, $$

$$ (a_1,\ldots,a_r)_n = (a_1)_n \cdots (a_r)_n. $$

Introduction: cont.

- The q-Pochhammer symbol (q-**shifted factorial**)

$$(a;q)_n := (1-a)(1-qa)(1-q^2a)\cdots(1-q^{n-1}a) = \prod_{k=0}^{n-1}(1-aq^k)$$

- The infinite q–Pochhammer symbol, and the theta function, $q \in \mathbb{C}^\dagger$,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad \theta(x;q) := (x,q/x;q)_\infty,$$

- Other adopted product notations:

$$(a_1,\ldots,a_r)_n := (a_1)_n\cdots(a_r)_n,$$

$$(a_1,\ldots,a_r;q)_n := (a_1;q)_n\cdots(a_r;q)_n,$$

$$(a_1,\ldots,a_r;q)_\infty := (a_1;q)_\infty\cdots(a_r;q)_\infty,$$

Basic hypergeometric series

- Note that for $a=0$, $(a;q)_k=(a;q)_\infty=1$.
- For $a=q^{-n}$, $n\in\mathbb{N}_0$, $(a;q)_\infty=0$.
- For $a=q^{-n}$, $n\in\mathbb{N}_0$, $(a;q)_k=0$ for $k>n$.
- Infinite series representation for nonterminating basic hypergeometric series

$$ {}_{r+1}\phi_s\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};z\right)=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_{r+1};q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k, $$

  divergent for $r<s$, convergent for $|z|<1$ if $r=s$ and entire if $s>r$.
- If one of the numerator parameters is of the form q^{-n}, $n\in\mathbb{N}_0$, then the infinite series terminates. This is fundamental to the study of q-orthogonal polynomials.

Well-poised and very-well poised basic hypergeometric series

- The nonterminating well poised basic hypergeometric series is defined as:

$$ {}_{r+1}\phi_r\left(\begin{matrix} a, a_2, \ldots, a_{r+1} \\ \frac{qa}{a_2}, \ldots, \frac{qa}{a_{r+1}} \end{matrix}; q, z\right). $$

- The nonterminating very-well poised bhs is defined as:

$$ {}_{r+1}W_r(a; a_4, a_5, \ldots, a_{r+1}; q, z) $$

$$:= {}_{r+1}\phi_r\left(\begin{matrix} a, \pm q\sqrt{a}, a_4, \ldots, a_{r+1} \\ \pm\sqrt{a}, \frac{qa}{a_4}, \ldots, \frac{qa}{a_{r+1}} \end{matrix}; q, z\right). $$

Jacobi polynomials: $P_n^{(\alpha,\beta)} : \mathbb{C} \to \mathbb{C}$

- Jacobi polynomials are orthogonal polynomials, orthogonal on the real interval $(-1,1)$. They can be defined for $n \in \mathbb{N}_0$ as

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c}-n, n+\alpha+\beta+1\\ \alpha+1\end{array};\frac{1-z}{2}\right).$$

- Orthogonality relation:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}\delta_{m,n}.$$

- They satisfy the parity relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

Jacobi polynomial special values

- The Jacobi polynomials have the following **two** special values

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!},$$

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}.$$

Jacobi polynomial Poisson kernel

- Poisson kernel for Jacobi polynomials: Bailey (1964), p. 102,

$$\sum_{n=0}^{\infty}\frac{n!(\alpha+\beta+1,\frac{\alpha+\beta+3}{2})_n}{(\alpha+1,\beta+1,\frac{\alpha+\beta+1}{2})_n}P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)t^{2n}=\frac{(1-t^2)}{(1+t^2)^{\alpha+\beta+2}}$$

$$\times F_4\left(\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2},\alpha+1,\beta+1,\frac{\sin^2\theta\sin^2\phi}{\frac12(t+t^{-1})},\frac{\cos^2\theta\cos^2\phi}{\frac12(t+t^{-1})}\right),$$

where

$$F_4(\alpha,\beta;\gamma,\gamma';x,y):=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,\beta)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

and $\sqrt{|x|}+\sqrt{|y|}<1$.

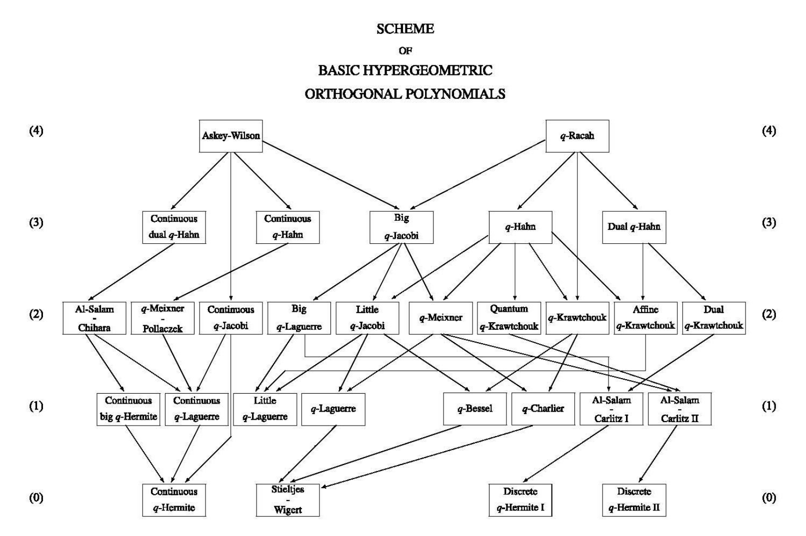
The Askey–Wilson polynomials

- The Askey–Wilson polynomials can be defined as

$$p_n(\tfrac{1}{2}(z+z^{-1});a,b,c,d|q) := a^{-n}(ab,ac,ad;q)_n \times {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az^{\pm} \\ ab, ac, ad \end{matrix}; q, q\right).$$

- They are orthogonal polynomials symmetric in the four parameters a, b, c, d.
- They are at the very top of the q-Askey scheme.

The q-Askey scheme of basic hypergeometric OPs



Continuous q-Jacobi polynomials: $P_n^{(\alpha,\beta)}(x|q)$

- Continuous q-Jacobi polynomials are orthogonal polynomials, orthogonal on the real interval $(-1,1)$. They can be defined for $n\in\mathbb{N}_0$ as

$$P_n^{(\alpha,\beta)}\left(\tfrac12(z+z^{-1})|q\right):=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\times{}_4\phi_3\left(\begin{array}{c}q^{-n},q^{\alpha+\beta+1+n},q^{\frac{\alpha}{2}+\frac14}z^{\pm}\\ q^{\alpha+1},-q^{\frac12(\alpha+\beta+1)},-q^{\frac12(\alpha+\beta+2)}\end{array};q,q\right)$$

- Orthogonality relation:

$$\int_{-1}^{1}P_m^{(\alpha,\beta)}(x|q)P_n^{(\alpha,\beta)}(x|q)w_q^{(\alpha,\beta)}(x|q)\mathrm{d}x=h_{n,m}^{(\alpha,\beta)}(q)\delta_{m,n}.$$

- They satisfy the parity relation

$$P_n^{(\alpha,\beta)}(-x|q)=(-q^{\frac12(\alpha-\beta)})^nP_n^{(\beta,\alpha)}(x|q).$$

Continuous q-Jacobi polynomial special values

- The continuous q-Jacobi polynomials have the following **four** special values

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12\alpha+\frac14}+q^{-\frac12\alpha-\frac14})|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n},$$

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12\alpha+\frac34}+q^{-\frac12\alpha-\frac34})|q)=q^{-\frac12 n}\frac{(q^{\alpha+1},-q^{\frac12(\alpha+\beta+3)};q)_n}{(q,-q^{\frac12(\alpha+\beta+1)};q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12\beta+\frac14}+q^{-\frac12\beta-\frac14})|q)=\left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12\beta+\frac34}+q^{-\frac12\beta-\frac34})|q)=\left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac12(\alpha+\beta+3)};q)_n}{(q,-q^{\frac12(\alpha+\beta+1)};q)_n}.$$

Poisson kernel for orthogonal polynomials

- Given an orthogonal polynomial $p_n(x;\mathbf{a})$ over set of parameters $\mathbf{a}$,

$$\int_a^b p_n(x;\mathbf{a})p_m(x;\mathbf{a})w(x;\mathbf{a})\mathrm{d}x = h_n(\mathbf{a})\delta_{m,n}$$

  the Poisson kernel with $x, y \in (a,b)$ is defined as

$$\sum_{n=0}^{\infty} h_n^{-1}(\mathbf{a})p_n(x;\mathbf{a})p_n(y;\mathbf{a})t^n.$$

- Note that as $t \to 1$ the Poisson kernel approaches the closure relation

$$\sum_{n=0}^{\infty} h_n^{-1}(\mathbf{a})p_n(x;\mathbf{a})p_n(y;\mathbf{a}) = \frac{\delta(x-y)}{w(x;\mathbf{a})}$$

- which is also the finite value for the Christoffel-Darboux formula

$$\sum_{n=0}^{k} h_n^{-1}(\mathbf{a})p_n(x;\mathbf{a})p_n(y;\mathbf{a})$$

Poisson kernel transformations from z, w special values

- Consider $x = \frac{1}{2}(z + z^{-1})$, $y = \frac{1}{2}(w + w^{-1})$ in the Poisson kernel.
- Then choose $z \in \{a, b, c, d\}$ and $w \in \{a, b, c, d\}$.
- By inserting special values of continuous q-Jacobi polynomials into its Poisson kernel, transformations for single basic nonterminating hypergeometric transformations with arbitrary argument are produced.
- Diagonal transformations produce unique transformations for a single basic nonterminating hypergeometric function.
- Transformation formulae for a single basic nonterminating hypergeometric function are produced in pairs corresponding to the off-diagonal elements.

Continuous q-Jacobi polynomial Poisson kernel

Gasper & Rahman (1986). Let $q \in \mathbb{C}^\dagger$, $t, \alpha, \beta \in \mathbb{C}$ such that $|t| < 1$. Then the Poisson kernel for continuous q-Jacobi polynomials is given by

$$K_t = \frac{(t^2, qbct; q)_\infty}{(qt^2, \frac{t}{bc}; q)_\infty} \sum_{n=0}^\infty \frac{(-bc, \pm\sqrt{q}bc, \frac{bc}{a}z^\pm, \frac{bc}{a}w^\pm; q)_n q^n}{(q, \frac{bc^2}{a}, \frac{b^2c}{a}, bc, \frac{bc}{a^2}, qbct^\pm; q)_n}$$

$$\times {}_{10}W_9\left(\frac{q^{-n}a^2}{bc}; \frac{q^{1-n}a}{b^2c}, \frac{q^{1-n}a}{bc^2}, q^{-n}, az^\pm, aw^\pm; q, q\right)$$

$$+ \frac{(t, \frac{at}{c}, b^2c^2, aw^\pm, bz^\pm, ctz^\pm, \frac{bct}{a}w^\pm; q)_\infty}{(ab, ac, bc, \frac{a}{c}, \frac{b^2c}{a}, \frac{bc^2t}{a}, \frac{bc}{t}, tz^\pm w^\pm; q)_\infty} \sum_{n=0}^\infty \frac{(-t, \pm\sqrt{q}t, \frac{bc^2t}{a}, tz^\pm w^\pm; q)_n q^n}{(q, qt^2, \frac{qt}{bc}, \frac{at}{c}, ctz^\pm, \frac{bct}{a}w^\pm; q)_n}$$

$$\times {}_{10}W_9\left(\frac{q^{n-1}bc^2t}{a}; q^nt, q^{n-1}bct, \frac{q^ntc}{a}, \frac{bc}{a}z^\pm, cw^\pm; q, q\right)$$

$$+ \frac{(t, \frac{ct}{a}, b^2c^2, cw^\pm, \frac{bc}{a}z^\pm, atz^\pm, btw^\pm; q)_\infty}{(ac, bc, \frac{bc^2}{a}, \frac{b^2c}{a}, \frac{c}{a}, abt, \frac{bc}{t}, tz^\pm w^\pm; q)_\infty} \sum_{n=0}^\infty \frac{(-t, \pm\sqrt{q}t, abt, tz^\pm w^\pm; q)_n q^n}{(q, qt^2, \frac{qt}{bc}, \frac{ct}{a}, atz^\pm, btw^\pm; q)_n}$$

$$\times {}_{10}W_9\left(q^{n-1}abt; q^nt, q^{n-1}bct, \frac{q^nat}{c}, bz^\pm, aw^\pm; q, q\right).$$

Generating functions from special values

- Generating functions are produced by replacing $z \in \{a, b, c, d\}$ or $w \in \{a, b, c, d\}$.

Generating functions

- $w=a$ generating function

$$\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta+1},q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q^{\beta+1},q^{\frac{\alpha+\beta+1}{2}};q)_n}P_n^{(\alpha,\beta)}(x|q)t^n$$

$$=\frac{(q^{2\alpha+1}t^2,-q^{\frac{3\alpha+\beta+5}{2}}t,-q^{\frac{\alpha+\beta+2}{2}}t,q^{\alpha+\frac{\beta}{2}+\frac{7}{4}}tz^{\pm};q)_\infty}{(q^{2\alpha+2}t^2,-q^{\alpha+1}t,-q^{\alpha+\beta+\frac{5}{2}}t,q^{\frac{\alpha}{2}+\frac{1}{4}}tz^{\pm};q)_\infty}$$

$$\times\,{}_8W_7\left(-q^{\alpha+\beta+\frac{3}{2}}t;q^{\frac{\beta-\alpha}{2}},q^{\frac{\alpha+\beta+3}{2}},-q^{\alpha+\frac{3}{2}}t,-q^{\frac{\beta}{2}+\frac{3}{4}}z^{\pm};q,-q^{\alpha+\frac{1}{2}}t\right)$$

- q to 1 limit

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1,\frac{\alpha+\beta+3}{2})_n}{(\beta+1,\frac{\alpha+\beta+1}{2})_n}P_n^{(\alpha,\beta)}(x)t^n=\frac{1-t^2}{(1+t^2-2tx)^{\frac{\alpha+\beta+3}{2}}}$$

$$\times\,{}_2F_1\left(\begin{matrix}\frac{\beta-\alpha}{2},\frac{\alpha+\beta+3}{2}\\ \beta+1\end{matrix};\frac{-2t(1+x)}{1+t^2-2tx}\right)$$

Generating functions

- $z=d$ generating function

$$\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta+1},\pm q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q^{\beta+1},\pm q^{\frac{\alpha+\beta+1}{2}};q)_n}P_n^{(\alpha,\beta)}(x|q)t^n$$

$$=\frac{(q^{\alpha+\beta+2}t^2,q^{\alpha+\beta+3}t;q)_\infty}{(q^{\alpha+\beta+3}t^2,t;q)_\infty}\,{}_5\phi_4\left(\begin{matrix}q^{\frac{\alpha+\beta+2}{2}},\pm q^{\frac{\alpha+\beta+3}{2}},q^{\frac{\alpha}{2}+\frac{1}{4}}w^{\pm}\\ q^{\alpha+1},-q^{\frac{\alpha+\beta+1}{2}},q^{\alpha+\beta+3}t,qt^{-1}\end{matrix};q,q\right)$$

$$+\frac{(q^{\alpha+\beta+2},q^{\alpha+1}t,-q^{\frac{\alpha+\beta+1}{2}}t,-q^{\frac{\alpha+\beta+2}{2}}t,q^{\frac{\alpha}{2}+\frac{1}{4}}w^{\pm};q)_\infty}{(q^{\alpha+1},-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha+\beta+2}{2}},t^{-1},q^{\frac{\alpha}{2}+\frac{1}{4}}tw^{\pm};q)_\infty}$$

$$\times\,{}_5\phi_4\left(\begin{matrix}q^{\frac{\alpha+\beta+2}{2}}t,\pm q^{\frac{\alpha+\beta+3}{2}}t,q^{\frac{\alpha}{2}+\frac{1}{4}}tw^{\pm}\\ q^{\alpha+1}t,q^{\alpha+\beta+3}t^2,-q^{\frac{\alpha+\beta+1}{2}}t,qt\end{matrix};q,q\right)$$

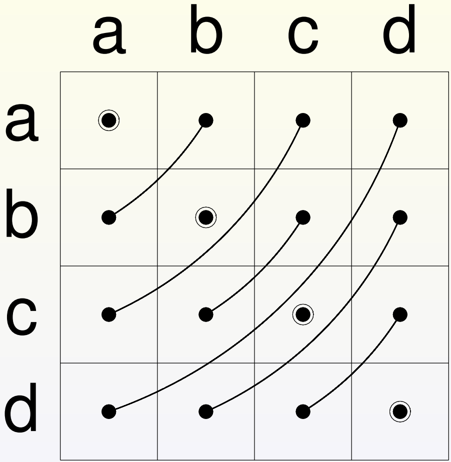
Generating functions

- q to 1 limit

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1, \frac{\alpha+\beta+3}{2})_n}{(\alpha+1, \frac{\alpha+\beta+1}{2})_n} P_n^{(\alpha,\beta)}(y) t^n = \frac{1-t^2}{(1-t)^{\alpha+\beta+3}}$$

$$\times\, {}_2F_1\left(\begin{matrix} \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \\ \alpha+1 \end{matrix} ; \frac{2t(y-1)}{(1-t)^2} \right)$$

Poisson kernel transformations



$$a = q^{\frac{\alpha}{2}+\frac{1}{4}},\ b = q^{\frac{\alpha}{2}+\frac{3}{4}},\ c = -q^{\frac{\beta}{2}+\frac{1}{4}},\ d = -q^{\frac{\beta}{2}+\frac{3}{4}}.$$

Alternate continuous q-Jacobi polynomials

- Another q-analogue of the Jacobi polynomials is given by

$$P_n^{(\alpha,\beta)}(x;q) = q^{-n\alpha}\frac{(-q^{\alpha+\beta+1};q)_n}{(-q;q)_n}P_n^{(\alpha,\beta)}(x|q^2).$$

This follows from the quadratic transformation

$$\begin{aligned}&{}_4\phi_3\left(\begin{matrix}q^{-2n}, q^{2n}a^2, qb^2, c^2\\ -a, -qa, q^2b^2c^2\end{matrix};q^2,q^2\right)\\ &\qquad = (bc)^n\frac{(-q,-\frac{a}{bc};q)_n}{(-a,-qbc;q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n}, q^na, \frac{qb}{c}, \frac{c}{b}\\ -q, -\frac{a}{bc}, qbc\end{matrix};q,q\right).\end{aligned}$$

and therefore has the following representation in terms of Askey–Wilson polynomials

$$P_n^{(\alpha,\beta)}(x;q) = \frac{q^{n/2}}{(q,-q,-q;q)_n}p_n(x;q^{\frac12},-q^{\frac12},q^{\alpha+1},-q^{\beta+1}|q).$$

Alternate continuous q-Jacobi polynomial special values

- The continuous q-Jacobi polynomials have the following **four** alternate special values

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}})|q^2)=(q^{\alpha})^n\,\frac{(q^{\alpha+1},-q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}})|q^2)=(-q^{\alpha})^n\,\frac{(-q^{\alpha+1},q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n},$$

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\alpha+\frac{1}{2}}+q^{-\alpha-\frac{1}{2}}|q^2)=\frac{(q^{\alpha+1},-q^{\alpha+1};q)_n}{(q,-q;q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\beta+\frac{1}{2}}+q^{-\beta-\frac{1}{2}})|q^2)=\left(-q^{\alpha-\beta}\right)^n\,\frac{(q^{\beta+1},-q^{\beta+1};q)_n}{(q,-q;q)_n}.$$

Future directions of work

- Generating functions/transformations, q to 1 limits.
- Rahman's continuous q-Jacobi polynomial Poisson kernel.
- Askey–Wilson polynomial Poisson kernel (these exist in the literature, but there are some typographical errors which need to be fixed).
- Higher multi-linear generating functions for Askey–Wilson polynomials.

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