Global Asymptotics of the Meixner Polynomials

Xiang-Sheng Wang

York University, Toronto, Canada

(This is a joint work with R. Wong.)

Outline

- Introduction
  1. Meixner polynomials
  2. Deift-Zhou nonlinear steepest-descent method
  3. Global asymptotics
- Global asymptotics of the Meixner polynomials
  1. Riemann-Hilbert problem
  2. Equilibrium measure
  3. Local asymptotics
  4. Global asymptotics
- Conclusion and discussion

The Meixner polynomials

For $\beta > 0$ and $0 < c < 1$, the Meixner polynomials are explicitly given by

$$M_n(z;\beta,c) = {}_2F_1\left(\begin{matrix}-n, -z\\ \beta\end{matrix}\middle| 1 - \frac{1}{c}\right) = \sum_{k=0}^{n} \frac{(-n)_k(-z)_k}{(\beta)_k k!}\left(1 - \frac{1}{c}\right)^k,$$

where $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}^*$.

The Meixner polynomials satisfy the discrete orthogonality condition

$$\sum_{k=0}^{\infty} \frac{c^k(\beta)_k}{k!} M_m(k;\beta,c) M_n(k;\beta,c) = \frac{c^{-n} n!}{(\beta)_n(1-c)^\beta}\delta_{mn}.$$

We are interested in finding large-n behavior of $M_n(z;\beta,c)$.

Zeros

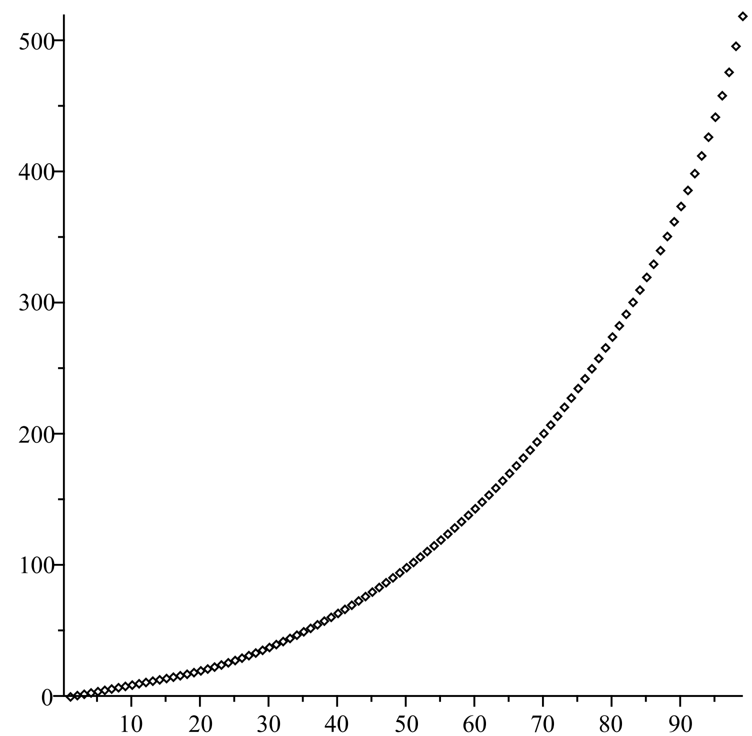


Figure 1: The zeros of $M_n(z;\beta,c)$ with $n = 100$, $\beta = 1.5$ and $c = 0.5$.

Saturated interval

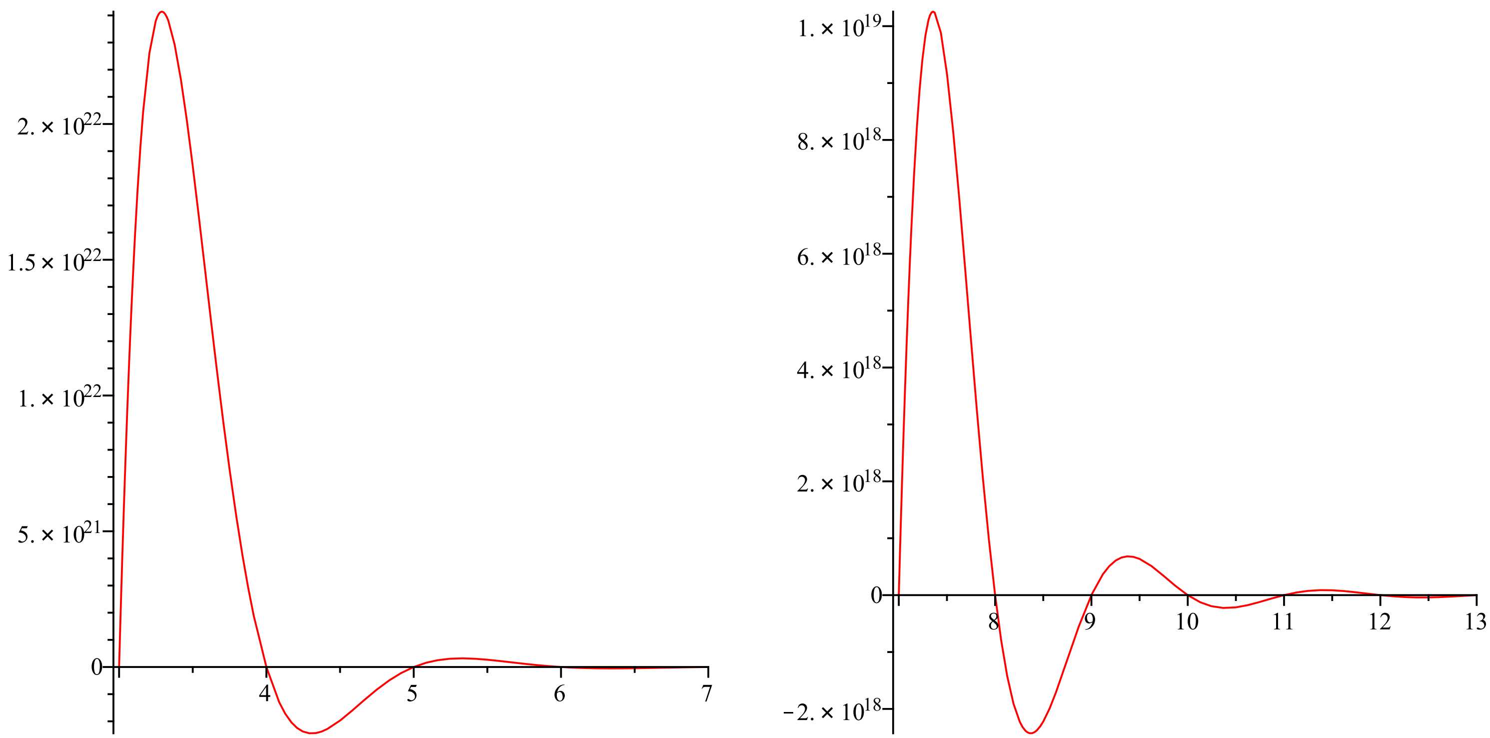


Figure 2: The first several zeros of $M_n(z;\beta,c)$ with $n = 100$, $\beta = 1.5$ and $c = 0.5$.

What have been done?

- Using probabilistic arguments, Maejima and Van Assche have given an asymptotic formula for $M_n(n\alpha;\beta,c)$ when $\alpha < 0$ and β is a positive integer. Their result is given in terms of elementary functions.

- Jin and Wong have applied the steepest-descent method for integrals to derive two infinite asymptotic expansions for $M_n(n\alpha;\beta,c)$. One holds uniformly for $0 < \varepsilon \le \alpha \le 1+\varepsilon$, and the other holds uniformly for $1-\varepsilon \le \alpha \le M < \infty$; both expansions involve the parabolic cylinder function and its derivative.

- Recently, Temme uses logarithm transformations to derive two uniform asymptotic formulas for $M_n(n\alpha;\beta,c)$ with α in $[0,\delta]$ and $[-\delta,0]$ respectively. The gamma function is used to describe asymptotic behavior of the Meixner polynomials near the origin.

What are we going to do?

- In view of Gauss's contiguous relations for hypergeometric functions, we may restrict our study to the case $1 \le \beta < 2$.

- Fixing any $0 < c < 1$ and $1 \le \beta < 2$, we intend to investigate the large-n behavior of $M_n(nz - \beta/2; \beta, c)$ for z in the whole complex plane.

- Our results are "global" in the sense that only two asymptotic formulas are needed to cover the whole complex plane.

- Our approach is based on the Deift-Zhou nonlinear steepest-descent method for oscillatory Riemann-Hilbert problems.

The Deift-Zhou nonlinear steepest-descent method

- Deift and Zhou (Ann. of Math. 1993): modified KdV equation.
- Deift et al. (CPAM 1999): orthogonal polynomials with respect to exponential weights.
- Baik et al. (Annals of Mathematics Studies 2007): orthogonal polynomials with respect to a general class of discrete weights.
- many other developments and applications $\cdots$

Local asymptotics and global asymptotics

- Local asymptotics (a and b are turning points, δ is a small positive number)
  1. negative real line: $(-\infty, -\delta]$ (Maejima and Van Assche)
  2. near the origin: $[-\delta, 0]$ and $[0, \delta]$ (Temme)
  3. saturated interval: $[\delta, a-\delta]$
  4. near left turning point: $[a-\delta, a+\delta]$
  5. oscillatory interval: $[a+\delta, b-\delta]$
  6. near right turning point: $[b-\delta, b+\delta]$
  7. exponential interval: $[b+\delta, \infty)$

- Global asymptotics (Jin and Wong): $[\delta, 1+\delta]$ and $[1-\delta, M]$.

- Global asymptotics (our improved results): $[0, 1]$ and $(-\infty, 0] \cup [1, \infty)$.

Global asymptotics via Riemann-Hilbert problem

- Jacobi polynomials: Wong and Zhang (Tran. AMS 2006)
- Krawtchouk polynomials: Dai and Wong (Chin. Ann. Math. Ser. B 2007)
- Hermite polynomials: Wong and Zhang (DCDS Ser. B 2007)
- Laguerre polynomials: Dai and Wong (Ramanujan J. 2008); Qiu and Wong (Numer. Algorithms 2008)
- Charlier polynomials: Ou and Wong (Anal. Appl. 2010)
- Discrete Chebyshev polynomials: Lin and Wong (in preparation)
- many other references $\cdots$

Riemann-Hilbert problem

- 1D $\to$ 2D (Fokas, Its and Kitaev): relate the Meixner polynomials with a 2×2 matrix-valued function which is the unique solution to an interpolation problem.

- Discrete $\to$ Continuous (Baik et al.): change the discrete interpolation problem to a continuous Riemann-Hilbert problem (RHP) whose unique solution can be expressed in terms of the solution to the basic interpolation problem.

Step 1: 1D $\to$ 2D

Define

$$P(z) := \begin{pmatrix} \pi_n(z) & \sum_{k=0}^{\infty} \frac{\pi_n(k)w(k)}{z-k} \\ \gamma_{n-1}^2 \pi_{n-1}(z) & \sum_{k=0}^{\infty} \frac{\gamma_{n-1}^2 \pi_{n-1}(k)w(k)}{z-k} \end{pmatrix},$$

where $\pi_n(z)$ is the monic Meixner polynomials. For any $k \in \mathbb{N}$, we have

$$\operatorname*{Res}_{z=k} P_{12}(z) = \pi_n(k)w(k) = P_{11}(k)w(k),$$

$$\operatorname*{Res}_{z=k} P_{22}(z) = \gamma_{n-1}^2 \pi_{n-1}(k)w(k) = P_{21}(k)w(k).$$

Thus,

$$\operatorname*{Res}_{z=k} P(z) = \lim_{z \to k} P(z) \begin{pmatrix} 0 & w(z) \\ 0 & 0 \end{pmatrix}.$$

Step 2: Discrete $\to$ Continuous (example)

Suppose

$$\operatorname*{Res}_{z=0} Q(z) = \lim_{z\to 0} Q(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Define

$$R(z) := \begin{cases} Q(z) \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix}, & \text{for any } z \in D(0,1) \setminus \{0\}; \\ Q(z), & \text{for any } z \in \mathbb{C} \setminus D(0,1). \end{cases}$$

We then have $R(z)$ analytic at $z = 0$ and

$$R_+(z) = R_-(z) \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}, \quad \text{for any } z \in \partial D(0,1).$$

Turning points and equilibrium measure

- Mhaskar-Rakhmanov-Saff (MRS) numbers (turning points)

$$a = \frac{1-\sqrt{c}}{1+\sqrt{c}}, \qquad b = \frac{1+\sqrt{c}}{1-\sqrt{c}}.$$

- Let x_i be the ith zeros of $M_n(nz - \beta/2; \beta, c)$, we have the following asymptotic zero distribution

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}(x) \rightharpoonup \rho(x) = \begin{cases} 1 & x \in [0,a]; \\ \frac{1}{\pi}\arccos\frac{x(b+a)-2}{x(b-a)} & x \in [a,b]; \\ 0 & \text{otherwise.} \end{cases}$$

Zero distribution

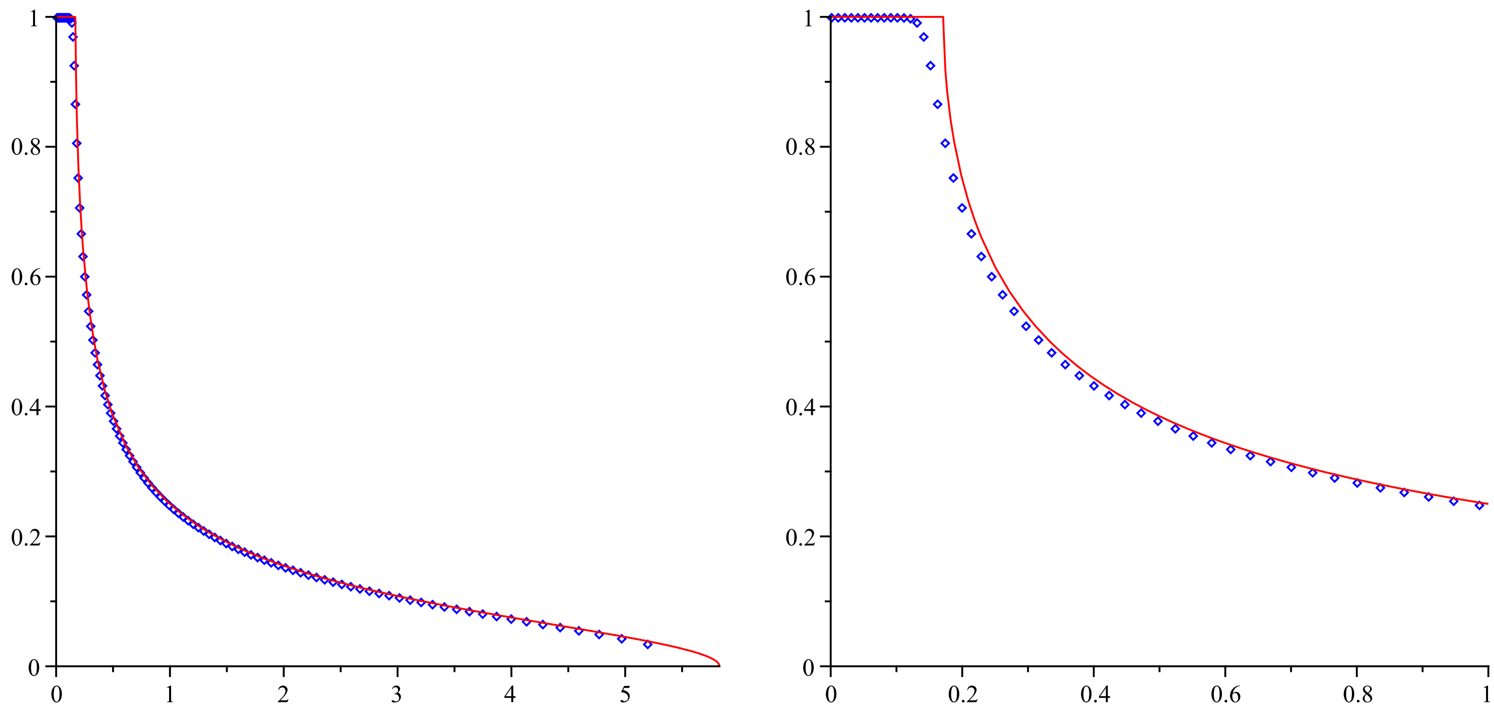


Figure 3: The zero distribution of $M_n(nz - \beta/2; \beta, c)$ with $n = 100$, $\beta = 1.5$ and $c = 0.5$. In this case the turning points are $a \approx 0.17157$ and $b \approx 5.82843$.

Local asymptotics: some local Riemann-Hilbert problems

- Local RHP near the turning points a and b: Airy parametrix (Deift et al., 1999).
- Local RHP near the interval (a, b): elementary function.
- Local RHP near the origin: gamma function.

Local RHP near the turning points a and b

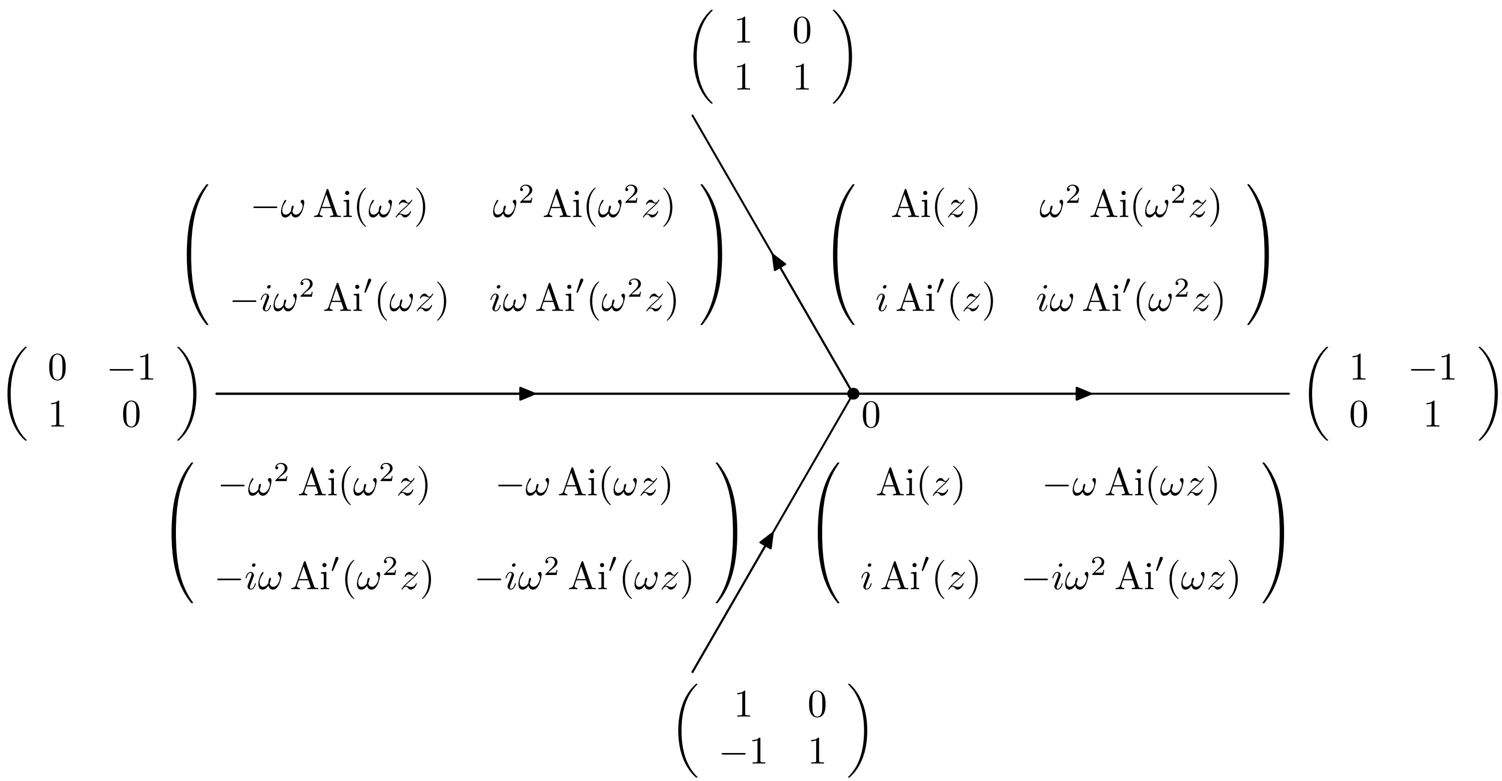


Figure 4: The Airy parametrix and its jump conditions.

Local RHP near the interval (a, b)

$$J_N(x) = \begin{cases} \begin{pmatrix} 0 & -(1-x)^{\beta-1} \\ (1-x)^{1-\beta} & 0 \end{pmatrix}, & \text{for any } x \in (a,1); \\ \\ \begin{pmatrix} 0 & -(x-1)^{\beta-1} \\ (x-1)^{1-\beta} & 0 \end{pmatrix}, & \text{for any } x \in (1,b). \end{cases}$$

$$N(z) = \begin{pmatrix} \dfrac{(z-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}} & \dfrac{-i(z-1)^{\frac{\beta-1}{2}}(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}} \\ \\ \dfrac{i(z-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{2-\beta}}{(z-a)^{1/4}(z-b)^{1/4}} & \dfrac{(z-1)^{\frac{\beta-1}{2}}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{2-\beta}}{(z-a)^{1/4}(z-b)^{1/4}} \end{pmatrix}.$$

Local RHP near the origin

(D1) $D(z)$ is analytic in $\mathbb{C} \setminus (-i\infty, i\infty)$;

(D2) $D_+(z) = D_-(z)[1 - e^{\pm 2i\pi(nz-\beta/2)}]$, for any $z \in (-i\infty, i\infty)$;

(D3) for $z \in \mathbb{C} \setminus (-i\infty, i\infty)$, $D(z) = 1 + O(|z|^{-1})$ as $z \to \infty$.

The solution is given by

$$D(z) = \begin{cases} \dfrac{e^{nz}\Gamma(nz - \beta/2 + 1)}{\sqrt{2\pi}(nz)^{nz+(1-\beta)/2}} & \operatorname{Re} z > 0; \\ \\ \dfrac{\sqrt{2\pi}(-nz)^{-nz+(\beta-1)/2}}{e^{-nz}\Gamma(-nz + \beta/2)} & \operatorname{Re} z < 0. \end{cases}$$

Local asymptotics: some notations

- The monic Meixner polynomials: $\pi_n(z) := (\beta)_n(1-\frac{1}{c})^{-n}M_n(z;\beta,c)$.
- Potential function $v(z) := -z\log c$ and Lagrange constant $l := 2\log\frac{b-a}{4} - 2$.
- For $z \in \mathbb{C} \setminus (-\infty, b]$,

$$\phi(z) := z\log\frac{\sqrt{bz-1}+\sqrt{az-1}}{\sqrt{bz-1}-\sqrt{az-1}} - \log\frac{\sqrt{z-a}+\sqrt{z-b}}{\sqrt{z-a}-\sqrt{z-b}}.$$

- For $z \in \mathbb{C} \setminus (-\infty, 0] \cup [a, \infty)$,

$$\widetilde{\phi}(z) := z\log\frac{\sqrt{1-az}+\sqrt{1-bz}}{\sqrt{1-az}-\sqrt{1-bz}} - \log\frac{\sqrt{b-z}+\sqrt{a-z}}{\sqrt{b-z}-\sqrt{a-z}}.$$

Local asymptotics: regions of approximation

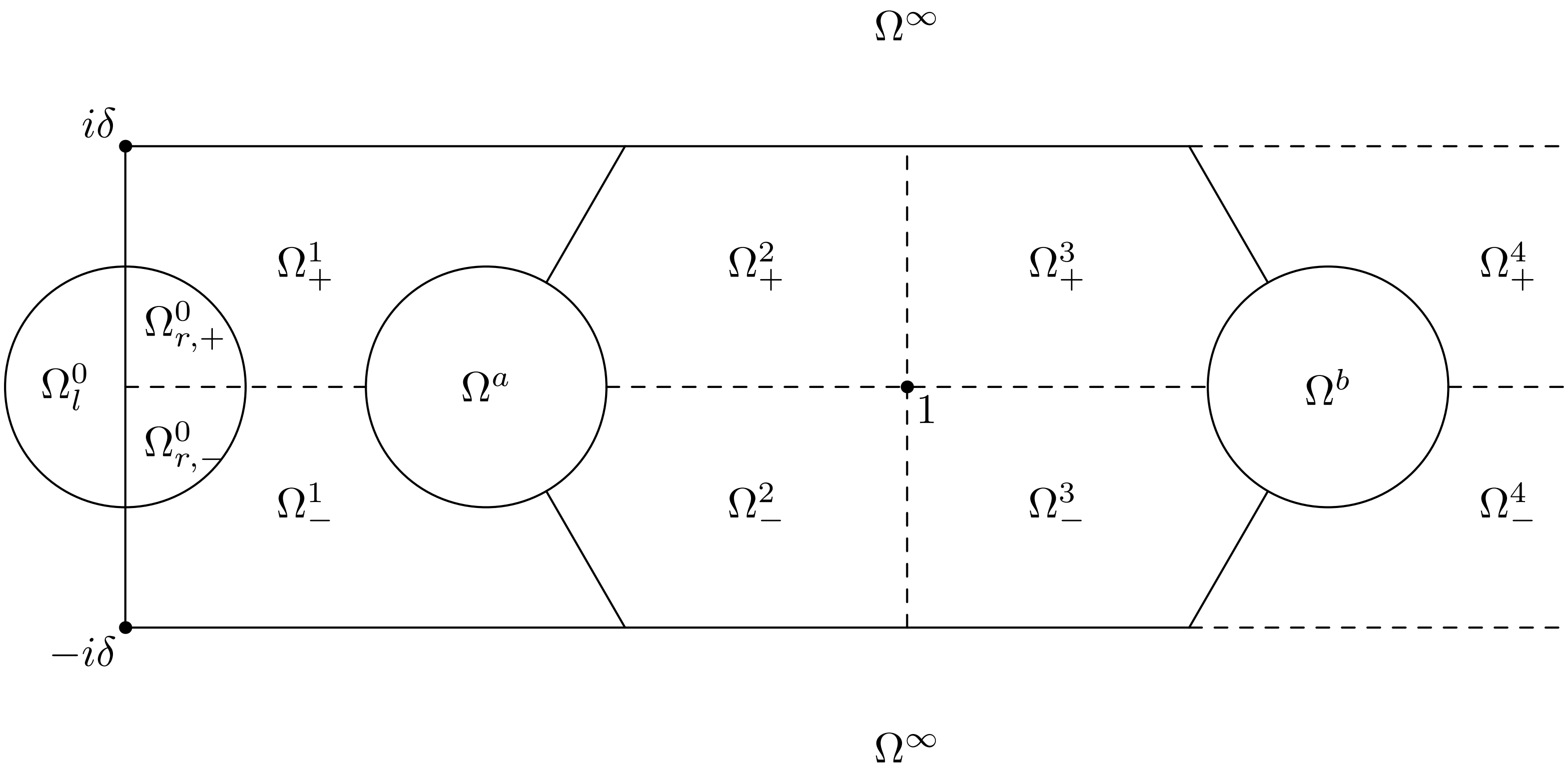


Figure 5: Local asymptotic regions.

Local asymptotics: saturated region

For $z \in \Omega_{\pm}^{1}$, we have

$$\pi_n(nz - \beta/2) \quad \sim \quad -2\sin(n\pi z - \beta\pi/2)(-n)^n e^{nv(z)/2 + nl/2 - n\widetilde{\phi}(z)} \times \frac{z^{(1-\beta)/2}\left(\frac{\sqrt{b-z}+\sqrt{a-z}}{2}\right)^{\beta}}{(a-z)^{1/4}(b-z)^{1/4}}.$$

Local asymptotics: oscillatory region

Let $z = \frac{b-a}{2}\cos u + \frac{b+a}{2} = -\frac{b-a}{2}\cos \widetilde{u} + \frac{b+a}{2}$. We have

$$\begin{aligned}\pi_n(nz-\beta/2) \quad &\sim \quad 2\cos[n\pi z - \beta\pi/2 + \pi/4 + \beta\widetilde{u}/2 \mp in\widetilde{\phi}(z)](-n)^n e^{nv(z)/2+nl/2} \\ &\times \frac{z^{(1-\beta)/2}\left(\frac{b-a}{4}\right)^{\beta/2}}{(z-a)^{1/4}(b-z)^{1/4}}\end{aligned}$$

for $z \in \Omega_\pm^2$, and

$$\begin{aligned}\pi_n(nz-\beta/2) \quad &\sim \quad 2\cos[\pi/4 - \beta u/2 \mp in\phi(z)]n^n e^{nv(z)/2+nl/2} \\ &\times \frac{z^{(1-\beta)/2}\left(\frac{b-a}{4}\right)^{\beta/2}}{(z-a)^{1/4}(b-z)^{1/4}}\end{aligned}$$

for $z \in \Omega_\pm^3$.

Local asymptotics: exponential region

For $z \in \Omega^4 \cup \Omega^\infty$, we have

$$\pi_n(nz - \beta/2) \sim n^n e^{nv(z)/2 + nl/2 - n\phi(z)} \times \frac{z^{(1-\beta)/2}\left(\frac{\sqrt{z-a}+\sqrt{z-b}}{2}\right)^\beta}{(z-a)^{1/4}(z-b)^{1/4}}.$$

Local asymptotics: near the origin

For $z \in \Omega_l^0$, we have

$$\pi_n(nz - \beta/2) \sim D(z) n^n e^{nv(z)/2 + nl/2 - n\phi(z)} \times \frac{(-z)^{(1-\beta)/2} (\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(b-z)^{1/4}(a-z)^{1/4}}.$$

For $z \in \Omega_{r,\pm}^0$, we have

$$\pi_n(nz - \beta/2) \sim -2\sin(n\pi z - \beta\pi/2) D(z)(-n)^n e^{nv(z)/2 + nl/2 - n\widetilde{\phi}(z)} \times \frac{z^{(1-\beta)/2} (\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(a-z)^{1/4}(b-z)^{1/4}}.$$

Local asymptotics: near left turning point

Let $\widetilde{F}(z) := \left[-\frac{3}{2}n\widetilde{\phi}(z)\right]^{2/3}$, we have for $z \in \Omega^a$,

$$\begin{aligned}
\pi_n(nz-\beta/2) \quad \sim \quad & (-n)^n\sqrt{\pi}e^{nv(z)/2+nl/2} \\
& \times\Bigg\{[\cos(n\pi z-\beta\pi/2)\,\mathrm{Ai}(\widetilde{F}(z)) - \sin(n\pi z-\beta\pi/2)\,\mathrm{Bi}(\widetilde{F}(z))] \\
& \quad\times\frac{(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta + (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\widetilde{F}(z)^{-1/4}} \\
& \quad+[\cos(n\pi z-\beta\pi/2)\,\mathrm{Ai}'(\widetilde{F}(z)) - \sin(n\pi z-\beta\pi/2)\,\mathrm{Bi}'(\widetilde{F}(z))] \\
& \quad\times\frac{(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta - (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\widetilde{F}(z)^{1/4}}\Bigg\}.
\end{aligned}$$

Local asymptotics: near right turning point

Let $F(z) := \left[\frac{3}{2}n\phi(z)\right]^{2/3}$, we have for $z \in \Omega^b$,

$$
\begin{aligned}
\pi_n(nz-\beta/2) \quad &\sim \quad n^n\sqrt{\pi}e^{nv(z)/2+nl/2} \\
&\times\left\{\frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^\beta+(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{-1/4}}\,\mathrm{Ai}(F(z))\right. \\
&\left.-\frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^\beta-(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4}}\,\mathrm{Ai}'(F(z))\right\}.
\end{aligned}
$$

Local asymptotics: regions of approximation

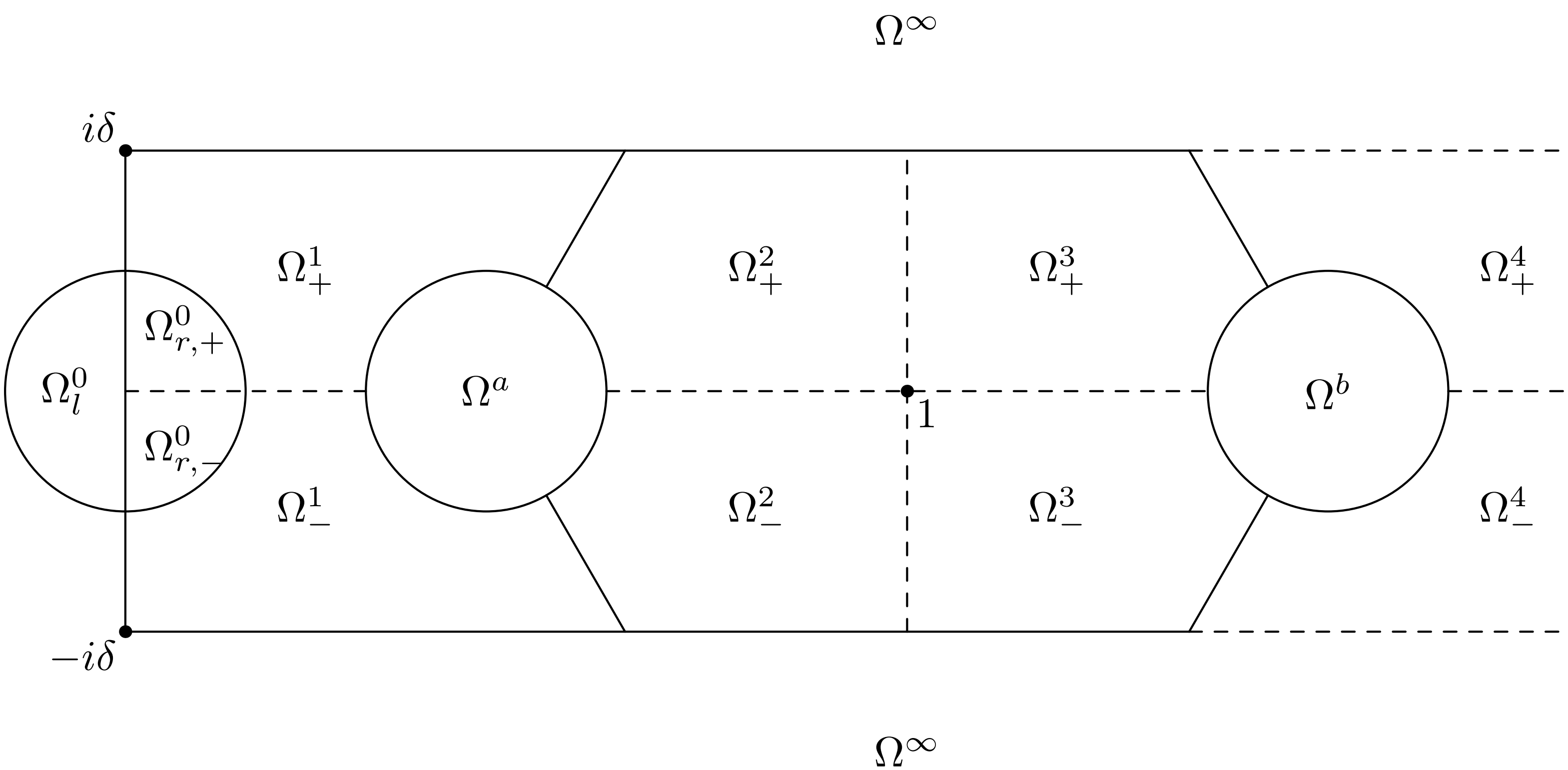


Figure 6: Local asymptotic regions.

Global asymptotics: regions of approximation

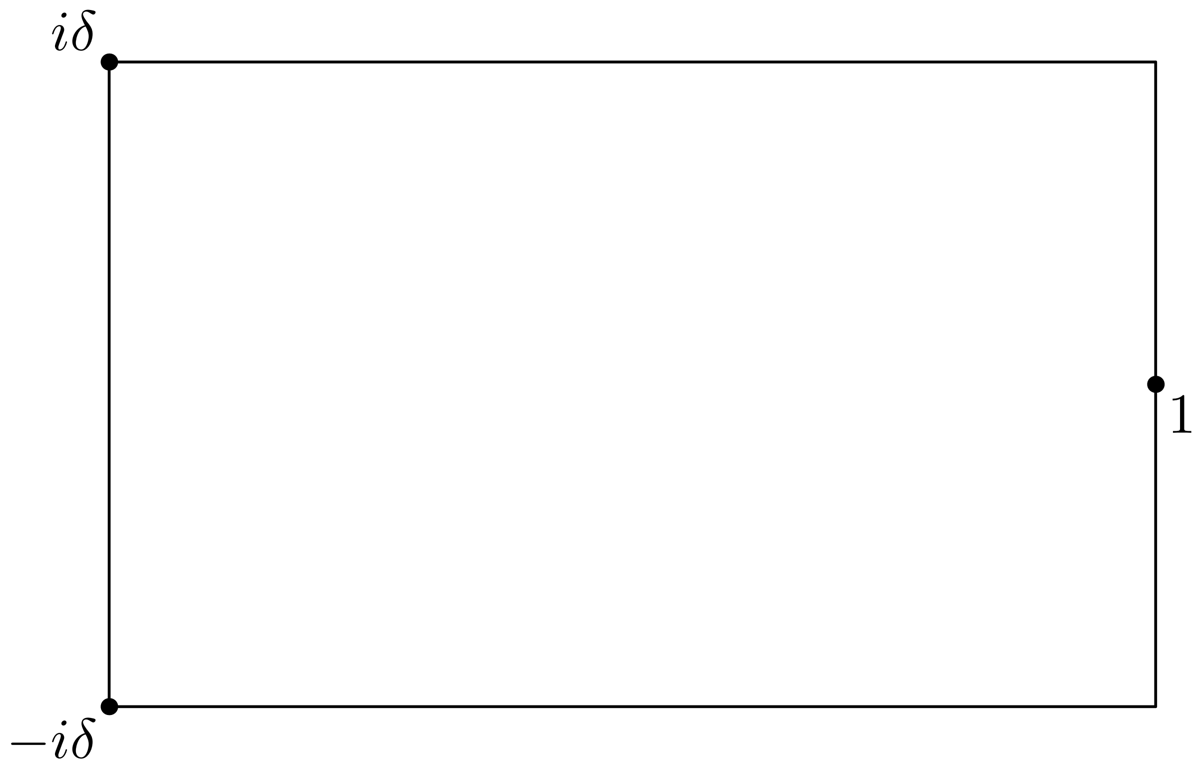


Figure 7: Global asymptotic regions.

Global asymptotics: outside the rectangle

For $\operatorname{Re} z \notin [0, 1]$ or $\operatorname{Im} z \notin [-\delta, \delta]$,

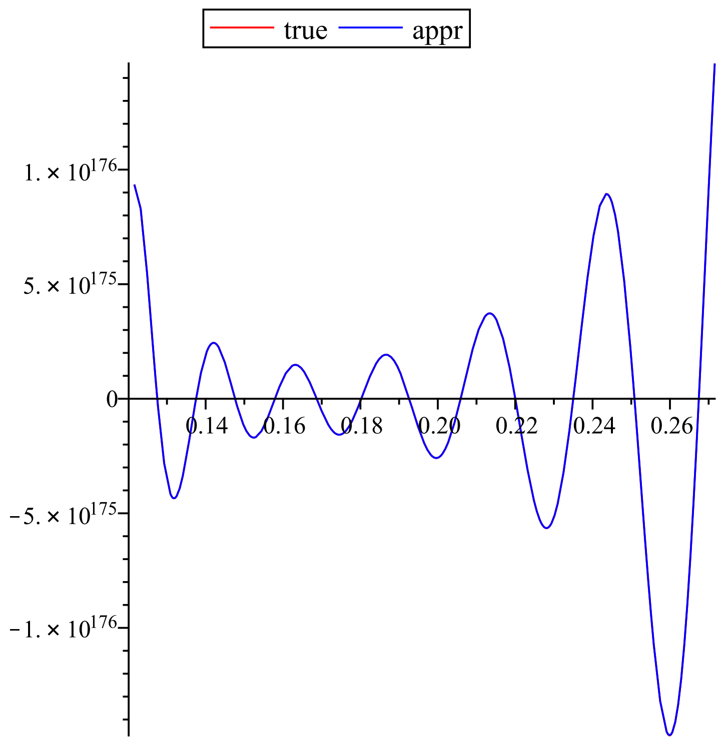
$$\begin{aligned}
\pi_n(nz - \beta/2) \quad &\sim \quad n^n \sqrt{\pi} D(z) e^{nv(z)/2 + nl/2} \\
&\times \Bigg\{ \frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^\beta + (\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{-1/4}} \operatorname{Ai}(F(z)) \\
&\quad - \frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^\beta - (\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4}} \operatorname{Ai}'(F(z)) \Bigg\}.
\end{aligned}$$

Global asymptotics: inside the rectangle

For $\operatorname{Re} z \in (0,1)$ and $\operatorname{Im} z \in (-\delta,\delta)$,

$$\begin{aligned}
\pi_n(nz-\beta/2) \quad \sim \quad & (-n)^n \sqrt{\pi} D(z) e^{nv(z)/2+nl/2} \\
& \times \Big\{ [\cos(n\pi z - \beta\pi/2)\,\mathrm{Ai}(\widetilde{F}(z)) - \sin(n\pi z - \beta\pi/2)\,\mathrm{Bi}(\widetilde{F}(z))] \\
& \qquad \times \frac{(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta + (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\widetilde{F}(z)^{-1/4}} \\
& \quad + [\cos(n\pi z - \beta\pi/2)\,\mathrm{Ai}'(\widetilde{F}(z)) - \sin(n\pi z - \beta\pi/2)\,\mathrm{Bi}'(\widetilde{F}(z))] \\
& \qquad \times \frac{(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta - (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\widetilde{F}(z)^{1/4}} \Big\}.
\end{aligned}$$

Numerical computation



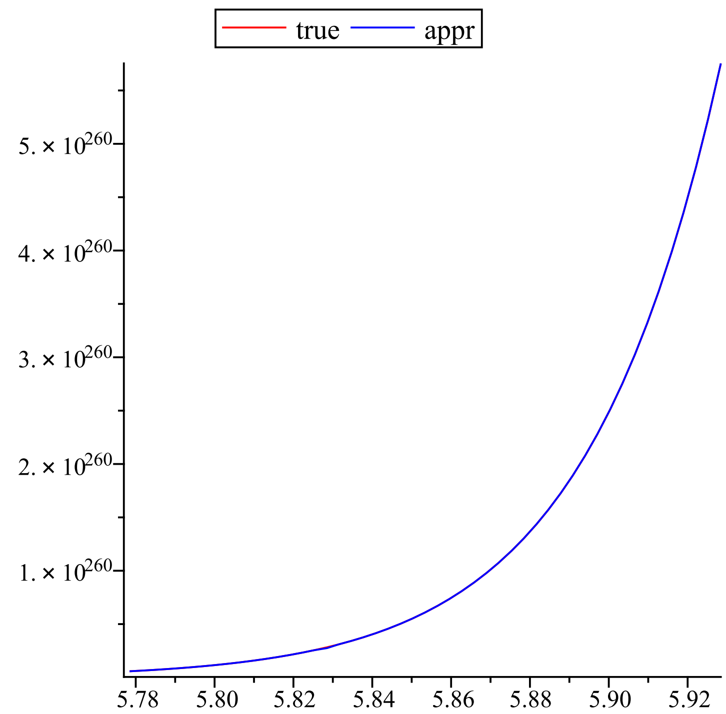


Figure 8: The true figure and approximate figure of $\pi_n(nz - \beta/2)$ for $n = 100$, $\beta = 1.5$ and $c = 0.5$. Here the turning points are $a \approx 0.17157$ and $b \approx 5.82843$.

Numerical computation

	True value	Appr. value (local)	Appr. value (global)
$z = -1$	1.99529×10^{233}	1.99473×10^{233}	1.99501×10^{233}
$z = -0.001$	8.36624×10^{187}	8.35137×10^{187}	8.35263×10^{187}
$z = 0.001$	3.07930×10^{187}	3.07272×10^{187}	3.07602×10^{187}
$z = 0.05$	-2.51701×10^{180}	-2.51507×10^{180}	-2.51523×10^{180}
$z = 0.171$	-9.12697×10^{174}	-9.12530×10^{174}	-9.11951×10^{174}
$z = 0.172$	-1.22035×10^{175}	-1.22003×10^{175}	-1.21926×10^{175}
$z = 2$	-4.71541×10^{201}	-4.70772×10^{201}	-4.71179×10^{201}
$z = 5.828$	2.78146×10^{259}	2.78231×10^{259}	2.78225×10^{259}
$z = 5.829$	2.86933×10^{259}	2.87018×10^{259}	2.87046×10^{259}
$z = 100$	2.16586×10^{399}	2.16586×10^{399}	2.16586×10^{399}

Table 1: The true values and approximate values of $\pi_n(nz - \beta/2)$ for $n = 100$, $\beta = 1.5$ and $c = 0.5$. Here the turning points are $a \approx 0.17157$ and $b \approx 5.82843$.

Future work

- Global asymptotics for a general class of discrete weight.
- The critical case when the turning point and the end point coalesce with each other.

Some pioneer works

- Local asymptotics for a general class of discrete weight with finite nodes (Baik et.al., 2007)
- Global asymptotics of the Krawtchouck polynomials (Dai-Wong, 2007)
- Global asymptotics for a general class of discrete weight with infinite nodes (Ou-Wong, 2010)
- Global asymptotics via recurrence relations (Wang-Wong, 2002; Li-Wong)
- Global asymptotics of discrete Chebyshev polynomials (Pan-Wong; Lin-Wong)
- $\cdots$

Thank you!