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Multiple orthogonal polynomials as special functions

Walter Van Assche

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Multiple orthogonal polynomials

definition

Polynomials of one variable but indexed by a multi-index $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ $(r \geq 1)$ satisfying orthogonality conditions with respect to r positive measure $\mu_1, \ldots, \mu_r$ on the real line.

They appear naturally in Hermite-Padé approximation to r functions

$$f_j(z) = \int \frac{d\mu_j(x)}{z - x}, \qquad 1 \leq j \leq r.$$

There are two types: type I and type II

$P_{\vec{n}}$ is a **monic** polynomial of degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ for which

$$\begin{aligned} \int P_{\vec{n}}(x) x^k \, d\mu_1(x) &= 0, \qquad k = 0, 1, \ldots, n_1 - 1 \\ &\vdots \\ \int P_{\vec{n}}(x) x^k \, d\mu_r(x) &= 0, \qquad k = 0, 1, \ldots, n_r - 1 \end{aligned}$$

$|\vec{n}|$ linear conditions for $|\vec{n}|$ unknowns.

Solution exists and unique: $\vec{n}$ is a *normal index* for type II.

type I multiple orthogonal polynomials

$(A_{\vec{n},1},\dots,A_{\vec{n},r})$ is a vector of r polynomials, with $A_{\vec{n},j}$ of degree n_j-1, for which

$$\begin{aligned} \int x^k \sum_{j=1}^r A_{\vec{n},j}\, d\mu_j(x) &= 0, \qquad k=0,1,\dots,|\vec{n}|-2 \\ \int x^{|\vec{n}|-1} \sum_{j=1}^r A_{\vec{n},j}\, d\mu_j(x) &= 1. \end{aligned}$$

$|\vec{n}|$ linear conditions for $|\vec{n}|$ unknowns.

Solution exists and unique: $\vec{n}$ is a *normal index* for type I ($\Leftrightarrow$ for type II).
Notation:

$$Q_{\vec{n}}(x) = \sum_{j=1}^r A_{\vec{n},j}(x) w_j(x), \qquad w_j(x) = \frac{d\mu_j(x)}{d\mu(x)}.$$

Determinantal point processes

what?

Surveys

- A. Borodin: *Determinantal point processes*, arXiv:0911.1153 [math.PR]
- T. Tao: http://terrytao.worldpress.com/2009/08/23/determinantal-processes
- A. Soshnikov: *Determinantal random point fields*, Russian Math. Surveys **55** (2000)
- R. Lyons: *Determinantal probability measures*, Publ. Math. Inst. Hautes Etudes Sci. **98** (2003)
- K. Johansson: *Random matrices and determinantal processes*, arXiv:math-ph/0510038

A point process on $\mathbb{R}$ is determinantal if there exists a kernel $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\text{Pr}\{\exists \text{ particle in each } (x_i, x_i + dx_i), 1 \leq i \leq n\}$$
$$= \det\big(K(x_i, x_j)\big)_{i,j=1}^{n} \, dx_1 \, dx_2 \ldots dx_n.$$

(provided these probabilities are positive, of course).

biorthogonal ensembles

If $(\phi_i)_{i=1,2,\ldots,n}$ and $(\psi_i)_{i=1,2,\ldots,n}$ are functions on $\mathbb{R}$ and

$$Z_n = \int_{\mathbb{R}^n} \det\big(\phi_i(x_j)\big)_{i,j=1}^n \det\big(\psi_i(x_j)\big)_{i,j=1}^n \, dx_1 \ldots dx_n,$$

then the point process with

$$P(x_1, \ldots, x_n) = Z_n^{-1} \det\big(\phi_i(x_j)\big)_{i,j=1}^n \det\big(\psi_i(x_j)\big)_{i,j=1}^n$$

is determinantal with

$$K(x,y) = \sum_{i=1}^n \sum_{j=1}^n (G^{-1})_{i,j} \phi_i(x) \psi_j(y),$$

and

$$G_{i,j} = \int_{\mathbb{R}} \phi_i(x) \psi_j(x) \, dx$$

For the Gaussian Unitary Ensemble (GUE) of Hermitian $n \times n$ matrices M with probability distribution

$$Z_n^{-1} \, e^{-2\mathrm{Tr} V(M)} \, dM$$

the eigenvalues are a determinantal point process with

$$K(x,y) = \sum_{i=0}^{n-1} p_i(x) p_i(y) e^{-V(x)-V(y)}$$

where $p_0(x), p_1(x), p_2(x), \ldots$ are the orthonormal polynomials for the weight $e^{-2V(x)}$:

$$\int p_i(x) p_j(x) e^{-2V(x)} \, dx = \delta_{m,n}$$

If A is a given Hermitian $n \times n$ matrix and the probability distribution is

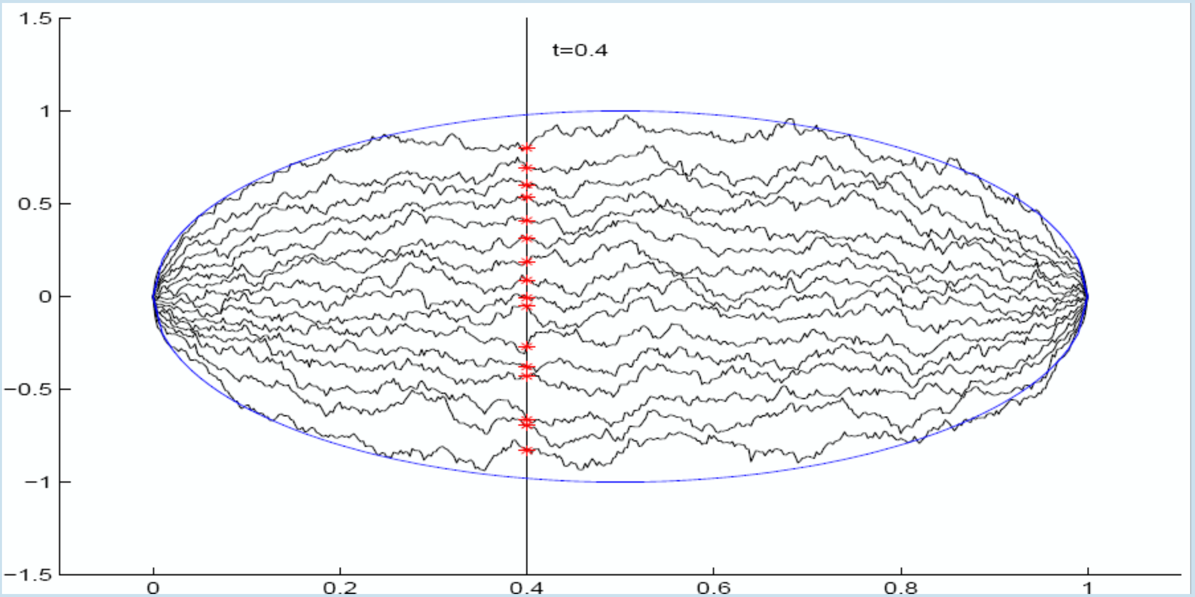
$$Z_n^{-1}\, e^{-\text{Tr}(V(M)+AM)}\, dM,$$

then the eigenvalues are a determinantal process with

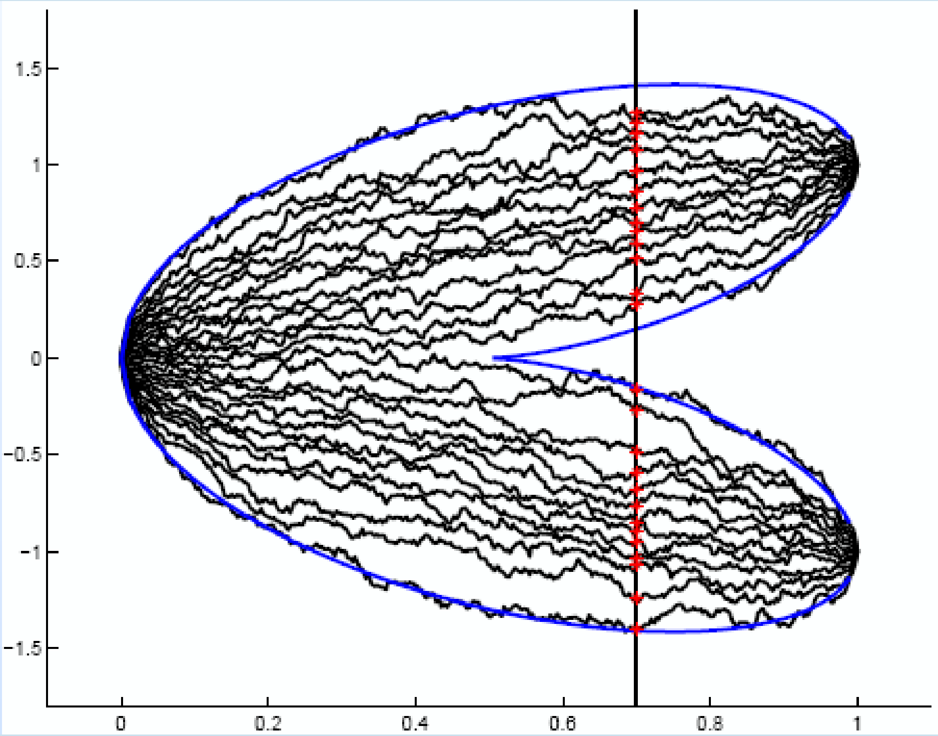
$$K(x,y) = \sum_{i=0}^{|\vec{n}|-1} P_{\vec{n}_i}(x) Q_{\vec{n}_{i+1}}(x)$$

where $P_{\vec{n}}$ and $Q_{\vec{n}}$ are multiple orthogonal polynomials for the measures $e^{-V(x)-a_j x}$ $(1 \leq j \leq r)$, with $a_1, \ldots, a_r$ the eigenvalues of A and n_j the multiplicity of the eigenvalues a_j. The multi-indices $(\vec{n}_0, \ldots, \vec{n}_n)$ are a path in $\mathbb{N}^r$ from $\vec{n}_0 = \vec{0}$ to $\vec{n}_{|\vec{n}|} = \vec{n}$ such that $\vec{n}_{i+1} - \vec{n}_i = \vec{e}_j$ for some $j \in \{1, \ldots, r\}$ and $\vec{e}_1, \ldots, \vec{e}_r$ are the unit vectors in $\mathbb{N}^r$.

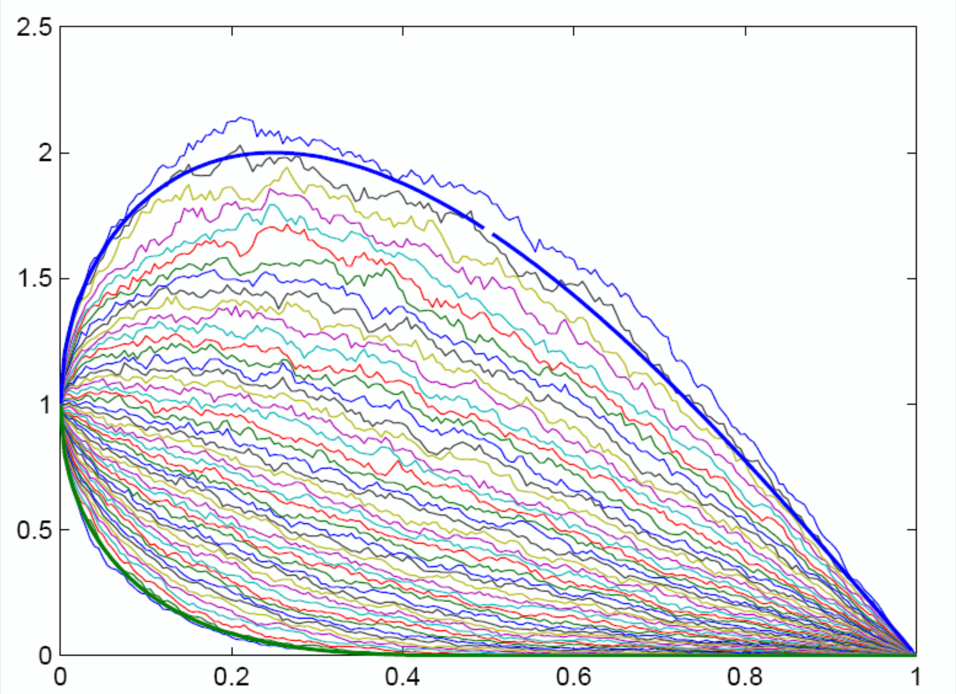
non-intersecting Brownian motions



non-intersecting Brownian motions



squared Bessel paths



$$Y(t) = X_1^2(t) + X_2^2(t) + \cdots + X_d^2(t)$$

where $(X_1(t), X_2(t), \ldots, X_d(t))$ is a d-dimensional Brownian motion starting from $(a_1, \ldots, a_d)$ and ending at $(0, 0, \ldots, 0)$. This is a biorthogonal ensemble which uses modified Bessel functions I_α with $d = 2(\alpha + 1)$.

Recurrence relations

nearest neighbor recurrence relations

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation.

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

Multiple orthogonal polynomials (with all multi-indices normal) satisfy a system of r recurrence relations

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \qquad 1 \leq k \leq r.$$

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_k}(x) + b_{\vec{n}-\vec{e}_k,k}Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}Q_{\vec{n}+\vec{e}_j}(x), \qquad 1 \leq k \leq r.$$

The recurrence coefficients satisfy some partial difference equations (Van Assche, 2011).
Suppose $1 \leq i \neq j \leq r$, then

$$\begin{aligned} b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n},j} &= b_{\vec{n}+\vec{e}_j,i} - b_{\vec{n},i} \\ \sum_{j=1}^{r} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{r} a_{\vec{n}+\vec{e}_i,k} &= \det \begin{pmatrix} b_{\vec{n}+\vec{e}_j,i} & b_{\vec{n},i} \\ b_{\vec{n}+\vec{e}_i,j} & b_{\vec{n},j} \end{pmatrix} \\ \frac{a_{\vec{n},i}}{a_{\vec{n}+\vec{e}_j,i}} &= \frac{b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}}{b_{\vec{n},j} - b_{\vec{n},i}} \end{aligned}$$

Reason: $P_{\vec{n}+\vec{e}_i+\vec{e}_j}(x)$ can be computed in two ways from the recurrence relations:

- first compute $P_{\vec{n}+\vec{e}_i}(x)$ and from there $P_{\vec{n}+\vec{e}_i+\vec{e}_j}(x)$
- first compute $P_{\vec{n}+\vec{e}_j}(x)$ and from there $P_{\vec{n}+\vec{e}_j+\vec{e}_i}(x)$

Christoffel-Darboux formula

For orthogonal polynomials there is the Christoffel-Darboux relation

$$\sum_{i=0}^{n-1} p_i(x)p_i(y) = a_n \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}$$

For multiple orthogonal polynomials there is a similar formula

$$\sum_{i=0}^{|\vec{n}|-1} P_{\vec{n}_i}(x)Q_{\vec{n}_{i+1}}(y) = \frac{P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x)Q_{\vec{n}+\vec{e}_j}(y)}{x-y}.$$

where $(\vec{n}_i)_{i=0,1,\ldots,|\vec{n}|}$ is a path in $\mathbb{N}^r$ from $\vec{n}_0 = \vec{0}$ to $\vec{n}_{|\vec{n}|} = \vec{n}$ such that for each $i \in \{0, 1, \ldots, |\vec{n}| - 1\}$ one has $\vec{n}_{i+1} - \vec{n}_i = \vec{e}_j$ for some $j \in \{1, 2, \ldots, r\}$ (Kuijlaars & Daems).

Various examples

multiple Hermite polynomials

$$\int_{-\infty}^{\infty} x^k H_{\vec{n}}(x) e^{-x^2+c_j x}\, dx = 0, \qquad k = 0,1,\ldots,n_j-1$$

where $c_i \leq c_j$ whenever $i \neq j$.
random matrices with external source (Kuijlaars & Bleher),
non-intersection Brownian motions (Kuijlaars, Daems, Delvaux, Bleher)
Rodrigues formula

$$e^{-x^2} H_{\vec{n}}(x) = (-1)^{|\vec{n}|} 2^{-|\vec{n}|} \left(\prod_{j=1}^{r} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x} \right) e^{-x^2}.$$

Recurrence relations: for $1 \leq k \leq r$

$$x H_{\vec{n}}(x) = H_{\vec{n}+\vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x).$$

$$\int_0^\infty x^k L_{\vec{n}}(x) x^{\alpha_j} e^{-x}\, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1$$

where $\alpha_1, \ldots, \alpha_r > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$.

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}(x) = \prod_{j=1}^{r} \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right) e^{-x}$$

$$x L_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k} L_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} L_{\vec{n}-\vec{e}_j}(x)$$

$$a_{\vec{n},j} = n_j(n_j + \alpha_j) \prod_{i=1, i\neq j}^{r} \frac{n_j + \alpha_j - \alpha_i}{n_j - n_i + \alpha_j - \alpha_i}$$

$$b_{\vec{n},j} = |\vec{n}| + n_j + \alpha_j + 1.$$

multiple Laguerre polynomials, second kind

$$\int_0^\infty x^k L_{\vec{n}}(x) x^\alpha e^{-c_j x}\, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1$$

with $c_1, \ldots, c_r > 0$ and $c_i \neq c_j$ whenever $i \neq j$.
Wishart ensembles in random matrix theory.

$$(-1)^{|\vec{n}|} \left(\prod_{j=1}^r c_j^{n_j} \right) x^\alpha L_{\vec{n}}(x) = \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}|+\alpha}$$

$$x L_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k} L_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j} L_{\vec{n}-\vec{e}_j}(x)$$

$$a_{\vec{n},j} = \frac{|\vec{n}| + \alpha)n_j}{c_j^2}, \qquad b_{\vec{n},j} = \frac{|\vec{n}| + \alpha + 1}{c_j} + \sum_{i=1}^r \frac{n_i}{c_i}$$

$$\int_0^1 x^k P_{\vec{n}}(x) x^{\alpha_j} (1-x)^\beta \, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1$$

with $\alpha_1, \ldots, \alpha_r, \beta > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$.

$$(-1)^{|\vec{n}|} \prod_{j=1}^{r} (|\vec{n}| + \alpha_j + \beta + 1)_{n_j} (1-x)^\beta P_{\vec{n}}(x) = \prod_{j=1}^{r} \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right) (1-x)^{|\vec{n}|+\beta}$$

multiple Jacobi polynomials

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x)$$

Let $q_r(x) = \prod_{j=1}^{r}(x-\alpha_j)$ and $Q_{r,\vec{n}}(x) = \prod_{j=1}^{r}(x-n_j-\alpha_j)$, then

$$a_{\vec{n},j} = \frac{q_r(-|\vec{n}|-\beta)q_r(n_j+\alpha_j)}{Q_{r,\vec{n}}(-|\vec{n}|-\beta)Q'_{r,\vec{n}}(n_j+\alpha_j)}$$

$$\frac{(n_j+\alpha_j)(|\vec{n}|+\beta)}{(|\vec{n}|+n_j+\alpha_j+\beta+1)(|\vec{n}|+n_j+\alpha_j+\beta)(|\vec{n}|+n_j+\alpha_j+\beta-1)}$$

$$b_{\vec{n},j} = \delta_{\vec{n}} - \delta_{\vec{n}+\vec{e}_j}, \qquad \delta_{\vec{n}} = -(|\vec{n}|+\beta)\frac{q_r(-|\vec{n}|-\beta)}{Q_{r,\vec{n}}(-|\vec{n}|-\beta)}.$$

multiple orthogonal polynomials and modified Bessel functions I

$$\begin{aligned}\int_0^\infty x^k P_{n,m}(x) x^{\nu/2} I_\nu(2\sqrt{x}) e^{-cx}\,dx &= 0, \qquad k = 0,1,\ldots,n-1\\ \int_0^\infty x^k P_{n,m}(x) x^{(\nu+1)/2} I_{\nu+1}(2\sqrt{x}) e^{-cx}\,dx &= 0, \qquad k = 0,1,\ldots,m-1\end{aligned}$$

with $\nu > -1$ and $c > 0$. If $p_{2n}(x) = P_{n,n}(x)$ and $p_{2n+1}(x) = P_{n+1,n}(x)$, then

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x)$$

with

$$b_n = \frac{c(2n+\nu+1)) + 1}{c^2}, \quad c_n = \frac{n(2+c(n+\nu))}{c^3}, \quad d_n = \frac{n(n-1)}{c^4}$$

and $y = p_n(x)$ satisfies

$$xy''' + (-2cx+\nu+2)y'' + (c^2x + c(n-\nu-2)-1)y' - c^2ny = 0.$$

multiple orthogonal polynomials and modified Bessel functions K

$$
\begin{aligned}
\int_0^\infty x^k P_{n,m}(x) x^{\alpha+\nu/2} K_\nu(2\sqrt{x})\, dx &= 0, \qquad k = 0,1,\ldots,n-1 \\
\int_0^\infty x^k P_{n,m}(x) x^{\alpha+(\nu+1)/2} K_{\nu+1}(2\sqrt{x})\, dx &= 0, \qquad k = 0,1,\ldots,m-1
\end{aligned}
$$

with $\alpha > -1$ and $\nu \geq 0$. Let $p_{2n}(x) = P_{n,n}(x)$ and $p_{2n+1}(x) = P_{n+1,n}(x)$

$$
xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x)
$$

$$
\begin{aligned}
b_n &= (n+\alpha+1)(3n+\alpha+2\nu) - (\alpha+1)(\nu-1), \\
c_n &= n(n+\alpha)(n+\alpha+\nu)(3n+2\alpha+\nu), \\
d_n &= n(n-1)(n+\alpha-1)(n+\alpha)(n+\alpha+\nu-1)(n+\alpha+\nu)
\end{aligned}
$$

and $y = p_n(x)$ satisfies

$$
x^2 y''' + x(2\alpha+\nu+3)y'' + [(\alpha+1)(\alpha+\nu+1) - x]y' + ny = 0.
$$

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