Difference and differential systems for Laguerre-Hahn orthogonal polynomials on the unit circle

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Outline

1. Basic results on orthogonal polynomials on the unit circle (OPUC) and notations.
2. Differential systems for Laguerre-Hahn OPUC and discrete Lax equations
3. Deformations in the Laguerre-Hahn class
    1. Motivation: the semi-classical case
    2. A non-semiclassical example

Basic results on OPUC

Let μ be a positive Borel measure with infinite support on the unit circle $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$ with moments

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} z^{-k} \, d\mu \,, \; k \in \mathbb{Z}, \; z = e^{i\theta}.$$

The *Carathéodory function of* μ

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \,.$$

Definition

Let $\phi_n \in \mathbb{P}$, $\deg(\phi_n) = n \,, \forall n \geq 0$. $\{\phi_n\}$ is said to be a *sequence of orthogonal polynomials on the unit circle* with respect to μ if

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_n(z) \overline{\phi_m(z)} \, d\mu = h_n \delta_{n,m} \,, \; z = e^{i\theta} \,, \quad h_n \geq 0 \,, \; n, m \geq 0 \,. \tag{1}$$

Remark: we shall take ϕ_n monic, i.e, $\phi_n(z) = z^n +$ *lower degree terms*. Then, $\{\phi_n\}$ is said to be the *monic orthogonal polynomial sequence* and it will be denoted by MOPS.

Note that

$$\phi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & \overline{c}_1 & \cdots & \overline{c}_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \cdots & \overline{c}_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \; n \in \mathbb{N}, \; \phi_0(z) = 1, \qquad (2)$$

where Δ_n is the minor of the Toeplitz matrix associated with μ, defined as

$$\Delta_n = \begin{vmatrix} c_0 & \cdots & c_n \\ \vdots & \ddots & \vdots \\ \overline{c}_n & \cdots & c_0 \end{vmatrix}, \; n \in \mathbb{N}, \; \Delta_0 = c_0, \; \Delta_{-1} = 1.$$

The *reflection parameters* $a_n = \phi_n(0), \; n \in \mathbb{N}$, satisfy

$$1 - |a_n|^2 = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}, \; n \in \mathbb{N}. \qquad (3)$$

The reversed polynomial, $f^*(z) = z^n \overline{f}(1/z)\,,\quad n = \deg(f)$.

$$f(z) = \sum_{k=0}^{n} f_k z^k \;\Rightarrow\; f^*(z) = \sum_{k=0}^{n} \overline{f}_{n-k} z^k .$$

Theorem (Szegő recurrence relations)

MOPS satisfy

$$\phi_n(z) = z\phi_{n-1}(z) + a_n \phi_n^*(z)\,,\quad n \geq 0 \tag{4}$$

$$\phi_n^* = \phi_{n-1}^*(z) + \bar{a}_n z \phi_{n-1}(z)\,,\quad n \geq 0\,, \tag{5}$$

$a_n = \phi_n(0),\quad |a_n| < 1\,.$

Conversely,

Theorem (Favard's Theorem on the unit circle)

Given a sequence of monic polynomials $\{\phi_n\}$, $\deg(\phi_n) = n$, *satisfying (4) or (5), there exists only one* μ *such that* $\{\phi_n\} \perp \mu$.

Associated polynomials of the second kind

$$\Omega_0(z) = 1,\ \Omega_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\, d\mu(\theta)\,,\ n \geq 1\,. \qquad (6)$$

Remark: $\{\Omega_n\}$ is an OPS, with $\Omega_n(0) = -\phi_n(0)$.

Matrix form for the recurrence relations

Let $Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}$, $\{\phi_n\}$ MOPS. Then,

$$\begin{cases} Y_n = \mathcal{A}_n Y_{n-1},\ \ \mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n\, z & 1 \end{bmatrix},\ \ n \geq 1 \\ Y_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{cases} \qquad (7)$$

$\mathcal{A}_n$ are the *transfer matrices*.

L. Golinskii and P. Nevai, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Commun. Math. Phys. **223** (2001), 223-259.

Functions of the second kind

$$Q_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\,d\mu(\theta)\,,\ n=0,1,\dots \tag{8}$$

Theorem (Peherstorfer and Steinbauer, 1995)

Let μ be a positive Borel measure, let F_μ be its Carathéodory function, let $\{\phi_n\}$ be a sequence of polynomials with $\deg(\phi_n)=n$.
$\{\phi_n\}\perp\mu$ *and Ω_n are the Second Kind polynomials iff*

$$\begin{cases} \Omega_n(z)+F(z)\phi_n(z)=Q_n(z) \\ \Omega_n^*(z)-F(z)\phi_n^*(z)=Q_n^*(z),\ \forall n\in\mathbb{N}. \end{cases} \tag{9}$$

F. Peherstorfer and R. Steinbauer, *Characterization of orthogonal polynomials with respect to a functional*, J. Comput. Appl. Math. **65** (1995), 339-355.

The Laguerre-Hahn class on the unit circle

Definition

A Carathéodory function F is Laguerre-Hahn if $\exists\ A \neq 0, B, C, D \in \mathbb{P}$:

$$AF' = BF^2 + CF + D$$

$\{\phi_n\} \perp F$ are said to be Laguerre-Hahn OPUC.

Particular cases:

$B = 0$ - Laguerre-Hahn affine class;

$B = 0\,,\ D$ specific polynomial - semi-classical class.

Remark: Laguerre-Hahn class includes linear fractional transformations of Laguerre-Hahn Carathéodory functions.

Examples

Let $\{\phi_n\} \perp F$, $a_n = \phi_n(0)$.

Perturbation of the reflection parameters of OPUC

- $\phi_n^K(0) := a_{n+K}\,,\ K \in \mathbb{N}\,,\ \ n \in \mathbb{N}\,.$

$$\{\phi_n^K\} \perp F^K = \frac{(\Omega_K - \Omega_K^*) + (\phi_K + \phi_K^*)F_\mu}{(\Omega_K + \Omega_K^*) + (\phi_K - \phi_K^*)F_\mu}\,. \tag{10}$$

F semi-classical $\Rightarrow$ F^K Laguerre-Hahn.

- $\phi_n^\lambda(0) := \lambda\, a_n\,,\ |\lambda| = 1\,,\ \ n \in \mathbb{N}\,.$

$$\{\phi_n^\lambda\} \perp F_\lambda = \frac{(\lambda - 1) + (1 + \lambda)F_\mu}{(1 + \lambda) + (\lambda - 1)F_\mu}. \tag{11}$$

F semi-classical $\Rightarrow$ F_λ Laguerre-Hahn.

Modifications of the measure of orthogonality

- $\tilde{\mu} := \mu + \tau\,,\ \tau$ a Bernstein-Szegö measure, $d\tau = \frac{1}{|G(e^{i\theta})|^2}\,d\theta,\ G \in \mathbb{P}$

$$F_{\tilde{\mu}} = \frac{P\,F_\mu + S}{P}\,,\quad P, S \in \mathbb{P}\,. \tag{12}$$

Then, F semi-classical $\Rightarrow$ $F_{\tilde{\mu}}$ Laguerre-Hahn affine.

Differential systems for Laguerre-Hahn OPUC and discrete Lax equations

Theorem

The following equations are equivalent:

$$AF' = BF^2 + CF + D \tag{13}$$

$$\begin{cases} A\phi_n' = (l_{n,1} - C/2)\phi_n + B\Omega_n - \Theta_{n,1}\phi_n^* \\ A\Omega_n' = (l_{n,1} + C/2)\Omega_n - D\phi_n + \Theta_{n,1}\Omega_n^* \\ AQ_n' = (l_{n,1} + BF + C/2)Q_n + \Theta_{n,1}Q_n^* \,, \end{cases} \tag{14}$$

$$\begin{cases} A(\phi_n^*)' = (l_{n,2} - C/2)\phi_n^* - B\Omega_n^* - \Theta_{n,2}\phi_n \\ A(\Omega_n^*)' = (l_{n,2} + C/2)\Omega_n^* + D\phi_n^* + \Theta_{n,2}\Omega_n \\ A(Q_n^*)' = \Theta_{n,2}Q_n + (l_{n,2} + BF + C/2)Q_n^* \,, \end{cases} \tag{15}$$

where $\deg(l_{n,i}, \Theta_{n,i}) \leq \text{máx}\{\deg(A), \deg(B), \deg(C), \deg(D)\}$.

Theorem

Let $Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}$, $\mathcal{Q}_n = \begin{bmatrix} -Q_n \\ Q_n^* \end{bmatrix}$. *Then,*

$$AF' = BF^2 + CF + D \;\Leftrightarrow\; \begin{cases} AY_n' = \mathcal{B}_n Y_n - Y_n\mathcal{C} \quad (\text{Sylvester eq.}) \\ \\ A\mathcal{Q}_n' = \left(\mathcal{B}_n + (BF + C/2)\, I\right) \mathcal{Q}_n \end{cases} \tag{16}$$

$\mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}$.

Furthermore,

$$\mathrm{tr}(\mathcal{B}_n) = nA\,, \quad n \in \mathbb{N}\,,$$

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_1) - A\sum_{k=1}^{n-1} l_{k,2}\,, \quad n \geq 2\,.$$

Consequently,

Theorem

Let w be a weight on the unit circle, $\{\phi_n\} \perp w$. The following statements are equivalent:

(a) w is semiclassical, $\dfrac{w'}{w} = \dfrac{C}{A}$, $C, A \in \mathbb{P}$.

(b)

$$A\tilde{Y}_n' = \left(\mathcal{B}_n - \frac{C}{2}I\right)\tilde{Y}_n\,, \quad \tilde{Y}_n = \begin{bmatrix} \phi_n & -Q_n/w \\ \phi_n^* & Q_n^*/w \end{bmatrix}, \tag{17}$$

where $\mathcal{B}_n$ is the corresponding matrix associated with $AF' = CF + D$.

Magnus, 1995 (for OPRL); Forrester & Witte, 2006 (for OPUC).

Discrete Lax equations

Corollary

Let $AF' = BF^2 + CF + D$, let $\mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix}$ be the corresponding transfer matrices, $a_n = \phi_n(0)$. Then,

$$A\mathcal{A}_n' = \mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1}\,, \;\; n \geq 2\,. \tag{18}$$

Proof:
Compatibility between the recurrence relation $Y_n = \mathcal{A}_n Y_{n-1}$ and

$$zAY_n' = \mathcal{B}_n Y_n - Y_n\mathcal{C}\,.$$

Remark: Eq. (18) produce difference equations for $a_n = \phi_n(0)$.

Deformations in L-H class

Problem: Given a $t-$dependence on A, B, C, D of

$$AF' = BF^2 + CF + D\,,$$

to describe the dynamics of $a_n(t) = \phi_n(0,t)$.

Motivation: Deformations of regular semi-classical weights on the unit circle (Forrester & Witte, 2006) as well as on the real line (Magnus, 1995).

$$\frac{w'}{w} = \frac{C}{A} \quad ('= d/dz) \tag{19}$$

where $\deg(C) < \deg(A), \;\; \deg(A) \geq 2, \;\; A = \prod_{j=1}^{m}(z - z_j(t)), \;\; z_i \neq z_j, \; i \neq j.$

Theorem (A. P. Magnus, 1995)

$\{\phi_n\}$ OPRL, $\alpha_{n+1}p_{n+1} = (x-\beta_n)p_n - \alpha_n p_{n-1}$, $\{\phi_n\}\perp w$, w regular semi-classical,
$\frac{w'}{w} = \frac{C}{A}$, $A = \prod_{j=1}^{m}(z - z_j(t))$.

$$\tilde{Y}_n' = \frac{\mathcal{B}_n - C/2\, I}{A}\tilde{Y}_n\,, \quad \tilde{Y}_n = \begin{bmatrix} p_n & \varepsilon_n/w \\ p_{n-1} & \varepsilon_{n-1}/w \end{bmatrix}, \quad \mathcal{B}_n = \begin{bmatrix} l_n & -\alpha_n\Theta_n \\ -\alpha_n\Theta_{n-1} & -l_n \end{bmatrix}, \tag{20}$$

$$H_n := \frac{\partial}{\partial t}\left(\tilde{Y}_n\right)\tilde{Y}_n^{-1} \tag{21}$$

Then, H_n satisfies

$$\frac{\partial}{\partial t}(\mathcal{B}_n/A) = H_n' + H_n\frac{\mathcal{B}_n}{A} - \frac{\mathcal{B}_n}{A}H_n\,. \tag{22}$$

Therefore,

$$\frac{\dot{\alpha}_n}{\alpha_n} = \frac{1}{2}\sum_{j=1}^{m}\frac{\Theta_n(z_j) - \Theta_{n-1}(z_j)}{A'(z_j)}\dot{z}_j\,, \quad \dot{\beta}_n = \sum_{j=1}^{m}\frac{l_{n+1}(z_j) - l_n(z_j)}{A'(z_j)}\dot{z}_j \tag{23}$$

Remark: (22) imply equations with the Painlevé property, that is, movable singular points can only be poles.

A.P. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comput. Appl. Math. 57 (1995), 215-237.

Theorem (P.J. Forrester and N.S. Witte, 2006)

$\{\phi_n\} \perp w$, w regular semi-classical, $\frac{w'}{w} = \frac{C}{A}$, $A = \prod_{j=1}^{m} (z - z_j(t))$.

$$\tilde{Y}_n' = \frac{\mathcal{B}_n - C/2\, I}{A} \tilde{Y}_n\,, \quad \tilde{Y}_n = \begin{bmatrix} \phi_n & -Q_n/w \\ \phi_n^* & Q_n^*/w \end{bmatrix}, \quad \mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}, \tag{24}$$

$$H_n := \frac{\partial}{\partial t} \left(\tilde{Y}_n \right) \tilde{Y}_n^{-1} \tag{25}$$

Then, H_n satisfies

$$\frac{\partial}{\partial t} (\mathcal{B}_n/A) = H_n' + H_n \frac{\mathcal{B}_n}{A} - \frac{\mathcal{B}_n}{A} H_n\,. \tag{26}$$

Therefore,

$$\frac{\dot{a}_n}{a_n} = \sum_{j=1}^{m} \frac{\dot{z}_j}{z_j} \frac{l_{n-1,1}(z_j) - C/2(z_j)}{A'(z_j)} \tag{27}$$

Proof:
Use

$$H_n = H_{n,\infty} - \sum_{j=1}^{m} \frac{\dot{z}_j}{z - z_j} \Re es\left(\frac{\mathcal{B}_n - C/2\, I}{A}\right)\Big|_{z_j} \tag{28}$$

in (26).

Theorem

Let $F : AF' = BF^2 + CF + D$, where A, B, C, D depend on t, and

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}, \quad Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}, \ \mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}.$$

Let $\mathcal{L}$ (nonsingular) be such that

$$A\mathcal{L}' = \mathcal{C}\mathcal{L}\,. \tag{29}$$

Then, H_n defined by

$$H_n = \dot{Y}_n Y_n^{-1} + Y_n \frac{\mathcal{E}}{A} Y_n^{-1}, \tag{30}$$

with $A\dot{\mathcal{L}} = \mathcal{E}\mathcal{L}$, satisfy

$$\frac{\partial}{\partial t}(\mathcal{B}_n/A) = H_n' + H_n \frac{\mathcal{B}_n}{A} - \frac{\mathcal{B}_n}{A} H_n\,. \tag{31}$$

Furthermore,

$$\dot{\mathcal{A}}_n = H_n \mathcal{A}_n - \mathcal{A}_n H_{n-1}\,, \ \ \mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n\, z & 1 \end{bmatrix}. \tag{32}$$

Lemma

Let $F = \dfrac{\alpha_1 + \beta_1 \tilde{F}}{\alpha_2 + \beta_2 \tilde{F}}$, $\alpha_i, \beta_i \in \mathbb{P}$, *where* $\tilde{F}$ *is semi-classical, associated with a weight* $\tilde{w}$, *let*

$$\tilde{Y}_n = \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ \tilde{\phi}_n^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}, \quad \tilde{\phi}_n \perp \tilde{w} \,. \tag{33}$$

Then,

$$H_n = \tilde{H}_n + \rho\, I, \;\; n \in \mathbb{N}\,, \tag{34}$$

where

$$\tilde{H}_n = \frac{\partial}{\partial t}\left(\tilde{Y}_n\right)\tilde{Y}_n^{-1}, \quad \rho = \frac{1}{2}\frac{\frac{\partial}{\partial t}(\tilde{w})}{\tilde{w}},$$

and I *is the identity matrix. Thus, the transfer matrices corresponding to* F *satisfy*

$$\dot{\mathcal{A}}_n = \tilde{H}_n \mathcal{A}_n - \mathcal{A}_n \tilde{H}_{n-1}\,. \tag{35}$$

Example: the sum of the Jacobi measure and the Lebesgue measure on the unit circle

Let

$$d\mu = w_{\alpha,\beta}\, d\theta + \frac{d\theta}{2\pi} \qquad (36)$$

where

$$w_{\alpha,\beta}(z) = t^{\alpha} z^{-\alpha-\beta}(z+1)^{2\beta_1}(z+1/t)^{2\alpha} \qquad (37)$$

$\beta = \beta_1 + i\beta_2, \beta_1, \beta_2, \alpha, \in \mathbb{R}, 2\beta_1 > -1, 2\alpha > -1, t = e^{i\phi}.$

$w_{\alpha,\beta}$ is real and positive on $\mathbb{T}$ and it is semi-classical, $\frac{w'_{\alpha,\beta}}{w_{\alpha,\beta}} = \frac{C}{A}$, with

$$A(z) = z(z+1)(z+1/t), \quad C(z) = (\alpha+\overline{\beta})z^2 + (\alpha - \beta + \frac{\overline{\beta}-\alpha}{t})z - \frac{\alpha+\beta}{t}. \qquad (38)$$

Remarks:

1. The Carathéodory function of μ is Laguerre-Hahn affine, non semi-classical.

2. The Toeplitz matrix of μ satisfies $\mathbf{T} = I + \mathbf{T}_{w_{\alpha,\beta}}$, where $\mathbf{T}_{w_{\alpha,\beta}}$ is the Toeplitz matrix of $w_{\alpha,\beta}$ and I is the identity matrix.

Differential system for $\{Y_n\} \perp \mu$

Let $\{\phi_n(z) = z^n + \tau_n z^{n-1} + \ldots + \gamma_n z + a_n\} \perp \mu$, let F be its Carathéodory function. Then,

$$AF' = CF + D_1 \,. \tag{39}$$

Thus,

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,, \quad \mathcal{C} = \begin{bmatrix} C/2 & -D_1 \\ 0 & -C/2 \end{bmatrix}, \ \mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}, \tag{40}$$

$$l_{n,1}(z) = (n + \frac{\alpha+\overline{\beta}}{2})z^2 + \xi_{n,1}\, z + \frac{2n+\alpha+\beta}{2t}\,, \tag{41}$$

$$l_{n,2}(z) = -\frac{\alpha+\overline{\beta}}{2} z^2 + \xi_{n,2}\, z - \frac{\alpha+\beta}{2t}\,, \tag{42}$$

$$\Theta_{n,1}(z) = -a_{n+1}(n+1+\alpha+\overline{\beta})z + \frac{a_n}{t}(n+\alpha+\beta)\,, \tag{43}$$

$$\Theta_{n,2}(z) = -\overline{a}_n(n+\alpha+\overline{\beta})z^2 + \frac{\overline{a}_{n+1}}{t}(n+1+\alpha+\beta)z\,, \tag{44}$$

$$\xi_{n,1} = -\tau_n + \frac{2n+\alpha-\beta}{2} + \frac{2n+\overline{\beta}-\alpha}{2t} - \overline{a}_n a_{n+1}(n+1+\alpha+\overline{\beta})\,,$$

$$\xi_{n,2} = \frac{\overline{\gamma_n}}{\overline{a}_n}(n-1+\alpha+\overline{\beta}) + \frac{2n+\alpha-\beta}{2} + \frac{2n+\overline{\beta}-\alpha}{2t} - (n+\alpha+\overline{\beta})\tau_n + \frac{\overline{a}_{n+1}}{\overline{a}_n\, t}(n+1+\alpha+\beta)\,.$$

Difference equations for a_n

Let $\{\phi_n(z) = z^n + \tau_n z^{n-1} + \ldots + \gamma_n z + a_n\} \perp \mu$.
Then,

$$a_{n+1}\overline{a}_n(n+1+\alpha+\overline{\beta})t = a_n\overline{a}_{n+1}(n+1+\alpha+\beta) + a_n\overline{a}_{n-1}(n-1+\alpha+\overline{\beta})t - a_{n-1}\overline{a}_n(n-1+\alpha+\beta)\,, \quad (45)$$

Furthermore,

$$\tau_n = \frac{n+\alpha-\beta}{2} + \frac{n+\overline{\beta}-\alpha}{2t} - \overline{a}_n a_{n+1}(n+1+\alpha+\overline{\beta}) + a_n\overline{a}_{n+1}\frac{n+1+\alpha+\beta}{2t} - a_{n-1}\overline{a}_n\frac{n-1+\alpha+\beta}{2t} + \frac{a_{n+1}(n+1+\alpha+\overline{\beta})}{2a_n} + \frac{a_{n-1}(n-1+\alpha+\beta)}{2ta_n}\,, \quad (46)$$

$$\gamma_n = a_n\overline{\tau}_n\frac{n+1+\alpha+\beta}{n-1+\alpha+\beta} - a_n\frac{n+\alpha-\overline{\beta}}{n-1+\alpha+\beta} - a_n t\frac{n-\alpha+\beta}{n-1+\alpha+\beta} + a_n^2\overline{a}_{n+1}\frac{n+1+\alpha+\beta}{n-1+\alpha+\beta} - a_{n+1}t\frac{n+1+\alpha+\overline{\beta}}{n-1+\alpha+\beta}\,. \quad (47)$$

Derivatives of $a_n(t)$

Let us denote the reflection parameters of the Jacobi weight $w_{\alpha,\beta}$ by $\tilde{a}_n$, $n \geq 1$. The reflection parameters of $\phi_n \perp \mu$ satisfy

$$\dot{a}_n = \frac{1}{t^3}\left(\tilde{a}_n(n+\alpha+\overline{\beta}) + \tilde{a}_{n-1}(n-1+\alpha+\beta)\right)\,,\ n \geq 2\,. \tag{48}$$

Proof: Note that $F = F_{\alpha,\beta} + 1$, where $F_{\alpha,\beta}$ is the Carathéodory function of $w_{\alpha,\beta}$.
Then, taking into account the previous Lemma, use $\tilde{H}_n = \frac{\partial}{\partial t}\left(\tilde{Y}_n^{(\alpha,\beta)}\right)\left(Y_n^{(\alpha,\beta)}\right)^{-1}$, with

$$\tilde{Y}_n^{(\alpha,\beta)} = \begin{bmatrix} \phi_n^{(\alpha,\beta)} & -Q_n^{(\alpha,\beta)}/w_{\alpha,\beta} \\ (\phi_n^{(\alpha,\beta)})^* & (Q_n^{(\alpha,\beta)})^*/w_{\alpha,\beta} \end{bmatrix},\ \phi_n^{(\alpha,\beta)} \perp w_{\alpha,\beta}\,,$$

in

$$\dot{\mathcal{A}}_n = \tilde{H}_n \mathcal{A}_n - \mathcal{A}_n \tilde{H}_{n-1}\,.$$

Then use the expansion of $\tilde{H}_n$ in terms of the residues of $\frac{1}{A}(\tilde{\mathcal{B}}_n - C/2\,I)$ at $z = z_j$, where $A = z(z+1)(z+\frac{1}{t})$,

$$\tilde{H}_n = \tilde{H}_{n,\infty} - \sum_{j=1}^{3} \frac{\dot{z}_j}{z-z_j} \Re es\left(\frac{1}{zA}(\tilde{\mathcal{B}}_n - C/2\,I)\right)|_{z=z_j}\,,$$

with $\tilde{H}_{n,\infty} = \begin{bmatrix} 0 & 0 \\ \dot{\overline{a}}_n & 0 \end{bmatrix}$.

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