Asymptotic Approximations of Orthogonal Polynomials by Using Three-Term Recurrence Relations

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Joint work with

H.Li (1992)
Z. Wang (2002-2005)
X.-S. Wang (2011)

Leonard Maximon (NIST; 27/5/2010):
"Suppose that I have a second order (three-term) homogeneous difference equation, with coefficients that are either linear or quadratic functions of the index, n. The equation for the Legendre polynomials is a good example. I assume further that I have specified the value of the solution for $n = 0$ and $n = 1$, so that in principal I can determine the solution for all n. The question is then, can one, knowing only the homogeneous equation and the first two values, determine (write down) the analytic expression for the asymptotic expansion (the first term in that expansion will do) for n very large."

My answer (31/5/2010):
"A simple answer to your question is that we don't know how to write down even the 1st term in the asymptotic expansion of the solution to a difference equation with given initial data. Even in the simple case of Legendre polynomials, it is not known how to tackle this problem."

J. Wimp and D. Zeilberger, *Resurrecting the Asymptotics of linear Recurrences*, J. Math. Anal. Appl. **111**(1985), 162-176
"Once on the forefront of mathematical research in America, the asymptotics of the solutions of linear recurrence equations is now almost forgotten, especially by the people who need it most, namely combinatorists and computer scientists."

J. Wimp, *Book Review*, Math. Comp., **56** January, 1991, 388-396
"There are still vital matter to be resolved in asymptotic analysis. At least one widely quoted theory, the asymptotic theory of irregular difference equations expounded by G. D. Birkhoff and W. R. Trjitzinsky [5, 6] in the early 1930's, is vast in scope; but there is now substantial doubt that theory is correct in all its particulars. The computations involved in the algebraic theory alone (that is, the theory that purports to show there are a sufficient number of solutions which formally satisfy the difference equation in question) are truly mindboggling."

A. Iserles, SIAM Rev. **42**(2002), pp.739-768
"An important observation, which I associate with Nick Trefethen, is that every hard mathematical construct becomes, subject to discretization, an even harder mathematical construct. Thus, for example, difference equations might be a handy and practical means to compute differential equations, but they are considerably more complicated to analyze."

Legendre Polynomials $P_n(x)$

1. Recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

2. Differential equation

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

$u_n(\theta) \triangleq (\sin\theta)^{\frac{1}{2}} P_n(\cos\theta)$

$$u_n''(\theta) + \left[(n+\frac{1}{2})^2 + \frac{1}{4\sin^2\theta}\right] u_n(\theta) = 0$$

3. Integral representation

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \{x + (x^2-1)^{\frac{1}{2}}\cos\theta\}^n d\theta, \qquad x > 1.$$

Asymptotic Formulas (as $n \to \infty$)

$$P_n(x) \quad \sim \quad \frac{1}{\sqrt{2n\pi}} \frac{\left\{x + (x^2-1)^{\frac{1}{2}}\right\}^{n+\frac{1}{2}}}{(x^2-1)^{\frac{1}{4}}}, \qquad x > 1;$$

$$P_n(\cos\theta) \quad = \quad \sqrt{\frac{2}{\pi n \sin\theta}} \cos\left\{\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right\} + O(n^{-3/2}), \qquad 0 < \theta < \pi$$

Question: Are there derivations of these asymptotic formulas, based on the recurrence relation for $P_n(x)$?

Difference equations:

$$y(n+2) + a(n)y(n+1) + b(n)y(n) = 0$$

$$a(n) = \sum_{s=0}^{\infty} \frac{a_s}{n^s}, \qquad B(n) = \sum_{s=0}^{\infty} \frac{b_s}{n^s}$$

$$\rho^2 + a_0\rho + b_0 = 0 \qquad \text{characteristic equation}$$

$$\rho_1, \rho_2 = \frac{-a_0 \pm \sqrt{a_0^2 - 4b_0}}{2}$$

Case (i) $\rho_1 \neq \rho_2$ (Birkhoff, 1911)

$$y_j(n) \sim \rho_j^n n^{\alpha_j} \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^s}, \qquad j = 1, 2,$$

$$\alpha_j = -\frac{a_1\rho_j + b_1}{2\rho_j^2 + \rho a_0} = \frac{a_1\rho_j + b_1}{a_0\rho_j + 2b_0}$$

Case (ii) $\rho_1 = \rho_2 = -\frac{1}{2}a_0$ but $2b_1 \neq a_0a_1$ (Adam, 1928)

$$y_\pm(n) \sim \rho^n e^{\pm\gamma\sqrt{n}} n^\alpha \sum_{s=0}^{\infty} (\pm 1)^s \frac{c_s}{n^{s/2}}$$

$$\alpha = \frac{1}{4} + \frac{b_1}{2b_0}, \qquad \gamma = 2\sqrt{\frac{a_0a_1 - 2b_1}{2b_0}}.$$

Exceptional case: $\rho_1 = \rho_2 = \rho = -\frac{1}{2}a_0, \qquad 2b_1 = a_0 a_1$

indicial polynomial

$$q(\alpha) = \alpha(\alpha - 1)\rho^2 + (a_1\alpha + a_2)\rho + b_2$$

$$q(\alpha_i) = 0, \qquad i = 1, 2.$$

Subcase (i) $\alpha_2 - \alpha_1 > 0$ and $\neq 1, 2, \cdots$

$$y_j(n) = \rho^n n^{\alpha_j} \sum_{s=0}^{\infty} \frac{a_{s,j}}{n^s}, \qquad a_{0,j} = 1 \qquad j = 1, 2$$

Subcase (ii) $\alpha_2 - \alpha_1 = 0, 1, 2, \cdots.$

$$y_1(n) = \rho^n n^{\alpha_1} \sum_{s=0}^{\infty} \frac{a_{s,1}}{n^s}$$

$$y_2(n) = \rho^n n^{\alpha_2} \sum_{s=0}^{\infty}{}' \frac{b_s}{n^s} + C y_1(n) \ln\ n,$$

where the prime on $\sum$ denotes that the term for $s = \alpha_2 - \alpha_1$ is absent, and

$C = 1$ when $\alpha_2 - \alpha_1 = 0$; $b_0 = 1$ when $\alpha_2 - \alpha_1 = 1, 2, \cdots$

More General Equations

$$y(n+2) + n^p a(n) y(n+1) + n^q b(n) y(n) = 0.$$

p, q are integers.

REFERENCE

C. R. Adams, *On the irregular cases of linear ordinary difference equations*, Trans. A.M.S., **30** (1928), pp. 507-541.

C. D. Birkhoff, *General Theory of linear difference equations*, Trans. A.M.S., **12** (1911), pp. 243-284.

C. D. Birkhoff, *Formal theory of irregular linear difference equations*, Acta Math., **54** (1930), pp. 250-246.

C. D. Birkhoff and W. J. Trjitzinsky, *Analytic theory of singular difference equations*, Acta Math., **60** (1932), pp. 1-89.

__________and H. Li, *Asymptotic expansion for second-order linear difference equations*, J. Comput. Appl. Math., **41** (1992), pp. 65-94.

__________and H. Li, *Asymptotic expansion for second-order linear difference equations II*, Stud. Appl. Math., **87** (1992), pp. 289-324.

R. Wong:" Their (B&T) papers have been considered for too complicated and even impenetrable."

Frank Olver: "the work of B&T set back all research into the asymptotic solution of difference equations for most of the 20th Century."

Legendre Polynomials (Cont'd.)

$$P_{n+2}(x) - \frac{2n+3}{n+2} x P_{n+1}(x) + \frac{n+1}{n+2} P_n(x) = 0$$

$$a(n) \sim -2x(1 - \frac{1}{2n} + \frac{1}{n^2} + \cdots)$$

$$b(n) \sim 1 - \frac{1}{n} + \frac{2}{n^2} + \cdots$$

$$\rho_1, \rho_2 = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\alpha_1 = \alpha_2 = -\frac{1}{2}$$

Case(i) $x \notin$ [-1,1]:

$$y_1(n;x) \sim (x + \sqrt{x^2-1})^n n^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_{s,1}}{n^s},$$

$$y_2(n;x) \sim (x - \sqrt{x^2-1})^n n^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_{s,2}}{n^s}.$$

Case(ii) $x \in (-1,1): \quad x = \cos\theta, \quad 0 < \theta < \pi.$

$$y_{\pm}(n;x) \sim (\cos\theta \pm i\sin\theta)^n n^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_{s,\pm}}{n^s}$$

$$y_1(n;x) \sim (\cos n\theta) n^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_{s,1}}{n^s},$$

$$y_2(n;x) \sim (\sin n\theta) n^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{c_{s,2}}{n^s}.$$

$$P_n(x) = C(x) y_1(n;x) + D(x) y_2(n;x)$$

How do you determine the coefficients $C(x)$ and $D(x)$?

An alternative derivation:

monic (Legendre) polynomials

$$\pi_n(x) := \frac{2^n n!}{(n+1)_n} P_n(x)$$

$$\pi_{n+1}(x) = x\pi_n(x) - \frac{n^2}{4n^2-1}\pi_{n-1}(x), \qquad n \geq 1$$

$$\pi_0(x) = 1, \;\; \pi_1(x) = x$$

Set

$$\pi_n(x) := \prod_{k=1}^{n} w_k(x)$$

$$w_{k+1}(x) = x - \frac{k^2}{4k^2-1}\frac{1}{w_k(x)}, \qquad k \geq 1,$$

$$w_1(x) = x.$$

Outside the interval $[-1, 1]$, we have

$$w_k(x) \sim \frac{x + \sqrt{x^2 - 1}}{2} \qquad \text{as } k \to \infty.$$

Put

$$w(x) := \frac{x + \sqrt{x^2 - 1}}{2}$$

and

$$u_k(x) := \frac{w_k(x)}{w(x)}.$$

$$u_{k+1}(x) = \frac{x}{w(x)} - \frac{k^2}{4k^2 - 1} \, \frac{1}{w^2(x) u_k(x)}, \qquad k \geq 1,$$

$$u_1(x) = \frac{x}{w(x)}.$$

Make the change of variable

$$t = t(x) := (x - \sqrt{x^2 - 1})^2.$$

Then

$$w^2(x) = \frac{1}{4t}, \qquad \frac{x}{w(x)} = 1 + t.$$

$$u_{k+1}(x) = 1 + t - \frac{4k^2 t}{4k^2 - 1}\,\frac{1}{u_k(x)}, \qquad k \geq 1,$$

$$u_1(x) = 1 + t.$$

Define $Q_0(t) := 1$ and

$$Q_n(t) := \prod_{k=1}^{n} u_k(x), \qquad n \geq 1.$$

We have

$$Q_1(t) = 1 + t$$

$$Q_{n+1}(t) = (1+t)Q_n(t) - \frac{4n^2 t}{4n^2 - 1} Q_{n-1}(t).$$

$$Q_n(t) = \sum_{j=0}^{n} \frac{(\frac{1}{2})_j (n-j+1)_j}{j!(n-j+\frac{1}{2})_j} t^j$$

$$Q_n(t) \to (1-t)^{-1/2} \qquad \text{as } n \to \infty.$$

$$\pi_n(x) = w(x)^n Q_n(t) \sim w(x)^n (1-t)^{-1/2}$$

For $x \notin [-1, 1]$,

$$\pi_n(x) \sim \left(\frac{x + \sqrt{x^2 - 1}}{2} \right)^n \left(\frac{x + \sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}} \right)^{1/2}, \qquad n \to \infty.$$

Inside the interval $(-1,1)$, we have already shown

$$P_n(x) \sim [C(x)\cos n\theta + D(x)\sin n\theta]\frac{1}{\sqrt{n}}, \qquad x = \cos\theta.$$

$$\pi_n(x) = \frac{2^n n!}{(n+1)_n} P_n(x) \sim \frac{\sqrt{\pi n}}{2^n} P_n(x) \sim \frac{\sqrt{\pi}}{2^n}(C(x)\cos n\theta + D(x)\sin n\theta) \tag{1}$$

This, in fact, holds for x in a neighborhood of $[-\delta,\delta], 0<\delta<1$ in the complex plane. For $x \in \mathbb{C}\backslash[-1,1]$,

$$\pi_n(x) \sim \left(\frac{x+\sqrt{x^2-1}}{2}\right)^n \left(\frac{x+\sqrt{x^2-1}}{2\sqrt{x^2-1}}\right)^{1/2}. \tag{2}$$

Note that $\theta = \arccos x$ and $0 < \text{Re}\ \theta < \pi$ implies $\text{Im}\ \theta < 0$ for $\text{Im}\ x > 0$.

Equation (1) gives

$$\pi_n(x) \sim \frac{\sqrt{\pi}}{2^n}\left(\frac{C(x)}{2} + \frac{D(x)}{2i}\right) e^{in\theta}.$$

Equation (2) gives

$$\pi_n(x) \sim \frac{1}{2} e^{in\theta} \left[\frac{e^{i(\theta-\pi/2)}}{2\sin\theta}\right]^{1/2}.$$

Therefore

$$\sqrt{\pi}\left(\frac{C(x)}{2} + \frac{D(x)}{2i}\right) = \frac{e^{i(\theta/2-\pi/4)}}{(2\sin\theta)^{1/2}}$$

Similarly, by matching (1) and (2) in the region Im $x < 0$, we obtain

$$\sqrt{\pi}\left(\frac{C(x)}{2} - \frac{D(x)}{2i}\right) = \frac{e^{-i(\theta/2-\pi/4)}}{(2\sin\theta)^{1/2}}.$$

Solving these two equations, we have

$$\sqrt{\pi}C(x) = \left(\frac{1+\sin\theta}{\sin\theta}\right)^{1/2} \qquad , \qquad \sqrt{\pi}D(x) = \left(\frac{1-\sin\theta}{\sin\theta}\right)^{1/2}.$$

$$\pi_n(x) \sim \frac{1}{2^n}\left[\cos n\theta\left(\frac{1+\sin\theta}{\sin\theta}\right)^{1/2} + \sin n\theta\left(\frac{1-\sin\theta}{\sin\theta}\right)^{1/2}\right]$$

$$\theta = \arccos x \qquad \text{with} \quad 0 < \operatorname{Re}\theta < \pi.$$

Hermite polynomials

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$$

$$\pi_n(x) = \frac{1}{2^n} H_n(x)$$

$$\pi_{n+1}(x) = x\pi_n(x) - \frac{n}{2}\pi_{n-1}(x), \qquad n \geq 1$$

$$\pi_0(x) = 1, \qquad \pi_1(x) = x.$$

$$\pi_n(x) = \prod_{k=1}^{n} \omega_k(x)$$

$$\omega_1(x) = x, \qquad \omega_{k+1}(x) = x - \frac{k}{2\omega_k(x)}$$

$x = x_n := \sqrt{2n}\, y$ with $y \in \mathbb{C}\backslash[-1,1]$,

$$\omega_k(x_n) = \frac{x_n + \sqrt{x_n - 2k}}{2}\left[1 + \frac{1}{2(x_n^2 - k)} + O\left(n^{-2}\right)\right]$$

$$\uparrow$$

uniform in $k = 1, \cdots, n$

$$\pi_n(\sqrt{2n}y) \quad \sim \quad \left(\frac{n}{2e}\right)^{n/2} \exp\left\{n\left[y^2 - y\sqrt{y^2-1} + \log\left(y + \sqrt{y^2-1}\right)\right]\right\}$$
$$\times \left(\frac{y + \sqrt{y^2-1}}{2\sqrt{y^2-1}}\right)^{1/2}, \qquad y \in C\backslash[-1,1].$$

$$p_n(x) := \frac{1}{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)} \pi_n(x)$$

$$\frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n + 1\right)} y p_n(\sqrt{2n}\, y) = p_{n+1}(\sqrt{2n} y) + p_{n-1}(\sqrt{2n} y)$$

Assume

$$p_n(\sqrt{2n} y) \sim n^{\alpha} [\rho(y)]^n \left\{ f(y) \cos[n\varphi(y)] + g(y) \sin[n\varphi(y)] \right\}.$$

From the 3-term recurrence relation, one finds

$$\rho(y) = e^{cy^2} \qquad \text{for some } c \in \mathbb{C}, \qquad \varphi(y) = \arccos y - y\sqrt{y^2 - 1}$$

$$\theta := \arccos y$$

$$f(y) = \frac{y^{2\alpha}}{(1-y^2)^{1/4}} \left(C_1 \cos\frac{\theta}{2} + C_2 \sin\frac{\theta}{2} \right)$$

$$g(y) = \frac{y^{2\alpha}}{(1-y^2)^{1/4}} \left(-C_1 \sin\frac{\theta}{2} + C_2 \cos\frac{\theta}{2} \right)$$

(I)

$$p_n(\sqrt{2n}y) \sim n^{\alpha} e^{ncy^2} y^{2\alpha}(1-y^2)^{-1/4} \left[C_1 \cos\left(n\varphi + \frac{\theta}{2}\right) + C_2 \sin\left(n\varphi + \frac{\theta}{2}\right) \right],$$

y in a small neighborhood of $[-1+\delta, 1-\delta]$, $\delta > 0$.

(II)

$$p_n(\sqrt{2n}y) \sim \frac{1}{\sqrt{2\pi}} \exp\left\{ n \left[y^2 - y\sqrt{y^2-1} + \log\left(y + \sqrt{y^2-1}\right) \right] \right\} \times \left(\frac{y+\sqrt{y^2-1}}{2\sqrt{y^2-1}} \right)^{1/2}, \qquad y \in \mathbb{C}\backslash[-1,1].$$

$\text{Im } y > 0 \Rightarrow \quad \alpha = 0,\ c = 1$ and

$$\frac{C_1}{2} + \frac{C_2}{2i} = \frac{e^{-i\pi/4}}{2\sqrt{\pi}}.$$

$\text{Im } y < 0 \Rightarrow \quad \alpha = 0,\ c = 1$ and

$$\frac{C_1}{2} - \frac{C_2}{2i} = \frac{e^{i\pi/4}}{2\sqrt{\pi}}.$$

$$\therefore \qquad C_1 = C_2 = \frac{1}{\sqrt{2\pi}}$$

$$\boxed{\alpha = 0, \quad c = 1, \quad C_1 = C_2 = \frac{1}{\sqrt{2\pi}}}$$

Uniform Asymptotic Expansions

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

$$P_{n+1}(x) - (A_n x + B_n)P_n(x) + P_{n-1}(x) = 0$$

Questions:

1. What is a turning point for a second-order linear difference equation?
2. How does Airy's function arise from a 3-term recurrence relation, when the function itself does not satisfy any differnece equation.
3. How the function ζ in $\mathrm{Ai}(\lambda^{2/3}\zeta)$ is obtained, when there is no corresponding transformation such as Langer's transformation for differnetial equations or the cubic transformation for integrals.

$$P_{n+1}(x) - (A_n x + B_n) P_n(x) + P_{n-1}(x) = 0$$

$$A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}$$

$$\theta \in \mathbb{R}, \qquad \alpha_0 \neq 0.$$

$$\nu := n + \tau_0$$

$\uparrow$

to be determined

$$A_n \sim \nu^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{\nu^s}, \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{\nu^s}$$

$$x := \nu^{\theta} t$$

Try

$$P_n = \lambda^n,$$

and let $n \to \infty$.

$$\lambda^2 - (\alpha'_0 t + \beta'_0)\lambda + 1 = 0 \qquad \text{characteristic equation}$$

Roots:

$$\lambda_\pm = \frac{1}{2}\left[(\alpha_0' t + \beta_0') \pm \sqrt{(\alpha_0' t + \beta_0')^2 - 4}\right]$$

Transition points:

$$\alpha_0' t_\pm + \beta_0' = \pm 2,$$

i.e. when the characteristic roots coincide!

Note: t_+ & t_- distinct.

Three cases:

(i) $\theta \neq 0$ & $t_+ \neq 0$ (turning point)

(ii) $\theta \neq 0$ and $t_+ = 0$

(iii) $\theta = 0$. (transition point)

Case (i): Formal solution

$$P_n(\nu^\theta t) = \mathrm{Ai}\left(\nu^{\frac{2}{3}}\zeta + \frac{\Phi}{\nu^{\frac{1}{3}}}\right) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\nu^s} + \mathrm{Ai}'\left(\nu^{\frac{2}{3}}\zeta + \frac{\Phi}{\nu^{\frac{1}{3}}}\right) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^{s+\frac{1}{3}}}$$

$$\zeta(t_+) = 0$$

$$x = \nu^\theta t \qquad \text{is fixed!}$$

$$n \to n+1 \text{ (or } \nu \to \nu+1) \qquad t \to t' = \left(1 + \frac{1}{\nu}\right)^{-\theta} t$$

$$n \to n-1 \text{ (or } \nu \to \nu-1) \qquad t \to t'' = \left(1 - \frac{1}{\nu}\right)^{-\theta} t$$

$$x = \nu^\theta t = (\nu+1)^\theta t' \qquad x = \nu^\theta t = (\nu-1)^\theta t''$$

$$\mathrm{Ai}\left(\nu^{\frac{1}{3}}\zeta + \frac{\Phi(\zeta)}{\nu^{\frac{1}{3}}}\right) \to \mathrm{Ai}\left[(\nu+1)^{\frac{2}{3}}\,\zeta\left(\left(1+\frac{1}{\nu}\right)^{-\theta} t\right) + \frac{\Phi\left(\zeta\left(\left(1+\frac{1}{\nu}\right)^{-\theta} t\right)\right)}{(\nu+1)^{\frac{1}{3}}}\right]$$

$$\zeta = \zeta(t), \qquad \Phi = \Phi(\zeta).$$

Lemma:

$$\mathrm{Ai}\left[(\nu+1)^{\frac{2}{3}}\,\zeta\left(\left(1+\frac{1}{\nu}\right)^{-\theta} t\right) + \frac{\Phi\left(\zeta\left(\left(1+\frac{1}{\nu}\right)^{-\theta} t\right)\right)}{(\nu+1)^{\frac{1}{3}}}\right]$$

$$\sim \mathrm{Ai}\left(\nu^{\frac{2}{3}}\zeta + \frac{\Phi}{\nu^{\frac{1}{3}}}\right)\sum_{s=0}^{\infty}\frac{G_s(\zeta,\Phi)}{\nu^s}$$

$$+\frac{1}{\nu^{\frac{1}{3}}}\mathrm{Ai}'\left(\nu^{\frac{2}{3}}\zeta + \frac{\Phi}{\nu^{\frac{1}{3}}}\right)\sum_{s=0}^{\infty}\frac{H_s(\zeta,\Phi)}{\nu^s}$$

$$(*) \qquad P_{n+1}(x) - (A_n x + B_n)P_n(x) + P_{n-1}(x) = 0$$

$$A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \qquad B_s \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s},$$

$$\alpha_0 t_{\pm} + \beta_0 = \pm 2, \qquad \text{turning points}$$

If $\theta \neq 0$ and $|\beta_0| < 2$ (i.e., $t_- < 0 < t_+$), choose

$$\tau_0 = -\frac{\alpha_1 t_+ + \beta_1}{(2 - \beta_0)\theta}$$

so that

$$\alpha_1' t_+ + \beta_1' = 0,$$

where α_1' and β_1' are the coefficients in

$$A_n \sim \nu^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s'}{\nu^s}, \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s'}{\nu^s},$$

and

$$\nu = n + \tau_0.$$

THEOREM A. (Wang and ________, Numer. Math., 2003)
If $\theta \neq 0$ and $|\beta_0| < 2$ (i.e., $t_- < 0 < t_+$), then $(*)$ has a pair of linearly independent solutions

$$P_n(x) = P_n(\nu^\theta t) \sim \left(\frac{4\zeta}{(\alpha_0 t + \beta_0)^2 - 4}\right)^{\frac{1}{4}} \times \left\{ \nu^{\frac{1}{6}} \mathrm{Ai}\left(\nu^{\frac{2}{3}}\zeta + \frac{\Phi}{\nu^{\frac{1}{3}}}\right) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\nu^s} - \nu^{-\frac{1}{6}} \mathrm{Ai}'\left(\nu^{\frac{2}{3}}\zeta + \frac{\Phi}{\nu^{\frac{1}{3}}}\right) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^s} \right\},$$

$$Q_n(x) = Q_n(\nu^\theta t) \sim \left(\frac{4\zeta}{(\alpha_0 t + \beta_0)^2 - 4} \right)^{\frac{1}{4}}$$
$$\times \left\{ \nu^{\frac{1}{6}} \mathrm{Bi}\left(\nu^{\frac{2}{3}} \zeta + \frac{\Phi}{\nu^{\frac{1}{3}}} \right) \sum_{s=0}^{\infty} \frac{\widetilde{A}_s(\zeta)}{\nu^s} \right.$$
$$\left. + \nu^{-\frac{1}{6}} \mathrm{Bi}'\left(\nu^{\frac{2}{3}} \zeta + \frac{\Phi}{\nu^{\frac{1}{3}}} \right) \sum_{s=0}^{\infty} \frac{\widetilde{B}_s(\zeta)}{\nu^s} \right\},$$

where $\zeta(t)$ and $\Phi(\zeta)$ are explicitly given analytic functions. These expansions hold uniformly for $0 < t < \infty$.

$$
\begin{aligned}
\frac{2}{3}[\zeta(t)]^{\frac{3}{2}} =& \alpha_0 t^{1/\theta} \int_{t_+}^{t} \frac{s^{-1/\theta}}{\sqrt{(\alpha_0 s + \beta_0)^2 - 4}} ds \\
& - \log \frac{\alpha_0 t + \beta_0 + \sqrt{(\alpha_0 t + \beta_0)^2 - 4}}{2}, \qquad t \geq t_+, \\
\frac{2}{3}[-\zeta(t)]^{\frac{3}{2}} =& \cos^{-1} \frac{\alpha_0 t + \beta_0}{2} \\
& - \alpha_0 t^{1/\theta} \int_{t}^{t_+} \frac{s^{-1/\theta}}{\sqrt{4 - (\alpha_0 s + \beta_0)^2}} ds, \qquad t < t_+.
\end{aligned}
$$

$$\Phi(\zeta) := -\frac{1}{\zeta^{\frac{1}{2}}} \int_{t_+}^{t} \frac{\alpha_1' \tau + \beta_1'}{2\theta\tau\zeta^{\frac{1}{2}} H_0(\zeta)} d\tau,$$

$$\alpha_1' = \alpha_1 + \theta\tau_0\alpha_0, \qquad \beta_1' = \beta_1$$

$$H_0(\zeta) = -\sqrt{\frac{(\alpha_0 t + \beta_0)^2 - 4}{4\zeta}}$$

Case (iii): $\theta = 0$ (transition point)

The characteristic roots again coincide when $x = x_{\pm}$, where

$$\alpha_0 x_{\pm} + \beta_0 = \pm 2.$$

Assume $\alpha_1 = \beta_1 = 0$ so that $\alpha_1 x_+ + \beta_1 = 0$

$$\tau_0 = -\frac{(\alpha_3 x_+ + \beta_3)}{2(\alpha_2 x_+ + \beta_2)}, \qquad N = n + \tau_0.$$

$$\nu = (\alpha_2' x_+ + \beta_2' + \frac{1}{4})^{1/2}.$$

$$\zeta^{\frac{1}{2}}(x) = \cosh^{-1}(\frac{\alpha_0 x + \beta_0}{2})$$

THEOREM B. (Wang and ________, Math. Comp., 2005)

If $\theta = 0$, then $(*)$ has a pair of linearly independent solutions

$$P_n(x) \sim \left(\frac{4\zeta}{(\alpha_0 x + \beta_0)^2 - 4} \right)^{\frac{1}{4}} \times \left[N^{\frac{1}{2}} \mathrm{I}_\nu(N\zeta^{1/2}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} \mathrm{I}_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right]$$

and

$$Q_n(x) \sim \left(\frac{4\zeta}{(\alpha_0 x + \beta_0)^2 - 4} \right)^{\frac{1}{4}} \times \left[N^{\frac{1}{2}} K_\nu(N\zeta^{1/2}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} - N^{\frac{1}{2}} \zeta^{\frac{1}{2}} K_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right].$$

There expansions hold uniformly for $x_- + \delta \le x < \infty$.