Rényi entropy and linearization of orthogonal polynomials

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Aim

Aim of this work: The calculation of the Rényi entropy of the probability density functions associated to the real families of classical orthogonal polynomials.

Rényi entropy of order q, ($q > 0$, $q \neq 1$):

$$R_q[\rho]$$

$\Longleftrightarrow$

L^q-norm:

$$\|\rho\|_q$$

For a density $\rho(x)$, $x \in \Delta \subseteq \mathbb{R}$:

$$R_q[\rho] = \frac{1}{1-q} \ln \|\rho\|_q^q = \frac{1}{1-q} \ln \int_\Delta (\rho(x))^q \, dx$$

Rényi entropy of orthogonal polynomials

Rakhmanov's density

The distribution of these polynomials along the orthogonality interval can be measured by means of the spreading properties of the normalized to unity Rakhmanov's density:

$$\rho_n(x) = \frac{1}{d_n^2}\omega(x)p_n^2(x)$$

Thus, we need to evaluate the Rényi entropy of these densities:

$$R_q[\rho_n] = \frac{1}{1-q}\ln\int_\Delta (\rho_n(x))^q\,dx = \frac{1}{1-q}\ln W_q[\rho_n]$$

where

$$W_q[\rho_n] = \|\rho_n\|_q^q = \int_\Delta (\rho_n(x))^q\,dx$$

is the entropic moment of order q.

Entropic moments of orthogonal polynomials

Entropic moments of classical orthogonal polynomials

$$W_q[\rho_n] = \int_\Delta (\rho_n(x))^q \, dx = \int_\Delta \left(\frac{1}{d_n^2}\right)^q (\omega(x))^q \left((p_n(x))^2\right)^q$$

Two possible approaches:

Power expansion

$$\left((p_n(x))^2\right)^q = \sum_{i=0}^{\infty} a_i x^i$$

(Combinatorial approach)

Linearization

$$\left((p_n(x))^2\right)^q = \sum_{i=0}^{\infty} c_i p_i(x)$$

(Algebraic approach)

Combinatorial approach

This method [2010,2011]* uses the expansion of a power of orthogonal polynomials, whose coefficients are given in terms of Bell polynomials:

$$(p_n(x))^r = \sum_{i=0}^{nr} \frac{r!}{(i+r)!} B_{i+r,r}\left(b_0, 2!b_1, \ldots, (i+1)!b_i\right) x^i$$

where $r \in \mathbb{N}$, b_i are the coefficients in the expansion $p_n(x) = \sum_{i=0}^{n} b_i x^i$, and the Bell polynomials are defined as

$$B_{m,l}(x_1, \ldots, x_{m-l+1}) = \sum_{\hat{\pi}(m,l)} \frac{m!}{j_1! \ldots j_{m-l+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{m-l+1}!}{(m-l+1)!}\right)^{j_{m-l+1}},$$

where the sum runs over all the partitions $\hat{\pi}(m,l)$ such that

$$j_1 + j_2 + \cdots + j_{m-l+1} = l, \quad \text{and} \quad j_1 + 2j_2 + \cdots + (m-l+1)j_{m-l+1} = m.$$

* PSM, J.S. Dehesa, et al; JCAM **233** (2010) 2136; J. Phys. A **43** (2010) 305203; JCAM **235** (2011) 1129.

Combinatorial approach

Notice now that, taking into account that $q > 0$, in general we have

$$\left((p_n(x))^2\right)^q = (p_n(x))^{2q}$$

only if $q \in \mathbb{N}$.

Then, we can evaluate the Rényi entropy $R_q[\rho_n]$ only for positive integer values of q.

Combinatorial approach

- In the case of the Hermite polynomials, the entropic moment of order $q \in \mathbb{N}$, $q > 1$, can be expressed as

$$W_q[\rho_n] = \sum_{i=0}^{nq} \frac{\Gamma\left(i+\frac{1}{2}\right)}{q^{i+\frac{1}{2}}} \frac{(2q)!}{(2i+2q)!} B_{2i+2q,2q}(b_0, 2!b_1, \ldots, (2i+1)!b_{2i}),$$

and the Rényi entropy is

$$R_q[\rho_n] = \frac{1}{1-q} \ln\left[\sum_{i=0}^{nq} \frac{\Gamma\left(i+\frac{1}{2}\right)}{q^{i+\frac{1}{2}}} \frac{(2q)!}{(2i+2q)!} B_{2i+2q,2q}(b_0, 2!b_1, \ldots, (2i+1)!b_{2i}),\right]$$

where b_i are the expansion coefficients of the orthonormal Hermite polynomials: $\tilde{H}_n(x) = \sum_{i=0}^{n} b_i x^i$.

- Similar results are obtained for the Laguerre and Jacobi polynomials.

Algebraic approach

The algebraic approach [2011]* is based on the linearization of the power of the classical orthogonal polynomials:

$$\left((p_n(x))^2\right)^q = \sum_{i=0}^{\infty} c_i p_i(x)$$

Then, let us study this linearization before attempting to calculate the Rényi entropy.

* PSM, A. Zarzo, J.S. Dehesa. Preprint (2011).

Linearization of Laguerre polynomials

We use the Srivastava linearization [Astrophys. Space Sci., 1998]:

$$x^{\mu}\left(L_n^{(\alpha)}(tx)\right)^r = \sum_{i=0}^{\infty} c_i(\mu, r, t, n, \alpha, \gamma) L_i^{(\gamma)}(x),$$

where the coefficients $c_i(\mu, r, t, n, \alpha, \gamma)$ are expressed as:

$$c_i(\mu, r, t, n, \alpha, \gamma) = (\gamma+1)_{\mu}\binom{n+\alpha}{n}^r \times F_A^{(r+1)}\left(\begin{matrix} \gamma+\mu+1; \overbrace{-n, \dots, -n}^{r}, -i \\ \underbrace{\alpha+1, \dots, \alpha+1}_{r}, \gamma+1 \end{matrix}; \overbrace{t, \dots, t}^{r}, 1\right),$$

where $F_A^{(r+1)}$ is a Lauricella function of type A of $r+1$ variables.

Lauricella functions

The Lauricella function of type A is defined as:

$$F_A^{(s)}\left(\begin{array}{c} a; b_1, \ldots, b_s \\ c_1, \ldots, c_s \end{array}; x_1, \ldots, x_s\right) = \sum_{j_1, \ldots, j_s = 0}^{\infty} \frac{(a)_{j_1+\cdots+j_s}(b_1)_{j_1}\cdots(b_s)_{j_s}}{(c_1)_{j_1}\cdots(c_s)_{j_s}} \frac{x_1^{j_1}\cdots x_s^{j_s}}{j_1!\cdots j_s!}$$

In our case, as $b_i = -n$, $\forall i$, this multiple hypergeometric sum is always finite.

Linearization of Hermite polynomials

The linearization of a power of Hermite polynomials such as

$$(H_n(tx))^r ,$$

for $r \in \mathbb{N}$, can be obtained from that of the Laguerre polynomials, by means of the relations

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{\left(-\frac{1}{2}\right)}\left(x^2\right),$$

and

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\left(\frac{1}{2}\right)}\left(x^2\right).$$

Then, the following three possible cases are in order.

Case 1: Even degree

In this case, the power of Hermite polynomials can be expressed as the following power of Laguerre polynomials:

$$(H_{2n}(tx))^r = A_{n,r}\left(L_n^{\left(-\frac{1}{2}\right)}(t^2x^2)\right)^r.$$

where

$$A_{n,r} = (-1)^{nr}2^{2nr}(n!)^r.$$

Now, the previous linearization of the Laguerre polynomials yields

$$\begin{aligned}(H_{2n}(tx))^r &= A_{n,r}\left(L_n^{\left(-\frac{1}{2}\right)}(t^2x^2)\right)^r \\ &= A_{n,r}\sum_{i=0}^{nr} c_i\left(0,r,t^2,n,-\frac{1}{2},-\frac{1}{2}\right)L_i^{\left(-\frac{1}{2}\right)}\left(x^2\right)\end{aligned}$$

Case 1: Even degree

Finally, we express the Laguerre polynomial $L_i^{\left(-\frac{1}{2}\right)}\left(x^2\right)$ in terms of a Hermite polynomial:

$$\begin{aligned}(H_{2n}(tx))^r &= A_{n,r}\sum_{i=0}^{nr} c_i\left(0,r,t^2,n,-\frac{1}{2},-\frac{1}{2}\right)L_i^{\left(-\frac{1}{2}\right)}\left(x^2\right)\\ &= A_{n,r}\sum_{i=0}^{nr} c_i\left(0,r,t^2,n,-\frac{1}{2},-\frac{1}{2}\right)\frac{1}{A_{i,1}}H_{2i}(x)\end{aligned}$$

$$= A_{n,r}\binom{n-\frac{1}{2}}{n}^r\sum_{i=0}^{nr} F_A^{(r+1)}\left(\begin{array}{c}\frac{1}{2};-n,\ldots,-n,-i\\ \frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}\end{array};t^2,\ldots,t^2,1\right)\frac{1}{A_{i,1}}H_{2i}(x),$$

which is the wanted linearization relation.

Case 2: Odd degree, odd value r

In this case, the power of Hermite polynomials can be expressed in terms of Laguerre polynomials like

$$(H_{2n+1}(tx))^r = B_{n,r} t^r x x^{r-1} \left(L_n^{\left(\frac{1}{2}\right)} \left(t^2 x^2\right) \right)^r$$
$$= B_{n,r} t^r x \left(x^2\right)^{\frac{r-1}{2}} \left(L_n^{\left(\frac{1}{2}\right)} \left(t^2 x^2\right) \right)^r$$

where

$$B_{n,r} = (-1)^{nr} 2^{(2n+1)r} n!^r.$$

Again, the previous linearization of the Laguerre polynomials yields

$$(H_{2n+1}(tx))^r = B_{n,r} t^r x \sum_{i=0}^{\frac{(2n+1)r-1}{2}} c_i \left(\frac{r-1}{2}, r, t^2, n, \frac{1}{2}, \frac{1}{2} \right) L_i^{\left(\frac{1}{2}\right)} \left(x^2\right)$$

Case 2: Odd degree, odd value r

Now, we express the Laguerre polynomial $L_i^{\left(\frac{1}{2}\right)}\left(x^2\right)$ in terms of a Hermite polynomial:

$$
\begin{aligned}
\left(H_{2n+1}(tx)\right)^r = B_{n,r}t^r \sum_{i=0}^{\frac{(2n+1)r-1}{2}} c_i\left(\frac{r-1}{2}, r, t^2, n, \frac{1}{2}, \frac{1}{2}\right)\frac{1}{B_{i,1}}H_{2i+1}(x) \\
= B_{n,r}t^r \left(\frac{3}{2}\right)_{\frac{r-1}{2}} \binom{n+\frac{1}{2}}{n}^r
\end{aligned}
$$

$$
\times \sum_{i=0}^{\frac{(2n+1)r-1}{2}} F_A^{(r+1)}\left(\begin{array}{c} \frac{r}{2}+1; -n, \ldots, -n, -i \\ \frac{3}{2}, \ldots, \frac{3}{2}, \frac{3}{2} \end{array}; t^2, \ldots, t^2, 1\right)\frac{1}{B_{i,1}}H_{2i+1}(x),
$$

that is the linearization relation we were looking for.

Case 3: Odd degree, even value r

Here, the linearization of the corresponding power of Laguerre polynomials is now in terms of polynomials of the type $L_i^{\left(-\frac{1}{2}\right)}\left(x^2\right)$:

$$
\begin{aligned}
\left(H_{2n+1}(tx)\right)^r &= B_{n,r} t^r x^r \left(L_n^{\left(\frac{1}{2}\right)}\left(t^2x^2\right)\right)^r \\
&= B_{n,r} t^r \left(x^2\right)^{\frac{r}{2}} \left(L_n^{\left(\frac{1}{2}\right)}\left(t^2x^2\right)\right)^r \\
&= B_{n,r} t^r \sum_{i=0}^{\frac{(2n+1)r}{2}} c_i\left(\frac{r}{2}, r, t^2, n, \frac{1}{2}, -\frac{1}{2}\right) L_i^{\left(-\frac{1}{2}\right)}\left(x^2\right)
\end{aligned}
$$

Case 3: Odd degree, even value r

We express the Laguerre polynomials $L_i^{\left(-\frac{1}{2}\right)}\left(x^2\right)$ in terms of even-degree Hermite polynomials to yield the final expression:

$$(H_{2n+1}(tx))^r = B_{n,r}t^r \sum_{i=0}^{\frac{(2n+1)r}{2}} c_i\left(\frac{r}{2}, r, t^2, n, \frac{1}{2}, -\frac{1}{2}\right)\frac{1}{A_{i,1}}H_{2i}(x)$$

$$= B_{n,r}t^r \left(\frac{1}{2}\right)_{\frac{r}{2}} \binom{n+\frac{1}{2}}{n}^r$$

$$\times \sum_{i=0}^{\frac{(2n+1)r}{2}} F_A^{(r+1)}\left(\begin{matrix} \frac{r+1}{2}; -n, \ldots, -n, -i \\ \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \end{matrix}; t^2, \ldots, t^2, 1\right)\frac{1}{A_{i,1}}H_{2i}(x),$$

that is the wanted linearization relation.

Linearization of Hermite polynomials: Summary

- Even degree:

$$(H_{2n}(tx))^r = A_{n,r}\binom{n-\frac{1}{2}}{n}^r \sum_{i=0}^{nr} F_A^{(r+1)}\left(\begin{array}{c} \frac{1}{2}; -n,\dots,-n,-i \\ \frac{1}{2},\dots,\frac{1}{2},\frac{1}{2} \end{array}; t^2,\dots,t^2,1\right)\frac{1}{A_{i,1}}H_{2i}(x)$$

- Odd degree, odd r:

$$(H_{2n+1}(tx))^r = B_{n,r}t^r\left(\frac{3}{2}\right)_{\frac{r-1}{2}}\binom{n+\frac{1}{2}}{n}^r$$

$$\times\sum_{i=0}^{\frac{(2n+1)r-1}{2}} F_A^{(r+1)}\left(\begin{array}{c} \frac{r}{2}+1; -n,\dots,-n,-i \\ \frac{3}{2},\dots,\frac{3}{2},\frac{3}{2} \end{array}; t^2,\dots,t^2,1\right)\frac{1}{B_{i,1}}H_{2i+1}(x),$$

- Odd degree, even r:

$$(H_{2n+1}(tx))^r = B_{n,r}t^r\left(\frac{1}{2}\right)_{\frac{r}{2}}\binom{n+\frac{1}{2}}{n}^r$$

$$\times\sum_{i=0}^{\frac{(2n+1)r}{2}} F_A^{(r+1)}\left(\begin{array}{c} \frac{r+1}{2}; -n,\dots,-n,-i \\ \frac{3}{2},\dots,\frac{3}{2},\frac{1}{2} \end{array}; t^2,\dots,t^2,1\right)\frac{1}{A_{i,1}}H_{2i}(x),$$

Linearization of Jacobi polynomials

In the case of the Jacobi polynomials, H.M. Srivastava [Astrophys. Space Sci., 1998] obtained the expression:

$$x^\mu \left(P_n^{(\alpha,\beta)}(1-2x)\right)^r = \binom{n+\alpha}{n}^r (\gamma+1)_\mu \sum_{i=0}^{\infty} \frac{(\gamma+\delta+2i+1)(-\mu)_i}{(\gamma+1)_i(\gamma+\delta+i+1)_{\mu+1}}$$

$$\times F^{2:2;\ldots,2}_{2:1;\ldots;1}\left(\begin{array}{c} \mu+1,\gamma+\mu+1 : \overbrace{-n,\alpha+\beta+n+1;\ldots;-n,\alpha+\beta+n+1}^{r} \\ \mu-i+1,\gamma+\delta+\mu+i+2 : \underbrace{\alpha+1;\ldots;\alpha+1}_{r} \end{array} ; \overbrace{1,\ldots,1}^{r}\right)$$

$$\times P_i^{(\gamma,\delta)}(1-2x)$$

where F is a Srivastava-Daoust generalized hypergeometric function of r variables.

We are interested in the case $\mu = 0$.

Srivastava-Daoust function

This function is defined as

$$F^{2:2;\dots,2}_{2:1;\dots;1}\left(\begin{array}{c} \mu+1,\gamma+\mu+1:\overbrace{-n,\alpha+\beta+n+1;\dots;-n,\alpha+\beta+n+1}^{r} \\ \mu-i+1,\gamma+\delta+\mu+i+2:\underbrace{\alpha+1;\dots;\alpha+1}_{r} \end{array};\overbrace{1,\dots,1}^{r}\right)$$

$$=\sum_{j_1,\dots,j_r=0}^{n}\frac{(\mu+1)_{j_1+\dots+j_r}(\gamma+\mu+1)_{j_1+\dots+j_r}}{(\mu-i+1)_{j_1+\dots+j_r}(\gamma+\delta+\mu+i+2)_{j_1+\dots+j_r}}$$

$$\times\frac{(-n)_{j_1}(\alpha+\beta+n+1)_{j_1}\cdots(-n)_{j_r}(\alpha+\beta+n+1)_{j_r}}{(\alpha+1)_{j_1}\cdots(\alpha+1)_{j_r}j_1!\cdots j_r!}$$

Notice that for $\mu=0$ this function may not be well defined.

Coefficient of the linearization for $\mu = 0$

The coefficient of the linearization formula can be expressed as

$$\binom{n+\alpha}{n}^{r}(\gamma+1)_{\mu}\frac{(\gamma+\delta+2i+1)(-\mu)_i}{(\gamma+1)_i(\gamma+\delta+i+1)_{\mu+1}}$$

$$\times\sum_{j_1,\ldots,j_r=0}^{n}\frac{(\mu+1)_{j_1+\cdots+j_r}(\gamma+\mu+1)_{j_1+\cdots+j_r}}{(\mu-i+1)_{j_1+\cdots+j_r}(\gamma+\delta+\mu+i+2)_{j_1+\cdots+j_r}}$$

$$\times\frac{(-n)_{j_1}(\alpha+\beta+n+1)_{j_1}\cdots(-n)_{j_r}(\alpha+\beta+n+1)_{j_r}}{(\alpha+1)_{j_1}\cdots(\alpha+1)_{j_r}j_1!\cdots j_r!}$$

The indetermination comes from the quotient $\dfrac{(-\mu)_i}{(\mu-i+1)_{j_1+\cdots+j_r}}$.

Coefficient of the linearization for $\mu = 0$

This indetermination can be fixed by considering the Slater formula (1966):

$$\frac{(-\mu)_i(\mu+1)_{j_1+\cdots+j_r}}{(\gamma+1)_i(\mu-i+1)_{j_1+\cdots+j_r}} = {}_2F_1\left(\begin{array}{c}\gamma+\mu+j_1+\cdots+j_r+1, -i\\ \gamma+1\end{array};1\right)$$

By inserting this result in the linearization coefficient and taking $\mu = 0$, we have

$$\binom{n+\alpha}{n}^r \frac{\gamma+\delta+2i+1}{\gamma+\delta+i+1} \sum_{j_1,\ldots,j_r=0}^{n} \sum_{j_{r+1}=0}^{i} \frac{(\gamma+1)_{j_1+\cdots+j_r+j_{r+1}}}{(\gamma+\delta+i+2)_{j_1+\cdots+j_r}}$$

$$\times \frac{(-n)_{j_1}(\alpha+\beta+n+1)_{j_1}\cdots(-n)_{j_r}(\alpha+\beta+n+1)_{j_r}(-i)_{j_{r+1}}}{(\alpha+1)_{j_1}\cdots(\alpha+1)_{j_r}(\gamma+1)_{j_{r+1}} j_1!\cdots j_r! j_{r+1}!}$$

However, this generalized hypergeometric series is not a Srivastava-Daoust function.

Linearization of Jacobi polynomials

Finally, the desired linearization formula can be written as

$$\left(P_n^{(\alpha,\beta)}(x)\right)^r = \sum_{i=0}^{nr} \tilde{c}_i(r,n,\alpha,\beta,\gamma,\delta) P_i^{(\gamma,\delta)}(x),$$

where

$$\begin{aligned}
&\tilde{c}_i(r,n,\alpha,\beta,\gamma,\delta) \\
&\quad = \binom{n+\alpha}{n}^r \frac{\gamma+\delta+2i+1}{\gamma+\delta+i+1} \sum_{j_1,\ldots,j_r=0}^{n} \sum_{j_{r+1}=0}^{i} \frac{(\gamma+1)_{j_1+\cdots+j_r+j_{r+1}}}{(\gamma+\delta+i+2)_{j_1+\cdots+j_r}} \\
&\qquad \times \frac{(-n)_{j_1}(\alpha+\beta+n+1)_{j_1}\cdots(-n)_{j_r}(\alpha+\beta+n+1)_{j_r}(-i)_{j_{r+1}}}{(\alpha+1)_{j_1}\cdots(\alpha+1)_{j_r}(\gamma+1)_{j_{r+1}} j_1!\cdots j_r! j_{r+1}!}
\end{aligned}$$

Rényi entropy of Hermite polynomials

Let us use the linearization of the Hermite polynomials to find a better expression for the Rényi entropy.

The entropic moments for these polynomials are defined as

$$W_q[\rho_n] = \int_{-\infty}^{\infty} (2^n n! \sqrt{\pi})^{-q} e^{-qx^2} \left((H_n(x))^2 \right)^q dx.$$

First, we perform the change of variable $\sqrt{q}x = y$, that yields

$$W_q[\rho_n] = \frac{(2^n n! \sqrt{\pi})^{-q}}{\sqrt{q}} \int_{-\infty}^{\infty} e^{-y^2} \left(H_n \left(\frac{y}{\sqrt{q}} \right) \right)^{2q} dx.$$

Rényi entropy of Hermite polynomials: Even degree

Now, the linearization of the Hermite polynomials with even degree yields

$$\left(H_{2n}\left(\frac{y}{\sqrt{q}}\right)\right)^{2q} = A_{n,2q}\binom{n-\frac{1}{2}}{n}^{2q} \times \sum_{i=0}^{2nq} F_A^{(2q+1)}\left(\begin{array}{c}\frac{1}{2}; -n, \ldots, -n, -i \\ \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\end{array}; \frac{1}{q}, \ldots, \frac{1}{q}, 1\right)\frac{1}{A_{i,1}}H_{2i}(y).$$

Once applied the previous integral, the only term different from zero is that with $i=0$. Thus,

Entropic moment of order q of Hermite polynomials: Even degree

$$W_q[\rho_{2n}] = (2^{2n}(2n)!\sqrt{\pi})^{-q}\sqrt{\frac{\pi}{q}}\frac{A_{n,2q}}{A_{0,1}}\binom{n-\frac{1}{2}}{n}^{2q} \times F_A^{(2q+1)}\left(\begin{array}{c}\frac{1}{2}; -n, \ldots, -n, 0 \\ \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\end{array}; \frac{1}{q}, \ldots, \frac{1}{q}, 1\right)$$

Rényi entropy of Hermite polynomials: Even degree

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Entropic moment of order q of Hermite polynomials: Even degree

$$W_q[\rho_{2n}] = (2^{2n}(2n)!\sqrt{\pi})^{-q}\sqrt{\frac{\pi}{q}}\frac{A_{n,2q}}{A_{0,1}}\binom{n-\frac{1}{2}}{n}^{2q} \times F_A^{(2q)}\left(\begin{array}{c}\frac{1}{2}; -n, \dots, -n \\ \frac{1}{2}, \dots, \frac{1}{2}\end{array}; \frac{1}{q}, \dots, \frac{1}{q}\right)$$

Rényi entropy of Hermite polynomials: Even degree

With this result, we can write the Rényi entropy:

Rényi entropy of Hermite polynomials: Even degree

$$R_q[\rho_{2n}] = \frac{1}{1-q} \ln \left[\left(2^{2n}(2n)!\sqrt{\pi}\right)^{-q} \sqrt{\frac{\pi}{q}} \frac{A_{n,2q}}{A_{0,1}} \binom{n-\frac{1}{2}}{n}^{2q} \right.$$
$$\left. \times F_A^{(2q)} \left(\begin{array}{c} \frac{1}{2}; -n, \ldots, -n \\ \frac{1}{2}, \ldots, \frac{1}{2} \end{array} ; \frac{1}{q}, \ldots, \frac{1}{q} \right) \right]$$

Rényi entropy of Hermite polynomials: Odd degree

The linearization of the Hermite polynomials with odd degree and even power yields

$$\left(H_{2n+1}\left(\frac{y}{\sqrt{q}}\right)\right)^{2q} = \frac{B_{n,2q}}{q^q}\left(\frac{1}{2}\right)_q \binom{n+\frac{1}{2}}{n}^{2q} \times \sum_{i=0}^{(2n+1)q} F_A^{(2q+1)}\left(\begin{array}{c} \frac{2q+1}{2}; -n, \ldots, -n, -i \\ \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \end{array}; \frac{1}{q}, \ldots, \frac{1}{q}, 1\right) \frac{1}{A_{i,1}} H_{2i}(y)$$

Again, only the term with $i=0$ survives for the entropic moment:

Entropic moment of order q of Hermite polynomials: Odd degree

$$W_q[\rho_{2n+1}] = (2^{2n+1}(2n+1)!\sqrt{\pi})^{-q}\sqrt{\frac{\pi}{q}}\frac{B_{n,2q}}{q^q A_{0,1}}\left(\frac{1}{2}\right)_q \binom{n+\frac{1}{2}}{n}^{2q} \times F_A^{(2q+1)}\left(\begin{array}{c} \frac{2q+1}{2}; -n, \ldots, -n, 0 \\ \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \end{array}; \frac{1}{q}, \ldots, \frac{1}{q}, 1\right)$$

Rényi entropy of Hermite polynomials: Odd degree

The linearization of the Hermite polynomials with odd degree and even power yields

$$\left(H_{2n+1}\left(\frac{y}{\sqrt{q}}\right)\right)^{2q} = \frac{B_{n,2q}}{q^q}\left(\frac{1}{2}\right)_q \binom{n+\frac{1}{2}}{n}^{2q}$$
$$\times \sum_{i=0}^{(2n+1)q} F_A^{(2q+1)}\left(\begin{array}{c} \frac{2q+1}{2}; -n, \ldots, -n, -i \\ \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \end{array}; \frac{1}{q}, \ldots, \frac{1}{q}, 1\right) \frac{1}{A_{i,1}} H_{2i}(y)$$

Again, only the term with $i = 0$ survives for the entropic moment:

Entropic moment of order q of Hermite polynomials: Odd degree

$$W_q[\rho_{2n+1}] = (2^{2n+1}(2n+1)!\sqrt{\pi})^{-q}\sqrt{\frac{\pi}{q}}\frac{B_{n,2q}}{q^q A_{0,1}}\left(\frac{1}{2}\right)_q \binom{n+\frac{1}{2}}{n}^{2q}$$
$$\times F_A^{(2q)}\left(\begin{array}{c} \frac{2q+1}{2}; -n, \ldots, -n \\ \frac{3}{2}, \ldots, \frac{3}{2} \end{array}; \frac{1}{q}, \ldots, \frac{1}{q}\right)$$

Rényi entropy of Hermite polynomials: Odd degree

The Rényi entropy has the expression:

Rényi entropy of Hermite polynomials: Odd degree

$$R_q[\rho_{2n+1}] = \frac{1}{1-q} \ln \left[(2^{2n+1}(2n+1)!\sqrt{\pi})^{-q} \sqrt{\frac{\pi}{q}} \frac{B_{n,2q}}{q^q A_{0,1}} \left(\frac{1}{2}\right)_q \right.$$
$$\left. \times \binom{n+\frac{1}{2}}{n}^{2q} F_A^{(2q)} \left(\begin{array}{c} \frac{2q+1}{2}; -n, \ldots, -n \\ \frac{3}{2}, \ldots, \frac{3}{2} \end{array} ; \frac{1}{q}, \ldots, \frac{1}{q} \right) \right]$$

Rényi entropy of Laguerre polynomials

Similar expressions are found for the Rényi entropy of the Laguerre polynomials:

Rényi entropy of Laguerre polynomials

$$L_q^R\left[\rho_n^{(\alpha)}\right] = \frac{1}{1-q}\ln\left[\left(\frac{n!}{\Gamma(\alpha+n+1)}\right)^q \frac{\Gamma(\alpha q+1)}{q^{\alpha q+1}}\binom{n+\alpha}{n}^{2q} \times F_A^{(2q)}\left(\begin{array}{c}\alpha q+1; -n, \dots, -n \\ \alpha+1, \dots, \alpha+1\end{array}; \frac{1}{q}, \dots, \frac{1}{q}\right)\right]$$

Rényi entropy of Jacobi polynomials

And finally, for the Jacobi polynomials:

Rényi entropy of Jacobi polynomials

$$R_q\left[\rho_n^{(\alpha,\beta)}\right] = \frac{1}{1-q}\ln\left[\frac{d_0^{(\alpha q,\beta q)}}{\left(d_n^{(\alpha,\beta)}\right)^q}\tilde{c}_0\left(2q,n,\alpha,\beta,\alpha q,\beta q\right)\right]$$

Rényi entropy of Jacobi polynomials

Where $d_n^{(\alpha,\beta)} = \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}$, and

$$\tilde{c}_0(2q,n,\alpha,\beta,\alpha q,\beta q) = \binom{n+\alpha}{n}^{2q} \sum_{j_1,\ldots,j_{2q}=0}^{n} \frac{(\alpha q+1)_{j_1+\cdots+j_{2q}}}{(\alpha q+\beta q+2)_{j_1+\cdots+j_{2q}}}$$

$$\times \frac{(-n)_{j_1}(\alpha+\beta+n+1)_{j_1}\cdots(-n)_{j_{2q}}(\alpha+\beta+n+1)_{j_{2q}}}{(\alpha+1)_{j_1}\cdots(\alpha+1)_{j_{2q}} j_1!\cdots j_{2q}!}$$

$$= \binom{n+\alpha}{n}^{2q}$$

$$\times F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{array}{c} \alpha q+1 : -n,\alpha+\beta+n+1;\ldots;-n,\alpha+\beta+n+1 \\ \alpha q+\beta q+2 : \alpha+1;\ldots;\alpha+1 \end{array} ;1,\ldots,1\right),$$

which is a Srivastava-Daoust function evaluated at unity.

Rényi entropy of Jacobi polynomials

Then, the final expression for the Rényi entropy of the Jacobi polynomials is

$$R_q\left[\rho_n^{(\alpha,\beta)}\right] = \frac{1}{1-q}\ln\left[\frac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right.$$

$$\times \frac{2^{\alpha q+\beta q+1}\Gamma(\alpha q+1)\Gamma(\beta q+1)}{(\alpha q+\beta q+1)\Gamma(\alpha q+\beta q+1)}\binom{n+\alpha}{n}^{2q}$$

$$\left.\times F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{array}{c}\alpha q+1:-n,\alpha+\beta+n+1;\ldots;-n,\alpha+\beta+n+1\\ \alpha q+\beta q+2:\alpha+1;\ldots;\alpha+2\end{array};1,\ldots,1\right)\right]$$

Conclusions

- We have obtained the Rényi entropies of the Rahkmanov densities associated to the Hermite, Laguerre and Jacobi orthogonal polynomials.

- These quantities have been obtained by means of two methods: The **combinatorial approach**, based on the Bell polynomials; and the **algebraic approach**, based on the linearization of the Hermite, Laguerre and Jacobi polynomials.

- With the algebraic approach, the Rényi entropy have been obtained in terms of Lauricella functions $F_A^{(2q)}$ for the Hermite and Laguerre polynomials, and in terms of a Srivastava-Daoust function for the Jacobi polynomial.

Conclusions

Differences between the combinatorial and algebraic approaches:

1. In the combinatorial approach: To evaluate the Bell polynomials, we need to find the partitions involved in their expressions.

   In the algebraic approach: We have simple multiple sums with a finite number of elements. (Much better for symbolic computation).

2. In the combinatorial approach: The final expression of the Rényi entropy, depends on q, the degree n and parameters of the polynomial, and all the coefficients of its power expansion.

   In the algebraic approach: The final expression of the Rényi entropy only depends on q, the degree n and the parameters of the polynomial.