Uniform Asymptotic Expansions for Linear Difference Equations with Turning Points

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Outline

Three term recurrence relations

Orthogonal polynomials on the real line (OPRL), monic, satisfy the three term recurrence relation

$$\pi_{n+1}(x) = (x + b_n)\pi_n(x) - c_n\pi_{n-1}(x). \qquad (1)$$

Let K_n be such that $\frac{K_{n+1}}{K_{n-1}} = c_n$, and set

$$P_n(x) = \frac{1}{K_n}\pi_n(x), \qquad A_n = \frac{K_n}{K_{n+1}}, \qquad B_n = \frac{K_n}{K_{n+1}}b_n.$$

$$P_{n+1}(x) + P_{n-1}(x) = (A_n x + B_n)P_n(x). \qquad (2)$$

Asymptotics for OPRL via difference equations

point-wise asymptotics

- Birkhoff (1911), Adams(1928), Wong & Li (1992a,1992b)
- Dingle & Morgan (1967a,1967b) WKB
- X.S. Wang & R. Wong (2011)

uniform asymptotics, turning point theory

- Costin-Costin (1996), Deift-McLaughlin (1998)
- Z. Wang & R. Wong (2003,2005) Airy-type, Bessel-type
- WKB: Geronimo-Smith-Van Assche (1992), Geronimo-Bruno-Van Assche (2004), Geronimo (2009, preprint)

Assume $P_n \sim \lambda^n$. Characteristic equation:

$$\lambda^2 + 1 = (A_n x + B_n)\lambda$$

The roots

$$\lambda_\pm = \frac{A_n x + B_n \pm \sqrt{(A_n x + B_n)^2 - 4}}{2}$$

coincide when

$$A_n x + B_n = \pm 2,$$

and the turning points are defined by

$$x_{1,2} \sim \frac{\pm 2 - B_n}{A_n}$$

Another point of view

Note that

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\sin n\theta$$

Assume $P_n(x) \sim C_1(x)\cos n\theta(x) + C_2(x)\sin n\theta(x)$, one has

$$\theta(x) = \arccos\frac{A_n x + B_n}{2}.$$

The $\arccos$ could be define outside $[-1,1]$ by analytic continuation, and the turning points are the two branch points (asymptotically)

$$x_{1,2} \sim \frac{\pm 2 - B_n}{A_n}.$$

$$P_{n+1}(x) + P_{n-1}(x) = (A_n x + B_n)P_n(x) \tag{3}$$

$$A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}. \tag{4}$$

rescale: $x = n^\theta t$
Turning points

$$t_{1,2} = \frac{\pm 2 - \beta_0}{\alpha_0}$$

Case (i) $\theta = 0$: Bessel type, Wang & Wong (2005)
Case (ii) $\theta \neq 0$ and $t_{1,2} \neq 0$: Airy type, Wang & Wong (2003)
Case (iii) $\theta \neq 0$ and $t_1 = 0$: Bessel type

Assume

$$P_n \sim \chi(\xi), \qquad \xi = n^2\zeta(t).$$

$\zeta(t)$: certain function with $\zeta(0) = 0$, increasing.
x is fixed $\Longrightarrow$

$$n \to n+1: \quad t \to t_+ = \left(1 + \frac{1}{n}\right)^{-\theta}, \quad \xi \to Q_+(\xi) = (n+1)^2\zeta(t_+)$$

$$n \to n-1: \quad t \to t_- = \left(1 - \frac{1}{n}\right)^{-\theta}, \quad \xi \to Q_-(\xi) = (n-1)^2\zeta(t_-)$$

$$P_{n+1}(x) \sim \chi(Q_+(\xi)), \qquad P_{n-1}(x) \sim \chi(Q_-(\xi)).$$

what is $\chi(\xi)$

$$Q_{\pm}(\xi) = \xi \pm \frac{2-\theta}{n}\xi + \frac{\theta^2 - 3\theta + 2}{n^2}\xi + O(n^{-3})$$

$$\chi''(\xi) + \frac{\theta - 1}{(\theta-2)\xi}\chi'(\xi) + \frac{1}{(\theta-2)^2\xi^2}\left(\frac{\alpha_0}{\zeta'(0)+\beta_2}\right)\chi(\xi) = 0 \quad (5)$$

solution:

$$\chi(\xi) = c_1 \xi^{\frac{1}{2-\theta}} J_\nu(b\xi^{1/2}) + c_2 \xi^{\frac{1}{2-\theta}} Y_\nu(b\xi^{1/2})$$

$$\nu = \sqrt{\frac{1+4\beta_2}{(\theta-2)^2}}, \qquad b = \sqrt{\frac{-4\alpha_0}{(\theta-2)^2\zeta'(0)}}$$

A lemma

Lemma

Let $Z_\nu(x)$ be any solution of the Bessel equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

Then we have

$$\left\{\frac{Q_\pm(\xi)}{\xi}\right\}^{\frac{1}{4}} Z_\nu\left\{Q_\pm(\xi)\right\} = Z_\nu(\xi^{\frac{1}{2}})G_\pm(\zeta, \frac{1}{n}) + \zeta^{\frac{1}{2}}Z_{\nu+1}(\xi^{\frac{1}{2}})H_\pm(\zeta, \frac{1}{n}),$$

$$\left\{\frac{Q_\pm(\xi)}{\xi}\right\}^{\frac{1}{4}} Z_{\nu+1}\left\{Q_\pm(\xi)\right\} = Z_\nu(\xi^{\frac{1}{2}})L_\pm(\zeta, \frac{1}{n}) + \zeta^{\frac{1}{2}}Z_{\nu+1}(\xi^{\frac{1}{2}})K_\pm(\zeta, \frac{1}{n}),$$

where $G_\pm, L_\pm \sim \cos\left\{(1 - \frac{\theta t}{2}\frac{\zeta'}{\zeta})\zeta^{\frac{1}{2}}\right\}$.

what is $\zeta(t)$

$$P_n(n^\theta t) \sim \zeta^{-\frac{\nu}{2}} Z_\nu(n\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s}{n^s} + \zeta^{-\frac{\nu+1}{2}} Z_{\nu+1}(n\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s}{n^s}$$

$$\Rightarrow \quad \cos\left\{(1 - \frac{\theta t}{2}\frac{\zeta'}{\zeta})\zeta^{\frac{1}{2}}\right\} = \frac{\alpha_0 t + \beta_0}{2}$$

When $0 < \theta < 2$

$$\zeta(t) = \cos^{-1}\left(\frac{\alpha_0 t + \beta_0}{2}\right) + \alpha_0 t^{\frac{1}{\theta}} \int_0^t \frac{\phi^{-\frac{1}{\theta}}}{\sqrt{4 - (\alpha_0\phi + \beta_0)^2}} d\phi \qquad (6)$$

for $0 \geq t < t_2 - \delta$, and $\cos^{-1}$ is defined for $t < 0$ by analytic continuation.

Bessel type expansion

Theorem

$\theta \neq 2$. (1) has two linearly independent solutions:

$$P_n(N^\theta t) = \left(\frac{4\zeta}{(\alpha_0' t + 2)^2 - 4}\right)^{\frac{1}{4}} \left[\zeta^{-\frac{\nu}{2}}(t) J_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{\widetilde{A}_s(\zeta)}{N^s} + \zeta^{-\frac{\nu+1}{2}}(t) J_{\nu+1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{\widetilde{B}_s(\zeta)}{N^s}\right]$$

$$Q_n(N^\theta t) = \left(\frac{4\zeta}{(\alpha_0' t + 2)^2 - 4}\right)^{\frac{1}{4}} \left[\zeta^{-\frac{\nu}{2}} Y_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{\widetilde{A}_s(\zeta)}{N^s} + \zeta^{-\frac{\nu+1}{2}}(t) Y_{\nu+1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{\widetilde{B}_s(\zeta)}{N^s}\right].$$

Example

Vanlessen (2007): Laguerre type, $w(x) = x^{\alpha}e^{-Q(x)}$, $\alpha > -1$, $Q(x)$ a polynomial with positive leading coefficient.

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1}p_{n-1}(x)$$

$$\frac{b_{n-1}}{n^{\frac{1}{m}}} \sim \frac{1}{4} + \frac{\alpha}{2hn} + O(n^{-2}), \qquad \frac{a_n}{n^{\frac{1}{m}}} \sim \frac{1}{2} + \frac{\alpha+1}{hn} + O(n^{-2}).$$

In our notations

$$\theta = \frac{1}{m}, \quad K_n \sim 1 - \frac{\alpha}{hn}, \quad A_n = \frac{K_n}{K_{n+1}}\frac{1}{b_n}, \quad B_n = \frac{K_n}{K_{n+1}}\frac{a_n}{b_n}.$$

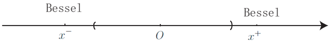
$$p_n(n^{\frac{1}{m}}t) \sim c(w(n^{\frac{1}{m}}t))^{\frac{1}{2}}\left[J_{\alpha}(n\zeta^{\frac{1}{2}}(t))A_0(\zeta) + J_{\alpha+1}(n\zeta^{\frac{1}{2}}(t))B_0(\zeta)\right]$$

$$P_{n+1}(x) + P_{n-1}(x) = (A_n x + B_n) P_n(x)$$

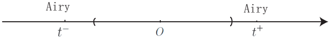
$$A_n \sim N^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{N^s}, \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{N^s}.$$

$\alpha'_1 t_{tp} + \beta'_1 = 0$, by chosing τ_0 in $N := n + \tau_0$

Case (i) $\theta = 0$: Bessel type, Wang & Wong (2005)



Case (ii) $\theta \neq 0$, $t^{\pm} \neq 0$: Airy type, Wang & Wong (2003)

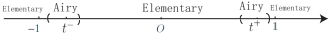


Case (ii) $\theta \neq 0$, $t^- = 0$: Bessel type

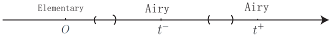
$\alpha_1 t_{tp} + \beta_1 \neq 0$, new rescaling

$$x = n^{\theta}\left(\tilde{t} + \frac{t_0}{n}\right),$$

t_0 is chosen so that $\alpha_1 \tilde{t}_{tp} + \beta_1 = 0$.
Case (iv) $\theta = 0$: Airy type, e.g. Pollaczek



Case (v) $\theta \neq 0$: Airy type, e.g. Mexiner, Hahn,



Other cases

$$B_n \sim n^{\phi} \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}, \qquad \phi \neq 0$$

involve Gamma function and Airy function, e.g. Charlier.

$$P_n(x) \sim \chi(\xi), \qquad \xi = n^{\sigma}\zeta(t),$$

σ: constant to be determined.

$$P_{n\pm 1}(x) \sim \chi(Q_{\pm}(\xi)) \sim \chi(\xi) + \cdots$$

Taylor expansion will yield the appropriate choice of σ and a ODE of $\chi(\xi)$, Airy, Bessel, etc.

Open questions

1. q-difference equations, e.g. Stieltjes-Wigert
2. higher order difference equations, multiple orthogonal polynomials (MOP)

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Thanks !