THE COTES-NEWTON FACTORIZATION OF $x^n \pm 1$

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The Cotes-Newton factorization formula is presented in the first chapter of the *NIST Handbook*. While Frank Olver and I were composing this chapter, I told him that I was researching the provenance and development of this and other key formulas, and he was helpful and encouraging. It was a real pleasure to work with Frank and I offer this bit of historical background in honor of his tremendous achievement in bringing the *NIST Handbook* to fruition in such accurate and accessible form.

Newton was the first mathematician to investigate the factorization of the binomials $x^n \pm 1$. It seems that his study of this very interesting problem arose in the course of his efforts to obtain series for π, similar to Leibniz's famous 1673 series. Note that Leibniz's series, contained in his letter for British mathematicians via Oldenburg, was first discovered by Madhava in the fourteenth century and may be written as

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

In his October, 1676 reply to Leibniz via Oldenburg, Newton presented an interesting variation of the Madhava-Leibniz formula:

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots . \tag{1}$$

In this letter, Newton made a cryptic remark about how he derived the result, but evidence suggests that Leibniz did not understand his meaning. However, it is clear from Newton's notes from around 1676 that he expanded $(1 + x^2)/(1 + x^4)$ as a series and integrated over $(0, 1)$ to obtain the series in (1). He then factored $x^4 + 1$ as

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \tag{2}$$

and applied partial fractions to evaluate

$$\int_0^1 \frac{1 + x^2}{1 + x^4}\, dx$$

as a sum of two arctangents.

In a 1702 paper, Leibniz factored

$$x^4 + a^4 = \left(x + a\sqrt{\sqrt{-1}}\right)\left(x - a\sqrt{\sqrt{-1}}\right)\left(x + a\sqrt{-\sqrt{-1}}\right)\left(x - a\sqrt{-\sqrt{-1}}\right)$$

and then wondered whether integrals such as

$$\int dx/(x^4 + a^4)\,, \int dx/(x^8 + a^8)\,, \text{ etc.}$$

could be evaluated in terms of logarithms and inverse trigonometric functions. This indicates that Leibniz had not deciphered Newton's comment on how he evaluated

(1).

In his factorization of $x^n \pm 1$, Newton used the method of undetermined coefficients; he learned this from his careful study of Descartes. In his old age, Newton described to de Moivre his method of reading Descartes: He followed the text until he came to a difficulty, and then began study from the beginning again; he found that by doing this, he could make progress beyond his stopping point, until he came to a new obstacle, sending him back to the beginning, and so on. Descartes discovered the technique of undetermined coefficients and used it to solve the quartic equation by first factorizing the quartic

$$x^4 + px^2 + qx + r = (x^2 - yx + c)(x^2 + yx + d)\,,$$

and then equating coefficients to show that y^2 satisfied a cubic. In a similar manner, Newton started with the factorization

$$(1 + nx + x^2)(1 - nx + px^2 - qx^3 + rx^4 - \cdots) = 1 \pm x^m\,, \tag{3}$$

when $m = 3, 4, \ldots, 12$. By equating coefficients, he found the algebraic equation satisfied by n as he eliminated the other unknowns p, q, r, etc. For example, when $m = 4$ in (3), Newton had $n^3 - 2n = 0$, so that $n = \pm\sqrt{2}$ when the coefficient of x^4 was $+1$ and $n = 0$ when the coefficient was -1. For the case $m = 8$, Newton gave the equation for n as

$$n^7 - 6n^5 + 10n^3 - 4n = 0$$

and he wrote down the solutions as

$$nn = 2 \quad \& \quad 2 \pm \sqrt{2}.$$

The values $n = \pm\sqrt{2}$ were involved in the factors of $x^8 - 1$ and the values $n = \pm\sqrt{2 + \sqrt{2}}$ were in the factors of $x^8 + 1$. Newton also observed that $2\cos(\pi/8) = \sqrt{2 + \sqrt{2}}$ and drew a figure of a right triangle with an angle of $22\frac{1}{2}$ degrees. Thus, Newton would have required just one more step to obtain the Cotes factorization of $x^n \pm 1$. Another curious thing in Newton's notes is that he wrote down the factors only when the coefficients of the factors could be expressed in terms of square roots. In his comments on Newton's notes, Whiteside reflected on whether Newton had thought about the values of m in $x^m - 1$, whose factors were expressible in terms of quadratic surds. This question is related to Gauss's work on the constructibility of regular polygons. Of course, there is no suggestion that Newton made this connection.

Now Newton did not publish any of this material and we may speculate that Cotes's interest in this factorization problem was aroused by Leibniz and Johann Bernoulli's 1702 papers on the integration of rational functions. They had not succeeded in determining the factors of $x^n + a^n$, needed to evaluate $\int dx/(x^n + a^n)$. In a May 5, 1716 letter to William Jones, Cotes wrote that he had resolved the factorization problem raised by Leibniz's paper. Cotes died shortly after this, before publishing his work. His cousin, Robert Smith, edited Cotes's unpublished manuscripts and had them printed as the *Harmonia Mensurarum* of 1722. Cotes had not given an explicit proof of the factorization formula; Henry Pemberton soon provided a geometric argument. We note that the *Harmonia* also presents without proof the formula

$$\log(\cos\theta + i\sin\theta) = i\theta\,. \tag{4}$$

As described in the *Harmonia*, the formula contains an error in sign; the $i = \sqrt{-1}$ appears on the other side of the equation.

In 1730, de Moivre published an analytic proof of a more general formula expressible in modern form as

$$x^{2n} - (2\cos n\theta)a^n x^n + a^{2n} = \prod_{k=0}^{n-1}\left(x^2 - 2\cos\left(\frac{2k\pi + n\theta}{n}\right)ax + a^2\right)\,. \tag{5}$$

This proof was based on a result de Moivre found in 1707: If l and x are cosines of arcs A and B of the unit circle and $A : B = n : 1$, then

$$x = \frac{1}{2}\sqrt[n]{l + \sqrt{l^2 - 1}} + \frac{1}{2}\frac{1}{\sqrt[n]{l + \sqrt{l^2 - 1}}}\,. \tag{6}$$

In modern form, this may be written as

$$\cos\theta = \frac{1}{2}\left((\cos n\theta + i\sin n\theta)^{1/n} + (\cos n\theta - i\sin n\theta)^{1/n}\right)\,. \tag{7}$$

De Moivre did not provide a verification of this result, but Daniel Bernoulli presented a proof in 1728. First, note de Moivre's derivation of the product formula (5). He set

$$z = \sqrt[n]{l + \sqrt{l^2 - 1}} \quad \text{or} \quad z^{2n} - 2lz^n + 1 = 0\,, \quad \text{where} \quad l = \cos n\theta\,.$$

Then, by (6),

$$x = (z + 1/z)/2 \quad \text{or} \quad z^2 - 2xz + 1 = 0, \quad \text{where} \quad x = \cos\theta\,.$$

So de Moivre concluded that $z^{2n} - 2lz^n + 1 = 0$, when $z^2 - 2xz + 1 = 0$; in other words, $z^2 - 2xz + 1$ was a factor of $z^{2n} - 2lz^n + 1$. To obtain the other $n-1$ factors, de Moivre gave a verbal argument, amounting to the formula

$$(\cos n\theta \pm i\sin n\theta)^{1/n} = \cos\left(\frac{2k\pi \pm n\theta}{n}\right) + i\sin\left(\frac{2k\pi \pm n\theta}{n}\right), \quad k = 0, 1, 2, \ldots\,.$$

This proved his factorization formula and he deduced Cotes's formulas by taking $\theta = 0$ and $\theta = \pi$.

In his 1728 paper on recurrent series, Daniel Bernoulli gave a result from which (6) can easily be derived. By a recurrent series, he meant a sequence $A_0, A_1, A_2, \ldots$ such that

$$a_0 A_{n+k} + a_1 A_{n+k-1} + \cdots + a_k A_n = 0, \quad n = 0, 1, 2, \ldots\,, \tag{8}$$

where $a_0, a_1, \ldots, a_k$ were given constants. Bernoulli provided a method, usually presented in modern textbooks, for solving the difference equation (8). He also introduced the method of writing the general solution as a linear combination of k special solutions. We note in passing that it took a decade of struggle before Euler realized that the same idea could be carried over to linear differential equations with constant coefficients. As an application of his method, Bernoulli considered the sequence $\sin\theta, \sin 2\theta, \sin 3\theta, \ldots$. Note that the addition formula for the sine function implies that for $A_n = \sin n\theta$,

$$A_{n+1} + A_{n-1} = \sin(n+1)\theta + \sin(n-1)\theta = 2\cos\,\theta\,\sin\,n\theta = 2\cos\,\theta A_n\,. \tag{9}$$

Taking $A_x = \lambda^x$ in (9) and dividing by λ^{n-1} produces

$$\lambda^2 - 2\cos\,\theta\,\lambda + 1 = 0\,.$$

The solutions of these equations are

$$\lambda_1 = \cos\,\theta + \sqrt{\cos^2\theta - 1} = \cos\,\theta + \sqrt{-1}\sin\,\theta,$$
$$\lambda_2 = \cos\,\theta - \sqrt{\cos^2\theta - 1} = \cos\,\theta - \sqrt{-1}\sin\,\theta\,;$$

note that $\lambda_1\lambda_2 = 1$. Using this and the initial values, one arrives at Bernoulli's result

$$\sin n\theta = \frac{1}{2\sqrt{-1}}\left((\cos\theta + \sqrt{\cos^2\theta - 1})^n - (\cos\theta + \sqrt{\cos^2\theta - 1})^{-n}\right)\,. \tag{10}$$

Similarly, from the sequence $1, \cos\theta, \cos 2\theta, \ldots$, one may obtain

$$\cos n\theta = \frac{1}{2}\left((\cos\theta + \sqrt{\cos^2\theta - 1})^n + (\cos\theta + \sqrt{\cos^2\theta - 1})^{-n}\right)\,. \tag{11}$$

De Moivre's (6) now follows immediately from (10) and (11). We remark that in 1717 de Moivre had presented his discovery of the method of generating functions to solve the difference equation (8).

Recall that Newton was interested in factorizing $1 + x^m$ to evaluate integrals of the form $\int x^n\,dx/(1+x^m)$ from which one could obtain various series for π. Of course, he was also interested in extending his table of integrals. Leibniz, Johann Bernoulli, and Cotes were also interested in the problem of integrating rational functions, but Euler was the first mathematician to write extensively on this subject. He devoted hundreds of pages to this topic, relating it to the gamma function, partial fractions expansions of trigonometric functions, and series for π and powers of π. In a 1744 paper, he applied his work to the evaluation of the beta integral

$$\int_0^\infty \frac{x^{m-1}\,dx}{1+x^{2n}}, \qquad m < 2n\,. \tag{12}$$

In a posthumous paper of 1785, he considered the more general integral

$$\int_0^\infty \frac{x^{m-1}\,dx}{1 - 2x^k\cos\theta + x^{2k}}\,. \tag{13}$$

Using de Moivre's factorization (5), he expressed the integrand in partial fractions

$$\frac{x^{m-1}}{1 - 2x^k\cos\theta + x^{2k}} = \sum_{s=0}^{k-1} \frac{A_s + B_s x}{1 - 2x\cos\left(\frac{2s\pi+\theta}{k}\right) + x^2}\,, \tag{14}$$

where, with $\omega_s = (2s\pi + \theta)/k$,

$$B_s = \frac{\sin(mw_s - \theta)}{k\sin\theta},\ A_s = -\frac{\sin((m-1)w_s - \theta)}{k\sin\theta}\,.$$

He noted that

$$\int \frac{(A_s + B_s x)\,dx}{1 - 2x\cos w_s + x^2} = \int \left(\frac{B_s(x - \cos\omega_s)}{1 - 2x\cos w_s + x^2} + \frac{(A_s + B_s\cos\omega_s)}{1 - 2x\cos w_s + x^2} \right) dx$$

$$= \frac{1}{2} B_s \log(1 - 2x\cos w_s + x^2) + \frac{A_s + B_s\cos\omega_s}{\sin w_s} \arctan\frac{x\sin w_s}{1 - x\cos w_s}\,. \tag{15}$$

Finally, Euler computed the value of the expression (15) at 0 and ∞ and summed it over s from 0 to $k-1$. The value at 0 may immediately be seen to be 0. However, the sum of the values at ∞ requires skill in dealing with trigonometric sums and Euler devoted several pages to this.

The sums facing Euler were

$$\frac{\log x}{k\sin\theta} \sum_{s=0}^{k-1} \sin(2s\alpha + \zeta)\,, \quad \text{where} \quad \alpha = \frac{m\pi}{k} \quad \text{and} \quad \zeta = \frac{(m-k)\theta}{k}\,,$$

and

$$\frac{1}{k\sin\theta} \sum_{s=0}^{k-1} (\pi - w_s)\cos(mw_s - \theta)\,.$$

His final result, obtained by clever use of the addition formula, was

$$\int_0^\infty \frac{x^{m-1}}{1 - 2x^k\cos\theta + x^{2k}}\,dx = \frac{\pi\sin\frac{m(\pi-\theta)+k\theta}{k}}{k\sin\theta\sin\frac{m\pi}{k}}\,. \tag{16}$$

Euler noted the special cases when $\theta = \pi/2$ and $\theta = \pi$:

$$\int_0^\infty \frac{x^{m-1}}{1 + x^{2k}}\,dx = \frac{\pi}{2k\sin\frac{m\pi}{2k}} \tag{17}$$

and

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1 + x^k)^2} = \frac{(1 - \frac{m}{k})\pi}{k\sin\frac{m\pi}{k}}\,.$$

Of course, the second particular case can be directly obtained from the first. Euler deduced some remarkable partial fractions expansions from (16); for example:

$$\frac{\pi\sin(n\eta/k)}{2k^2\sin(n\pi/k)} = \frac{\sin\eta}{k^2 - n^2} - \frac{2\sin 2\eta}{4k^2 - n^2} + \frac{3\sin 3\eta}{9k^2 - n^2} - \frac{4\sin 4\eta}{16k^2 - n^2} + \cdots\,.$$

This series may also be viewed as the Fourier expansion of the function on the left-hand side.

Dedekind's 1852 doctoral thesis, written under Gauss's supervision, succeeded in streamlining the evaluation of (17). Dedekind noted that his evaluation avoided the tedious computations at ∞ contained in textbooks. He started with the partial fractions expansion

$$\frac{x^{m-1}}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^{n} \frac{\zeta^{(2k-1)m}}{x - \zeta^{2k-1}}\,,$$

where $\zeta = e^{\pi i/n}$. Integration gave him

$$\int_0^\infty \frac{x^{\frac{m}{n}-1}}{x+1}dx = n\int \frac{x^{m-1}}{x^n + 1}\,dx = -\sum_{k=1}^{n} \zeta^{m(2k-1)} \log(\zeta^{2k-1} - x)\,. \tag{18}$$

At $x = 0$, the sum without the negative sign would be

$$\sum_{k=1}^{n} \zeta^{m(2k-1)} \log(\zeta^{2k-1}) = \frac{\pi i}{n} \sum_{k=1}^{n} (2k-1)\zeta^{m(2k-1)} = \frac{\pi}{\sin(m\pi/n)} \,. \tag{19}$$

To evaluate (18) at ∞, Dedekind rewrote the expression as

$$-\sum_{k=1}^{n} \zeta^{m(2k-1)} \log\left(\frac{\zeta^{2k-1}}{x} - 1\right) - \log\, x \sum_{k=1}^{n} \zeta^{m(2k-1)} \,.$$

The sum multiplying $\log x$ is easily seen to be zero. Therefore, the value at ∞ must be

$$-\log(-1) \sum_{k=1}^{n} \zeta^{m(2k-1)} = 0 \,.$$

Thus, (19) yields the value of the beta integral

$$\int_0^\infty \frac{x^{\frac{m}{n}-1}\, dx}{1+x} \,.$$

Notes on the Literature

For Descartes's work on the quartic, see Descartes (1954), pp. 180–192. To read Newton's factorization method, see Newton (1967–81), vol. 4, pp. 205–213. Euler (1911–2000), I-18, pp. 190–208 contains his 1785 paper, discussed above. For Euler's earlier work on integration of rational functions, see I-17. The original source for Cotes's papers is his *Harmonia Mensararum*, Cotes (1722). Gowing (1983) presents a detailed discussion of Cotes's mathematical work; see p. 50 for equation (4) and pp. 67–79 for the factorization formulas. Leibniz (1971), vol. 5, pp. 350–362 contains his 1702 paper on integration of rational functions. For de Moivre's factorization, see the first few pages of de Moivre (1730). Smith (1959), vol. II, pp. 440–454 gives an English translation of the relevant portions of de Moivre's work. See Bernoulli (1982–96), vol. 2, pp. 49–64 for his paper on difference equations.

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