Advances on the Liouville-Green approximation for differential and difference equations

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dedicated to Frank Olver

Collaborators: M. Vianello, F. Locatelli, T. Soldà; F. Pistellato, M. Vergine, and B. Bucci.

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- 8. The LG approximation for systems of difference equations.
- 9. A further application: to the numerical solution of rapidly oscillatory equations.

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Several ways to present the WKB (or LG) approximation: Often, especially physicists, start looking for solutions , of the form

$$y(x) \sim \exp\left\{\frac{1}{\varepsilon}\sum_{n=0}^{\infty}\varepsilon^n S_n(x)\right\} \quad \text{as } \varepsilon \to 0.$$

Inserting *formally* this into equation

$$\varepsilon^2 y'' - q(x)y = 0$$

leads then, to lowest order, to the two basis solutions

$$y(x) \sim q^{-1/4}(x)\, \exp \pm \left\{ \frac{1}{\varepsilon} \int_{x_0}^{x} q^{1/2}(\xi)\, d\xi \right\}.$$

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This comes solving the so-called Eikonal equation

$$S_0'^2 = q(x),$$

yielding

$$S_0(x) = \pm \int_{x_0}^{x} q^{1/2}(\xi)\, d\xi,$$

and

$$2S_0' S_1' + S_0'' = 0,$$

from which

$$S_1(x) = -\frac{1}{4} \log q(x) + const.$$

• 2. Olver's approach provided a *rigorous* justification (for 2nd order linear ODEs), and precise *error bounds*. He stressed the double asymptotic nature, with respect to x and ε.

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Instead of repeating his classical results, here we give a unified treatment for second-order linear differential *and* difference equations (SIMA '94):

Theorem (SIMA '94). Consider the linear Volterra integral equation

$$\varepsilon(x) = \int_x^{+\infty} K(x,t)\,[\phi(x,t) + \varepsilon(t)]\,d\mu, \quad x \in [x_0, +\infty),$$

$x_0 \in \mathbf{R}$ and μ a complex (finite) measure. Assume that, for each fixed $x \in [x_0, +\infty)$,

(i) $K(x,\cdot)$, $\phi(x,\cdot)$ are μ-measurable complex-valued functions;

(ii) $|K(x,t)|$, $|K(x,t)\phi(x,t)| \leq h(x,t)$ $|\mu|$-a.e. for $t \geq x$, where $h(x,\cdot) \in L^1([x,+\infty);\mu)$, and, moreover,

$$V(x) := \int_x^{+\infty} h(x,t)\,d|\mu|$$

is nonincreasing and $\lim_{x\to+\infty} V(x) < 1$.

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is nonincreasing and $\lim_{x\to+\infty} V(x) < 1$. Then, $\exists$! solution $\varepsilon(x)$ for $x > x_1 := \inf\{x : x \ge x_0, V(x) < 1\}$, and the estimate holds:

$$|\varepsilon(x)| \le \frac{V(x)}{1 - V(x)}.$$

Specializing K, ϕ ($\equiv 1$, in general), and μ, we recover:

- (i) the classical Olver's results for ODEs;
- (ii) the case that we called "finite moments perturb.s";
- (iii) the analogous cases for difference equations.

We had studied cases (iii) directly in earlier works (JCAM '92, JMAA '92), and later (JAT '99).

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We'll go back later to difference eq.s.

• 3. The LG approximation for systems of ODEs.

Theorem (asymptotically constant case), JMAA 2007.

Consider the matrix differential equation $Y'' = Q(t)Y$, $Q, Y \in \mathbf{M}_n$, where

$$Q(t) = f(t)A + G(t),$$

with $f(t) > 0$ in $[a, +\infty)$, $a \geq 0$, $f \in C^2([a, +\infty))$, $G \in C^0([a, +\infty); \mathbf{M}_n)$, and $A \in \mathbf{M}_n$ is constant and nonsingular, and such that its real negative eigenvalues (if any) generate Jordan blocks of dimension 1. Denote by $A^{1/2}$ one of the solutions of the matrix equation $X^2 = A$, whose eigenvalues all have nonnegative real parts, and by $A^{-1/2}$ its inverse (it is intended that $A^{-1/2} \equiv (A^{1/2})^{-1}$).

Set

$$\varphi(t) := \int_a^t f^{1/2}(s)ds,$$

and

$$V_1(t) := \int_t^{+\infty} f^{-1/2}(s) \cdot \|A^{-1/2}K_1(s)\|\, ds,$$

where

$$K_1(t) := \left(\frac{5}{16}f^{-2}f'^2 - \frac{1}{4}f^{-1}f''\right) I - G_1(t),$$

$$G_1(t) := e^{\varphi(t)A^{1/2}}G(t)e^{-\varphi(t)A^{1/2}}.$$

(I) Then, if $V_1(a) < \infty$, there exists a C^2 solution, $Y_1(t)$, to equation above on $[a, +\infty)$ which can be represented as

$$Y_1(t) = f^{-1/4}(t) e^{-\varphi(t) A^{1/2}} [I + E_1(t)],$$

with

$$\|E_1(t)\| \le e^{c\, V_1(t)} - 1, \quad f^{-1/2}(t) \cdot \|A^{-1/2} E_1'(t)\| \le \frac{d}{c} \left(e^{c\, V_1(t)} - 1 \right),$$

where

$$d := \max_{0 \le t < +\infty} \|e^{-2tA^{1/2}}\|, \qquad c := \frac{1}{2}(1 + d).$$

(II) Similarly, defining

$$V_2(t) := \int_a^t f^{-1/2}(s) \cdot \|A^{-1/2} K_2(s)\| \, ds$$

where

$$K_2(t) := \left(\frac{5}{16} f^{-2} f'^2 - \frac{1}{4} f^{-1} f'' \right) I - G_2(t),$$

$$G_2(t) := e^{-\varphi(t) A^{1/2}} G(t) e^{\varphi(t) A^{1/2}},$$

if the relation $V_2(+\infty) < \infty$ holds, then there exists a C^2 solution, $Y_2(t)$, to the equation above, on $[a, +\infty)$,

$$Y_2(t) = f^{-1/4}(t) e^{\varphi(t) A^{1/2}} \left[I + E_2(t) \right],$$

with

$$\|E_2(t)\| \leq e^{c\, V_2(t)} - 1, \quad f^{-1/2}(t) \cdot \|A^{-1/2} E_2'(t)\| \leq \frac{d}{c} \left(e^{c\, V_2(t)} - 1 \right),$$

(III) When both conditions, $V_1(a) < +\infty$ and $V_2(+\infty) < +\infty$, hold, and $\alpha(-A^{1/2}) < 0$, and $\varphi(t) \to +\infty$ as $t \to +\infty$, a solution $\tilde{Y}_2(t)$ exists on a suitable subinterval $[\tilde{a}, +\infty)$ of $[a, +\infty)$, such that $\tilde{Y}_2(t) \sim e^{\varphi(t)A^{1/2}}$ as $t \to +\infty$. More precisely, we have

$$\tilde{Y}_2(t) = e^{\varphi(t)A^{1/2}} \left[I + \tilde{E}_2(t) \right],$$

with

$$\|\tilde{E}_2(t)\| \le \frac{1+K}{1-K} \left\{ c \left[\tilde{V}_2(+\infty) - \tilde{V}_2(t) \right] + \frac{p(2t)}{2} \tilde{V}_2(+\infty)\, e^{-t\alpha(A^{1/2})} \right\}$$

where $K := \exp\{c\,\tilde{V}_2(+\infty)\} - 1$, $\tilde{V}_2(t) := \int_{\tilde{a}}^{t} \|A^{-1/2}K_2(s)\|\, ds$ (with $\tilde{a}$ chosen in such a way that $K < 1$), is an estimate of $E_2(+\infty)$. The pair $(Y_1(t), \tilde{Y}_2(t))$ is a basis for solutions to equation above, on $[\tilde{a}, +\infty)$.

Theorem (almost-diagonal case), Asympt. Anal. 2006.
Consider the differential equation

$$Y'' = Q(t)Y, \qquad t \in [a, +\infty),$$

with $a \geq 0$, Q and Y $n \times n$ matrix-valued functions, whose entries are in general complex, where

$$Q(t) = D(t) + G(t), \quad D \in C^2([a, +\infty); \mathbf{M}_n),\ G \in C^0([a, +\infty); \mathbf{M}_n),$$

with $D(t) := \mathrm{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))$ a nonsingular diagonal matrix, and stipulate that below, in $D^{1/2}(t)$, we choose $\mathrm{Re}\, \lambda_j^{1/2}(t) \geq 0$ for all j's, while $\lambda_j^{-1/2}$ means $(\lambda_j^{1/2})^{-1}$, and $D^{1/4}(t)$ the square root of $D^{1/2}(t)$, obtained choosing $\mathrm{Re}\, \lambda_j^{1/4}(t) \geq 0$ for every j.

Setting

$$V_1(t) := \int_t^{+\infty} \|D^{-1/2}(s)K_1(s)\|\, ds,$$

where

$$K_1(t) := \Gamma(t) - G_1(t), \quad \Gamma(t) := \frac{5}{16} D^{-2}(D')^2 - \frac{1}{4} D^{-1} D'',$$

$$G_1(t) := D^{1/4}(t)\, e^{\int_a^t D^{1/2}(s)\, ds} G(t)\, e^{-\int_a^t D^{1/2}(s)\, ds} D^{-1/4}(t),$$

assume that $V_1(a) < \infty$.

Then, there exists on the whole halfline $[a, +\infty)$ a C^2 solution, say $Y_1(t)$, which can be represented as

$$Y_1(t) = D^{-1/4}(t)\, e^{-\int_a^t D^{1/2}(s)\, ds}\, [I + E_1(t)],$$

where

$$\|E_1(t)\| \leq e^{c\, V_1(t)} - 1, \qquad \|D^{-1/2}(t) E_1'(t)\| \leq e^{c\, V_1(t)} - 1,$$

and $c = 1$. When the eigenvalues of $D(t)$ are all real positive, the previous estimates can be improved setting $c = 1/2$.

Defining

$$V_2(t) := \int_a^t \|D^{-1/2}(s)K_2(s)\|\, ds,$$

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$$K_2(t) := \Gamma(t) - G_2(t),$$

$$G_2(t) := D^{1/4}(t)\, e^{-\int_a^t D^{1/2}(s)\, ds} G(t)\, e^{\int_a^t D^{1/2}(s)\, ds} D^{-1/4}(t),$$

assuming that $V_2(+\infty) < \infty$, there exists on $[a, +\infty)$ a C^2 solution to equation above,

$$Y_2(t) = D^{-1/4}(t)\, e^{\int_a^t D^{1/2}(s)\, ds}\, [I + E_2(t)],$$

where

$$\|E_2(t)\| \le e^{c\, V_2(t)} - 1, \qquad \|D^{-1/2}(t)E_2'(t)\| \le e^{cV_2(t)} - 1,$$

and c is defined as above.

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• 5. Abstract equations. (Something done, something probably not yet.)

• 6. LG approximations for second-order scalar difference equations.
Consider

$$Y_{n+2} + A_n Y_{n+1} + B_n Y_n = 0, \qquad A_n, B_n \in \mathbf{C},\ n \in \mathbf{Z}_\nu,$$

with
(i) $A_n \neq 0$ for all $n \in \mathbf{Z}_\nu$;
(ii)

$$\frac{B_n}{A_n A_{n-1}} \to L, \quad L \in \mathbf{C} \setminus [1/4, +\infty),\ L \neq 0, \ \text{ as } n \to \infty;$$

(iii)

$$\sum_{n=\nu+1}^{\infty} \left| \frac{B_n}{A_n A_{n-1}} - L \right| < \infty.$$

This can be taken into the "canonical" form

$$\Delta^2 y_n + q_n y_n = 0, \quad n \in \mathbf{Z}_\nu,$$

with

$$q_n := -1 + \frac{4B_n}{A_n A_{n-1}}, \quad n \geq \nu + 1,$$

upon setting

$$Y_n := \alpha_n y_n, \quad \alpha_n := \alpha_{\nu+1} \prod_{k=\nu}^{n-2} \left(-\frac{A_k}{2} \right), \; n \geq \nu + 2$$

It is convenient to write

$$q_n = a + g_n, \quad a := 4L, \;\; g_n := 4 \left(\frac{B_n}{A_n A_{n-1}} - L \right).$$

When a is real, the three following cases are important:

1. $a > 0$, and $\sum_{n=\nu}^{\infty} |g_n| < \infty$;
2. $a = 0$, and $\sum_{n=\nu}^{\infty} n^k |g_n| < \infty$, $k = 1$ or 2
   ("finite moments perturbations", JMAA '92);
3. $a < 0$, $a \neq -1$, and $\sum_{n=\nu}^{\infty} |g_n| < \infty$.

In case 1 (JCAM '92), all real solutions are *oscillatory*, and there exist $n_0 \in \mathbf{Z}_\nu$ and two linearly independent solutions,

$$y_n^{(j)} = (\lambda_j)^n \left[1 + \epsilon_n^{(j)}\right], \quad j = 1, 2, \; n \geq n_0,$$

where

$$\lambda_1 = 1 + ia^{1/2}, \quad \lambda_2 = \overline{\lambda_1},$$

are the roots of the characteristic polynomial associated with the unperturbed difference equation (that with $g_n \equiv 0$), and

$$|\epsilon_n^{(j)}| \leq \frac{V_n}{1 - V_n}, \; j = 1, 2, \; n \geq n_0,$$

$$V_n := [a(1 + a)]^{-1/2} \sum_{k=n}^{\infty} |g_k|.$$

When g_n is real-valued, $y_n^{(1)}$ and $y_n^{(2)}$ are complex conjugate. Moreover, $n_0 = \min\{n : n \in \mathbf{Z}_\nu, V_n < 1\}$.

In case 2 (JMAA '92), assuming, e.g., $\sum_{n=\nu}^{\infty} n^2 |g_n| < \infty$, $\exists n_0 \in \mathbf{Z}_\nu$ such that there are two linearly independent solutions for $n \geq n_0$ of the form

$$y_n^{(1)} = 1 + \varepsilon_n^{(1)}, \quad y_n^{(2)} = n + \varepsilon_n^{(2)},$$

with

$$|\varepsilon_n^{(j)}| \leq \frac{V_n^{(j)}}{1 - V_n^{(1)}}, \quad \text{for } n \geq n_1, \ j = 1, 2,$$

$$V_n^{(1)} := \sum_{j=n}^{\infty} (j - n + 1)\, |g_j|, \quad V_n^{(2)} := \sum_{j=n}^{\infty} j\,(j - n + 1)\, |g_j|,$$

$n_1 := \min\{n : n \in \mathbf{Z}_\nu, V_n^{(1)} < 1\}$; n_0 is the smallest index $\geq \nu$ such that $g_n \neq -1$ for all $n \geq n_0$ (note that $n_0 \leq n_1$).

An LG theory for Case 3, apparently, was missing in the literature (up to about 1992).

Set $\alpha = -\beta < 0$, $\beta \neq 1$, and assume $\sum^{\infty} |g_n| < \infty$. Then $\exists n_1 \in \mathbf{Z}_\nu$ and two linearly independent solutions to $\Delta^2 y_n + q_n y_n = 0$,

$$y_n^- = (\lambda_-)^n [1 + \varepsilon_n], \; n \geq n_1; \quad y_n^+ \sim (\lambda_+)^n, \; n \to \infty,$$

where $\lambda_\pm = 1 \pm \sqrt{\beta}$ are the roots of the characteristic equation associated to the unperturbed difference equation, and

$$|\varepsilon_n| \leq \frac{V_n}{1 - V_n}, \quad V_n := \frac{1}{\sqrt{\beta}|\sqrt{\beta} - 1|} \sum_{k=n}^{\infty} |g_k|, \quad n \geq n_1,$$

being $n_1 = \min\{n \in \mathbf{Z}_\nu : V_n < 1\}$. When g_n is real, $y_n^\pm$ are real.

An LG theory for Case 3, apparently, was missing in the literature (up to about 1992).
In this case, others than before, *qualitative* asymptotics (but no precise error bounds) could be obtained by Poincarś or Perron's theorems.

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where $\lambda_\pm = 1 \pm \sqrt{\beta}$ are the roots of the characteristic equation associated to the unperturbed difference equation, and

$$|\varepsilon_n| \leq \frac{V_n}{1 - V_n}, \quad V_n := \frac{1}{\sqrt{\beta}|\sqrt{\beta} - 1|} \sum_{k=n}^{\infty} |g_k|, \quad n \geq n_1,$$

being $n_1 = \min\{n \in \mathbf{Z}_\nu : V_n < 1\}$. When g_n is real, $y_n^\pm$ are real.

In JAT '99, we considered the general case of *complex* A_n, B_n, hence L (or a, besides g_n).

7. Applications to orthogonal polynomials (OPs).

Recall: All OPs on the real line with respect to a positive Borel measure satisfy a three-term linear recurrence relation like (e.g., for the monic form)

$$P_{n+2}(x) - (x - \gamma_n)P_{n+1}(x) + \delta_n P_n(x) = 0, \quad n = -1, 0, 1, 2, \dots,$$

$$P_{-1}(x) \equiv 0, \quad P_0(x) \equiv 1,$$

with γ_n real and $\delta_n > 0$. Here x is a fixed parameter, while we are interested in $n \to \infty$.

The "oscillatory class" ($a > 0$) includes several important examples of OPs.

7.1. Legendre polynomials (JCAM '92).

$$A_n \equiv A_n(x) = -\frac{2n+3}{n+2}\,x, \quad B_n = \frac{n+1}{n+2},$$

hence

$$q_n(x) = -1 + \frac{4}{x^2}\frac{(n+1)^2}{(2n+3)(2n+1)}.$$

Here $x \neq 0$, but recall the multiplicative transformation $Y_n = \alpha_n y_n$, with

$$\alpha_n \equiv \alpha_n(x) = \frac{(2n-1)!!}{(2n)!!}\left(\frac{x}{2}\right)^n, \quad \nu = -1, \alpha_0 = 1.$$

Then

$$\lim_{n\to\infty} q_n(x) = -1 + \frac{1}{x^2} =: a(x) \text{ (finite and } > 0 \text{ for } 0 < |x| < 1),$$

$$\sum_{n=-1}^{\infty} |q_n(x) - a(x)| = \frac{1}{x^2}\sum_{n=-1}^{\infty}\frac{1}{(2n+3)(2n+1)} < \infty.$$

The theory we developed in JCAM '92 yields the basis:

$$Y_n^{(j)} = (\operatorname{sgn} x)\,\frac{(2n-1)!!}{(2n)!!}\,\exp\left\{(-1)^{j+1}\, i\, n\,\theta(x)\right\}\,\left[1+\varepsilon_n^{(j)}(x)\right],\ \ j=1,2,$$

for $n \geq n_0(x) := \min\left\{n : (1-x^2)^{-1/2}/2(2n+1) < 1\right\}$ (recall that $g_j(x) := q_j(x) - a(x)$) for each fixed x, with $0 < |x| < 1$, where

$$\left|\varepsilon_n^{(j)}(x)\right| \leq \frac{V_n(x)}{1-V_n(x)}, \quad V_n(x) = \frac{(1-x^2)^{-1/2}}{2(2n+1)},\ \ n \geq n_0(x).$$

Note that $V_n(x) = \mathcal{O}(n^{-1})$ as $n \to \infty$.

Here $n_0(x)$ and the error estimates can be given *uniformly* in x, for x in any given but fixed compact subset K of $(-1,1)$. Then $\varepsilon_n^{(j)}(x)$ can be shown to be continuous in K. The basis functions $Y_n^{(j)}$ are however discontinuous at $x=0$ for all odd n.

We can then represent

$$P_n(x) = c_1(x)Y_n^{(1)}(x) + c_2(x)Y_n^{(2)}(x), \quad n \geq n_0(x),$$

and also

$$= (\operatorname{sgn} x)^n \frac{(2n-1)!!}{(2n)!!} A(x) \left[\cos(n\theta(x) + \delta(x)) + E_n(x)\right].$$

Comparing with Darboux asymptotic formula

$$P_n(\cos\phi) = \left(\frac{2}{\sin\phi}\right)^{1/2} \frac{(2n-1)!!}{(2n)!!} \cos\left[\left(n + \frac{1}{2}\right)\phi - \frac{\pi}{4}\right] + \mathcal{O}\left(n^{-3/2}\right),$$

where $x = \cos\phi$, $0 < \phi < \pi$, we can identify $A(x)$, $\delta(x)$.

The result is

$$|A(x)| = \left(\frac{2}{\sin\phi}\right)^{1/2} = \frac{\sqrt{2}}{(1-x^2)^{1/4}}, \quad \delta(x) = \frac{1}{2}\phi(x) - \frac{\pi}{4} \pmod{\pi},$$

and then the Darboux formula *with* an upper bound for the *absolute error*

$$\frac{(2n-1)!!}{(2n)!!}\,|A(x)| \cdot |E_n(x)|$$

$$\leq \frac{\sqrt{2}}{(1-x^2)^{1/4}}\,\frac{(2n-1)!!}{(2n)!!}\,\frac{1}{2(2n+1)(1-x^2)^{1/2}-1} \quad n \geq n_0(x),$$

which is of order of $n^{-3/2}$. (The *relative error* was already given in terms of $E_n(x)$ above.)

7.2. Pollaczek polynomials (JCAM '92).

We can do something similar for a subfamily of the Pollaczek polynomials $P_n^\lambda(x; a, b, c)$, which contains the ultraspherical polynomials.

The parameters should meet in any case the limitations: $a \geq |b|$, $2\lambda + c > 0$, $c \geq 0$, or $a \geq |b|$, $2\lambda + c \geq 1$, $c > -1$.

We have:

$$A_n \equiv A_n(x) = -2\,\frac{(n + \lambda + a + c + 1)x + b}{n + c + 2}, \quad B_n = \frac{n + 2\lambda + c}{n + c + 2}.$$

We are able to treat the subfamily $P_n^\lambda(x;0,0,c)$, i.e., that with $a=b=0$). With $c=0$ we obtain the ultraspherical polynomials (when $\lambda=1/2$ we recover the Legendre polynomials). We obtain

$$P_n^\lambda(x;0,0,c) = (\operatorname{sgn} x)^n \frac{(\lambda+c)_n}{(c+1)_n} A(x) \left[\cos(n\theta(x)+\delta(x)) + E_n(x)\right].$$

where

$$|E_n(x)| \le |\varepsilon_n^1(x)| \le \frac{V_n}{1-V_n},$$

$$V_n = \frac{|\lambda|\,|1-\lambda|}{(1-x^2)^{1/2}} \sum_{k=n}^{\infty} \frac{1}{|(k+\lambda+c)(k+\lambda+c+1)|}.$$

Again, $A(x)$ and $\delta(x)$ are continuous for $0<|x|<1$.
In the special case $a=b=c=0$ of *ultraspherical* OPs, we can identify $A(x)$ and $\delta(x)$, obtaining

$$A(x) = \frac{2^{1-\lambda}}{(1-x^2)^{\lambda/2}}, \quad \delta(x) = (\operatorname{sgn} x)\left(\lambda\phi - \frac{1}{2}\lambda\pi\right).$$

7.3. Blumenthal-Nevai (or BN) class of OPs (MAA '96).

Again, if explicit *qualitative* representations are available, the LG approach allows to obtain precise error *bounds*.
Recall the oscillatory case (JCAM '92). Assuming A_n and B_n real, hence g_n and α_n real, every solution can be represented as

$$Y_n = A\alpha_n\rho^n[\cos(n\theta + \eta) + E_n], \quad n \geq n_0,$$

A and η being two real parameters, and

$$\rho := |\lambda^+| = (1 + a)^{1/2}, \quad \theta := \arg\, \lambda^+,$$

$$|E_n| \leq |\varepsilon_n^+| = \mathcal{O}(V_n), \quad V_n := [a(1 + a)]^{-1/2} \sum_{k=n}^{\infty} |g_k|,$$

provided that

$$\lim_{n\to\infty} \frac{B_n}{A_n A_{n-1}} =: L = \frac{1}{4}(a + 1) > \frac{1}{4},$$

and

$$(*) \qquad \sum_{k=n}^{\infty} \left| \frac{B_n}{A_n A_{n-1}} - L \right| < \infty.$$

Now we focus on

$$P_{n+2}(x) - (x - \gamma_n)P_{n+1}(x) + \delta_n P_n(x) = 0, \;\; n = -1, 0, 1, 2, \ldots,$$

with $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$, γ_n real, and $\delta_n > 0$.

0.2cm

The "Blumenthal-Nevai class" is the class of monic OPs for which both limits,

$$\lim_{n\to\infty} \gamma_n = \gamma, \quad \lim_{n\to\infty} \delta_n = \delta,$$

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$$\lim_{n\to\infty} \gamma_n = \gamma, \quad \lim_{n\to\infty} \delta_n = \delta,$$

exist and are finite.

This case includes, e.g., Jacobi and Pollaczek OPs.

Details concerning the (monic) Jacobi OPs are very cumbersome (in MAA '96 we used *Mathematica* to express the error control term, $V_n^{(\alpha,\beta)}(x)$, as a function of the parameters α and β).

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We stress that error estimates could be established for Jacobi OPs *also* in the range $-1 < \alpha, \beta < -1/2$.

It can be shown that the smallest and the largest limit-points of the spectrum are

$$\sigma := \gamma - 2\sqrt{\delta}, \quad \tau := \gamma + 2\sqrt{\delta}.$$

Blumenthal had shown that the zeros of the $P_n(x)$ are dense in $[\sigma, \tau]$, and Nevai proved that this interval is a subset of the spectrum.

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Many people then contributed. Here we confine our interest to the asymptotic behavior as $n \to \infty$ for x in the essential spectrum, $[\sigma, \tau]$.

With the previous symbols,

$$L \equiv L(x) = \frac{\delta}{(x-\gamma)^2} > \frac{1}{4} \quad \forall x \in (\sigma, \tau) \setminus \{\gamma\}.$$

Note that $\delta \geq 0$, but $\delta = 0$ would give $L = 0$ rather than $> 1/4$.

Also, being

$$\alpha \equiv \alpha(x) = \prod_{k=\nu}^{n-2} \left(\frac{x-\gamma_k}{2} \right),$$

provided that $x \neq \gamma$, we can choose $\nu = -1$ as long as $x \neq \gamma_k$, for each k; otherwise, we can take $\nu = m + 1$, where $m := \max\{k : \gamma_k = x\}$. In any case, $\nu = \nu(x)$.

The convergence condition (*) above becomes, for the BN class,

$$(**) \qquad \sum_{n=\nu+1}^{\infty} \left| \frac{\delta_n}{(x-\gamma_n)(x-\gamma_{n-1})} - \frac{\delta}{(x-\gamma)^2} \right| < \infty.$$

Also in this case, an asymptotic representation like

$$P_n(x) = A(x) \left(\prod_{k=\nu}^{n-2} \left(\frac{x-\gamma_k}{|x-\gamma|} \sqrt{\delta} \right) \right) \left(\frac{2\sqrt{\delta}}{|x-\gamma|} \right)^{\nu+1} \times$$

$$\times \left[\cos(n\theta(x) + \eta(x)) + E_n(x) \right], \quad n \geq n_0(x),$$

holds, with

$$\theta(x) := \arctan \left[\left(\frac{4\delta}{(x-\gamma)^2} - 1 \right)^{1/2} \right].$$

Setting $x = \gamma + 2\sqrt{\delta}\cos\phi$, $0 < \phi < \pi$, $\pi \neq 0$, the latter becomes $\theta(x) = \phi$ if $\gamma < x < \gamma + 2\sqrt{\delta}$ (i.e., $0 < \phi < \pi/2$), and $\theta(x) = \pi - \phi$ if $\gamma - 2\sqrt{\delta} < x < \gamma$ (i.e., $\pi/2 < \phi < \pi$).

Moreover, the error term above can be estimated as

$$|E_n(x)| \leq \frac{V_n(x)}{1 - V_n(x)},$$

with

$$V_n(x) := \left[\delta\left(\delta - \frac{(x-\gamma)^2}{4}\right)\right]^{-1/2} \sum_{k=n}^{\infty} \left| \frac{\delta_k (x-\gamma)^2}{(x-\gamma_k)(x-\gamma_{k-1})} - \delta \right|.$$

Here $n_0(x)$ is the smallest integer $\geq \nu(x)$ s.t. $V_n(x) < 1$.

Again, whenever the functions $A(x)$ and $\eta(x)$ can be identified, the asy. relation above yields an asy. representation with a *precise error bounds*.

In any case, it provides qualitative information on the asy. behavior of the OPs in the BN subclass characterized by (**).

7.4. Ultraspherical functions of the second kind (JAT '99).

An asymptotic representation with a precise error bound could be obtained for such functions.

They are *recessive* solutions to the difference equation satisfied by the ultraspherical OPs, and play a role in providing error estimates in Gaussian (Gauss-Gegenbauer) quadratures of analytic integrands,

• 8. The LG approximation for *systems* of *difference* equations (done in case of asymptotically constant matrix coefficient; to be completed in case of asymptotically diagonal matrix coefficient.)

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These cases parallel those of systems of ODEs. The case corresponding to the case of asymptotically constant coefficient, as well as that corresponding to the "finite moments perturbation", was studied in JMAA 2008. (The other is in progress.)

Some applications to *orthogonal matrix polynomials* can be done.

• 9. A further application: Numerical solution of rapidly oscillatory ODEs.

In Math. Comp. ’90 and in some Proc.s vol. in ’90, a method was developed to compute zeros of solutions, and actually even a basis of sol.s, to 2nd order linear ODEs.

• 9. A further application: Numerical solution of rapidly oscillatory ODEs.

In Math. Comp. '90 and in some Proc.s vol. in '90, a method was developed to compute zeros of solutions, and actually even a basis of sol.s, to 2nd order linear ODEs.

This method turned to be very effective even whe the sol.s oscillate extremely rapidly.

The method exploits the fact that the equation $y'' + q(x)y = 0$ has a basis of solutions of the form $y_1(x) = |\alpha'(x)|^{-1/2} \sin \alpha(x)$, $y_2(x) = |\alpha'(x)|^{-1/2} \cos \alpha(x)$, where $\alpha(x)$, called *a phase*, is a solution of the so-called Kummer equation,

$$\alpha'^2(x) = q(x) - \frac{1}{2} \{\alpha, x\} .$$

Here

$$\{\alpha, x\} := \frac{\alpha'''(x)}{\alpha'(x)} - \frac{3}{2} \left(\frac{\alpha''(x)}{\alpha'(x)} \right)^2$$

is the "Schwarzian derivative" of $\alpha(x)$ with respect to x.

A theory, developed by O. Borůvka, provides a number of properties of such phase functions. In particular, being $\alpha(x)$ strictly *monotonic*, the oscillatory behavior of every solution is reduced to describing a much better behaved function. In particular, zeros of every solution can be computed efficiently.

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The algorithm we developed allows to exploit efficiently *symbolic manipulators* such as *Mathematica*.

The algorithm:

The algorithm: We first set $\phi := (\alpha')^2$, tranforming the Kummer equation into

$$\phi(x) = q(x) + [\phi, x]\,,$$

where

$$[\phi, x] := -\frac{1}{4}\frac{\phi''(x)}{\phi(x)} + \frac{5}{16}\left(\frac{\phi'(x)}{\phi(x)}\right)^2$$

replaces the Schwarzian. This suggests the *iterative algorithm*

$$\phi_0(x) = q(x), \quad \phi_{n+1}(x) = q(x) + [\phi_n, x]\,,\ n = 0, 1, 2, \ldots,$$

for $\phi_n(x) := (\alpha_n'(x))^2$.

We proved convergence, in a suitable sense, of the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ to an appropriate solution $\phi(x)$. These results concern some classes of coefficients $q(x)$ of the original differential equation, namely when $q(x)$ tends to a positive constant, or to be polynomial, or exponential, as $x \to +\infty$.

A *phase* α is any C^3-solution of the equation $\tan\alpha(x) = v(x)/u(x)$, where (u, v) denotes a basis for the ODE. Such a function has the property that $\alpha'(x) \neq 0$, and one obtains

$$\alpha'(x) = \frac{W}{u^2(x) + v^2(x)},$$

where W is the (constant) Wronksian of (u, v). The Kummer ODE above is a close-form equation satisfied by any α.

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Then, casting the problem in the complex domain (to control derivatives), we can go to te iterative schemse (for $\alpha_n(x)$, or, more conveniently, for $\phi_n(x)$).

It is noteworthy that in 1950 Frank derived an asymptotic-numerical method to compute zeros of certain Special Functions, starting from the equation $y(x, \alpha) := u(x) \cos \alpha - v(x) \sin \alpha = 0$. In particular, he considered the Bessel equation.

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It turns out that the generic zero, say $x = \rho(\alpha)$, of one of such solution can be computed from

$$\rho'^2 = \frac{1}{q(\rho)} \left(1 - \frac{1}{2} \{\rho, \alpha\} \right).$$

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$$\rho'^2 = \frac{1}{q(\rho)} \left(1 - \frac{1}{2} \{\rho, \alpha\} \right).$$

It can be shown that the function $x = \rho(\alpha)$ is the *inverse* of the phase function $\alpha(x)$, which solves Kummer's equation.

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Crucial is the estimate of $[\phi, x] - [\psi, x]$. Note that:

• This is also useful in simplifying the algorithm, e.g., using symbolical manipulations.

• Neglecting $\{\alpha, x\} = -2[\phi, x]$ in the appropriate iterative scheme, being $\phi(x) \approx q(x)$, amounts – loosely speaking – to neglect $[q(x), x]$ ($\approx [\phi(x), x]$), cf. the LG (or WKB) theorem. In fact, this amounts to neglect $|q'(x)/q(x)|$ and $|q''(x)/q(x)|$ with respect to 1.

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and I wished him still a long career.
This was 22 years ago, so it seems that this happened!

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I'd like to renew my wishes now ...

Thank you for your attention!