Asymptotics for Singular Perturbation Problems

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Singular perturbations

- Matched expansions
- Multiple Scales
- Renormalization Group[1]

Instead eliminate the secular terms by introducing *a-priori* slowly-varying coefficients

[1] Chen, Goldenfeld and Oono Phys. Rev. E **54** (1),1996

Rayleigh equation

The Rayleigh oscillator has the form

$$\ddot{y} + y = \epsilon \dot{y}(1 - \frac{1}{3}\dot{y}^2), \quad \epsilon \to 0+, \quad t \geq 0. \tag{1}$$

Introduce the two-timing ansatz

$$y(t, \tau, \epsilon) = A(\tau, \epsilon)e^{it} + \epsilon B(\tau, \epsilon)e^{3it} + \epsilon^2 C(\tau, \epsilon)e^{5it} + O(\epsilon^3) + c.c. \tag{2}$$

for $\tau = \epsilon t$ where A, B, C etc. are functions of order one that must be determined. We substitute into (1) and collect the coefficients of the various harmonics. For example, the coefficients of the first, third and fifth harmonics are

$$
\begin{aligned}
2i\frac{dA}{d\tau} - iA(1-|A|^2) + \epsilon\left(\frac{d^2A}{d\tau^2} - A^2\frac{dA^*}{d\tau} - \frac{dA}{d\tau}(1-2|A|^2) - 3iA^{*2}B\right) & \\
+\epsilon^2\left(18iA|B|^2 + 2iAA'A^{*'} + 6A^{*'}A^*B - B'A^{*2} - i(A')^2A^*\right) + O(\epsilon^3) &= 0, \\
-8B - \frac{1}{3}iA^3 + \epsilon\left(6i\frac{dB}{d\tau} + 6i|A|^2B - A'A^2 - 3iB\right) + O(\epsilon^2) &= 0, \\
-24C - 3iA^2B + O(\epsilon) &= 0.
\end{aligned}
$$

Rearrange and solve for the amplitude equation and the coefficients of the ansatz to obtain

$$\frac{dA}{d\tau} = \frac{1}{2}A(1-|A|^2) + \epsilon\frac{i}{16}A(|A|^4-2) + O(\epsilon^3). \tag{3}$$

$$B = -\frac{i}{24}A^3 + \epsilon\frac{1}{64}A^3(3|A|^2-2) + O(\epsilon^2) \tag{4}$$

$$C = -\frac{1}{192}A^5 + O(\epsilon) \tag{5}$$

so that the *uniformly valid* asymptotic expansion reads

$$y(t,\epsilon) = Ae^{it} - \epsilon\frac{i}{24}(Ae^{it})^3 + \epsilon^2\left(\frac{1}{64}(3|A|^2-2)(Ae^{it})^3 - \frac{1}{192}(Ae^{it})^5\right) + c.c., \tag{6}$$

which coincides with known Renormalization Group results[2]

[2] E.Kirkinis, *Journal of Mathematical Physics* **49** 073518 (2008)

Non-linear boundary-layer equation

Consider the boundary layer problem for $y = y(x, \epsilon)$,

$$\epsilon y'' + 2y' + y^2 = 0, \quad y(0) = 0, y(1) = 1. \tag{7}$$

The slowly-varying amplitudes depend on the variable x while the fast 'oscillations' on the stretched variable $X = x/\epsilon$. Considering also the form of the nonlinear term we introduce the ansatz

$$y(x, \epsilon) = A(x, \epsilon) + B(x, \epsilon)e^{-2\frac{x}{\epsilon}} + \epsilon C(x, \epsilon)e^{-4\frac{x}{\epsilon}} + \epsilon^2 F(x, \epsilon)e^{-6\frac{x}{\epsilon}} + O(\epsilon^3). \tag{8}$$

Substituting into (7) and collecting the coefficients of the powers $e^{-k\frac{x}{\epsilon}}$ we obtain

$$
\begin{aligned}
2\frac{dA}{dx} + A^2 + \epsilon\frac{d^2A}{dx^2} + O(\epsilon^2) &= 0, && (9)\\
-2\frac{dB}{dx} + 2AB + \epsilon\frac{d^2B}{dx^2} + O(\epsilon^2) &= 0, && (10)\\
-8C + B^2 + \epsilon(2AC - 6\frac{dC}{dx}) + \epsilon^2\frac{d^2C}{dx^2} + O(\epsilon^3) &= 0, && (11)\\
24F + 2BC + \epsilon\left(2AF - 10\frac{dF}{dx}\right) + O(\epsilon^2) &= 0. && (12)
\end{aligned}
$$

Rel. (9) and (10) immediately lead to the amplitude equations

$$
\begin{aligned}
\frac{dA}{dx} &= -\frac{1}{2}A^2 - \epsilon\frac{1}{4}A^3 + O(\epsilon^2),\\
\frac{dB}{dx} &= AB + \epsilon\frac{1}{4}A^2B + O(\epsilon^2).
\end{aligned}
$$

Rel. (11) can be solved for C

$$C(1+\frac{1}{4}\epsilon A) = -\frac{1}{8}B^2 - \epsilon\frac{3}{16}AB^2 + O(\epsilon^2) \tag{13}$$

which leads to

$$C = -\frac{1}{8}B^2 - \epsilon\frac{5}{32}AB^2 + O(\epsilon^2) \tag{14}$$

while (12) leads to

$$F = \frac{1}{96}B^3 + O(\epsilon). \tag{15}$$

Thus we obtain the *uniformly valid* asymptotic expansion

$$y(x;\epsilon) = A + Be^{-2\frac{x}{\epsilon}} - \epsilon\frac{1}{8}(Be^{-2\frac{x}{\epsilon}})^2 + \epsilon^2\left(-\frac{5}{32}A(Be^{-2\frac{x}{\epsilon}})^2 + \frac{1}{96}(Be^{-2\frac{x}{\epsilon}})^3\right) + O(\epsilon^3),$$

(cf. R.E. O'Malley Jr. and E.Kirkinis, Studies in Applied Mathematics, (4) **124** 383-410, (2010))

Delay problems

Consider the linear oscillator with delay

$$\ddot{x}(t) + \omega^2 x(t) + \epsilon x(t-r) = 0, \quad \epsilon \to 0+ \quad \omega, r \in R. \tag{16}$$

We introduce the ansatz (which is our *uniformly valid* asymptotic expansion as well)

$$x(t) = A(\tau,\epsilon)e^{i\omega t} + A^*(\tau,\epsilon)e^{-i\omega t} \tag{17}$$

for $\tau = \epsilon t$. Substituting into (16) we obtain
$\epsilon^2 A''(\tau) + 2\epsilon i\omega A'(\tau) = -\epsilon A(\tau - \epsilon r)e^{-i\omega r}$ and rearranging

$$A'(\tau) = \frac{1}{2\omega} i A(\tau - \epsilon r)e^{-i\omega r} + \epsilon \frac{i}{2\omega} A''(\tau) + O(\epsilon^3). \tag{18}$$

This leads to the first order amplitude equation

$$A'(\tau) = \frac{i}{2\omega} A(\tau - \epsilon r)e^{-i\omega r} + O(\epsilon).$$

Differentiating one more time and substituting into (18) we obtain the second order amplitude equation

$$A'(\tau) = \frac{i}{2\omega}A(\tau - \epsilon r)e^{-i\omega r} - \epsilon\frac{i}{8\omega^3}e^{-2i\omega r}A(\tau - 2\epsilon r) + O(\epsilon^2). \tag{19}$$

The above is a general form of the amplitude equations and higher order corrections may be similarly obtained. To obtain the corresponding RG result[3],[4] one can McLaurin-expand the above about τ:

$$A'(\tau) = \frac{i}{2\omega}A(\tau)e^{-i\omega r} + \epsilon\left(\frac{r}{4\omega^2} - \frac{i}{8\omega^3}\right)e^{-2i\omega r}A(\tau) + O(\epsilon^2). \tag{20}$$

[3] Goto *Prog. Theor. Phys.* Vol. 118 211-227 (2007)

[4] E.Kirkinis Asymptotic Analysis, (1)-(2) **67** 1-16, (2010)