Uniform Asymptotic Expansions of Meixner-Pollaczek Polynomials with Varying Parameters

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This is a joint work with J. Wang and R. Wong

Dedicate to Professor Frank Olver for his contributions to the advancement of special functions

Introduction

- In this talk, we concern with the uniform asymptotics of the Meixner-Pollaczek (MP) polynomials as the degree n tends to infinity.
- The Meixner-Pollaczek polynomials were first discovered by Meixner (1934) and later studied by Pollaczek (1950). The major properties were discussed by Chihara (1978), Koekoek and Swarttouw (1998). Certainly, we can find the MP polynomials in DLMF.

Introduction

- The Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ with parameters $\lambda > 0$ and $\phi \in (0,\pi)$ can be defined by the hypergeometric functions

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left({-n, \lambda + ix \atop 2\lambda} ; 1 - e^{-2i\phi} \right).$$

- They are orthogonal on the real line with respect to the weight function

$$w(x;\lambda,\phi) = |\Gamma(\lambda + ix)|^2 \exp\{(\pi - 2\phi)x\},$$

and we have the orthogonality

$$\int_{-\infty}^{+\infty} P_m^{(\lambda)}(x;\phi) P_n^{(\lambda)}(x;\phi) w(x;\lambda,\phi) dx = \frac{\Gamma(n+2\lambda)}{(2\sin\phi)^{2\lambda} n!} \delta_{mn}.$$

Introduction

The asymptotic analysis of the MP polynomials $P_n^{(\lambda)}(x;\phi)$ as $n \to \infty$.

- Y.Chen and M.Ismail (1997) investigated the asymptotic behaviors of the extreme zeros of the MP polynomials, and also the asymptotic distribution of zeros in symmetric case.
- X.Li and R.Wong (2001) obtained an asymptotic expansion of the MP polynomials in terms of the parabolic cylinder functions which is valid uniformly in the interval $[-nM, nM]$ for a given $M > 0$. They also obtained the improved asymptotic behaviors of the zeros.
- I.V.Krasovsky (2003) also investigated the asymptotic distribution of zeros of MP polynomials on the approach of the semiclassical WKB analysis of difference equations.

Introduction

- The aim of our work is to derive asymptotic expansions of the MP polynomials $P_n^{(\lambda)}(z;\phi)$ in the complex plane with varying large parameter λ, say $\lambda = \lambda_n \sim nA$ for some constant $A > 0$.
- The uniform asymptotics of orthogonal polynomials with varying weights was investigated by many authors, e.g. P.Deift and his collaborators for varying exponential weights. Many of these works focused on the weights with a varying large parameters, in particular, on the Laguerre polynomials $L_n^{\alpha_n}$ and Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$.

Introduction

Here are some references.

- C.Bosbach and W.Gawronski (1998), A.B.J.Kuijlaars and K.McLaughlin (2001), A.Aptekarev and R.Khabibullin (2007) et.al. for the Laguerre polynomials;
- C.Bosback and W.Gawronski (1999), A.B.J.Kuijlaars and A.Martinez-Finkelshtein (2004), A.Martinez-Finkelshtein and R.Orive (2005), R.Wong and W.J.Zhang (2006) et.al. for the Jacobi polynomials.
- V.S.Buyarov, J.S.Dehesa, A.Martinez-Finkelshtein and E.B.Saff (1999) discussed the asymptotics of information entropy both for Jacobi and Laguerre polynomials.

Introduction

- Our uniform asymptotic expansions are for $P_n^{(\tau A)}(\tau z, \phi)$ with $\tau = n + \frac{1}{2}$. The result is given as follows.
  In a bounded region (a "rectangle" containing the support of the equilibrium measure), the expansion involves the parabolic cylinder functions;
  In an unbounded region (outside of the "rectangle"), the expansion involves the elementary functions. These two regions are overlapped and the union of them covers the whole plane.
- Our method is the Riemann-Hilbert approach developed by P.Deift and X.Zhou. This powerful method has been already successfully applied in the asymptotic analysis for many orthogonal polynomials.

Fundamental Riemann-Hilbert problem

- Let $\pi_n(z)$ be the monic polynomials of the MP polynomials. Let $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ be the 2×2 matrix-valued function

$$Y(z) = \begin{pmatrix} \pi_n(z) & C[\pi_n w](z) \\ c_n \pi_{n-1}(z) & c_n C[\pi_{n-1} w](z) \end{pmatrix},$$

where $c_n = -2\pi i (2\sin\phi)^{2(n+\lambda-1)}/[(n-1)!\Gamma(n+2\lambda-1)]$, and

$$C[f](z) := \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(x)}{x-z}dx, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

is the Cauchy transform of f.

Fundamental Riemann-Hilbert problem

- From the well-known result of Fokas, Its and Kitaev, $Y(z)$ satisfies the following Riemann-Hilbert problem (RHP):

(Y_a) $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

(Y_b) for $x \in \mathbb{R}$,

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x; \lambda, \phi) \\ 0 & 1 \end{pmatrix};$$

(Y_c) for $z \in \mathbb{C} \setminus \mathbb{R}$ and $z \to \infty$,

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Fundamental Riemann-Hilbert problem

- Now we set $\lambda = \lambda_n = \tau A$ where $\tau = n + \frac{1}{2}$ and $A > 0$, and make a rescale transform

$$U(z) = \begin{pmatrix} \tau^{-n} & 0 \\ 0 & \tau^{n} \end{pmatrix} Y(\tau z).$$

Then $U(z)$ satisfies a RHP similar to $Y(z)$ but with the jump matrix

$$\begin{pmatrix} 1 & w_n(x) \\ 0 & 1 \end{pmatrix},$$

where $w_n(x)$ is the weight function with varying parameter $\lambda = \tau A$, that is,

$$w_n(x) = w(\tau x; \tau A, \phi).$$

Fundamental Riemann-Hilbert problem

- The weight function $w_n(x)$ has an analytic continuation

$$w_n(z) = \Gamma(\tau A + i\tau z)\Gamma(\tau A - i\tau z)\exp\{(\pi - 2\phi)\tau z\}$$

  which has singularities at $z = \pm(k/\tau + A)i$, $(k = 0, 1, 2, \ldots)$.
- The difficult in our arguments is that $w_n(z)$ is quite complicate which involves the Gamma functions.

Equilibrium measures

- The equilibrium measure $\mu_n(x)dx$ related to the weight function $w_n(x)$ is supported on the interval $[\alpha_n, \beta_n]$, where the constants α_n, β_n are known as the Mhaskar-Rakhmanov-Saff numbers (MRS numbers).
- Let

$$G(z) := \frac{1}{\pi i} \int_{\alpha_n}^{\beta_n} \frac{\mu_n(s)}{s-z} ds, \quad z \in \mathbb{C} \setminus [\alpha_n, \beta_n]$$

  be the Cauchy transform of $\mu_n(x)$.
- Then

$$G_+(x) + G_-(x) = -\frac{i}{\pi\tau} \frac{d}{dx} \log w_n(x), \quad x \in (\alpha_n, \beta_n).$$

Equilibrium measures

- Let

$$h(z) = -\frac{d}{dz}\log w_n(z) = i[\psi(\tau A - i\tau z) - \psi(\tau A + i\tau z)] - (\pi - 2\phi),$$

where $\psi(z) = d\log\Gamma(z)/dx = \Gamma'(z)/\Gamma(z)$.

- From the Plemelj formula, we get that

$$G(z) = \frac{\sqrt{(z-\alpha_n)(z-\beta_n)}}{2\pi\tau^2 i}\int_{\alpha_n}^{\beta_n}\frac{h(s)}{\sqrt{(s-\alpha_n)(s-\beta_n)}}\frac{1}{s-z}ds.$$

- Then, the equilibrium measure $\mu_n(x)$ is given by

$$\mu_n(x) = \operatorname{Re} G_+(x), \quad x \in [\alpha_n, \beta_n].$$

Equilibrium measures

- The MRS numbers α_n, β_n can be determined by

$$\int_{\alpha_n}^{\beta_n} \frac{i[\psi(\tau A - i\tau s) - \psi(\tau A + i\tau s)] - (\pi - 2\phi)}{\sqrt{(s-\alpha_n)(s-\beta_n)}} ds = 0,$$
$$\int_{\alpha_n}^{\beta_n} \frac{i[\psi(\tau A - i\tau S) - \psi(\tau A + i\tau s)] - (\pi - 2\phi)}{\sqrt{(s-\alpha_n)(s-\beta_n)}} s ds = 2\tau\pi.$$

- The asymptotic expansions of α_n, β_n as $n \to \infty$ can be obtained by the use of the asymptotic formula of $\psi(z)$

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} z^{-2k},$$

for $|z| \to \infty$ in $|\arg z| < \pi$, where B_n are the Bernoulli numbers.

Equilibrium measures

- The result expansions of α_n, β_n are given by

$$\alpha_n \sim \sum_{j=0}^{\infty} \frac{a_j}{n^j}, \qquad \beta_n \sim \sum_{j=0}^{\infty} \frac{b_j}{n^j},$$

where the first coefficients are

$$a_0 = \frac{(A+1)\cot\phi - \sqrt{2A+1}}{\sin\phi},$$
$$b_0 = \frac{(A+1)\cot\phi + \sqrt{2A+1}}{\sin\phi},$$

and a_j, b_j can be determined iteratively.

Equilibrium measures

- Let $\sigma(z) = \sqrt{(z-\alpha_n)(z-\beta_n)}$, $z \in \mathbb{C} \setminus [\alpha_n, \beta_n]$, $\sigma(z) \sim z, z \to \infty$, $C(x) := (x-\alpha_n)\sqrt{\beta_n^2 + A^2} - (x-\beta_n)\sqrt{\alpha_n^2 + A^2}$, and $D(x) := 2\sqrt{(\beta_n - x)(x-\alpha_n)}\,\mathrm{Im}\,\sqrt{(-iA-\beta_n)(iA-\alpha_n)}]$.
- From $\mu_n(x) = \mathrm{Re}\, G_+(x)$, we can get for $x \in [\alpha_n, \beta_n]$,

$$\mu_n(x) = \frac{1}{2\pi}\log\frac{C(x)+D(x)}{C(x)-D(x)} + \frac{\sqrt{(x-\alpha_n)(\beta_n - x)}}{4\pi\tau}F_n(x),$$

where

$$F_n(x) \sim \frac{1}{(x+iA)\sigma(-iA)} + \frac{1}{(x-iA)\sigma(iA)} + \sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}}{k\tau^{2k-1}}\omega_k(x),$$

$$\omega_k(x) = \frac{1}{(2k-1)!}\left[\frac{i}{(s-x)\sigma(s)}\right]^{(2k-1)}\Bigg|_{iA}^{-iA}.$$

Equilibrium measures

- For the symmetric case $\alpha_n = -\beta_n$ (i.e. $\phi = \pi/2$), the asymptotic behavior of $\mu_n(x)$ reduces to

$$\mu_n(x) \sim \frac{1}{2\pi} \log \frac{\sqrt{{\beta_n}^2 + A^2} + \sqrt{{\beta_n}^2 - x^2}}{\sqrt{{\beta_n}^2 + A^2} - \sqrt{{\beta_n}^2 - x^2}},$$

which is very similar to that already obtained by Y. Chen and M. Ismail in 1997.

The mapping $\phi_n(z)$

As pointed by P.Deift et.al., it is important to introduce an auxiliary functions $\phi_n(z)$ related to the weight function $w_n(z)$ or μ_n.

- Let $$\nu_n(z) = \pi i G(z) + \tfrac{1}{2}h(z),$$ $$z \in \mathbb{C} \setminus ([\alpha_n, \beta_n] \cup \{z = \pm(A + k/\tau)i : k = 0, 1, 2, \dots\}).$$

  Then $\nu_{n,\pm}(x) = \pm\pi i \mu_n(x)$ for $x \in (\alpha_n, \beta_n)$.

- $\nu_n(z)$ has asymptotics

$$\nu_n(z) \sim \frac{i}{2} \log \frac{C(z) + D(z)}{C(z) - D(z)} + \frac{\sigma(z)}{4\tau} F_n(z)$$

  uniform valid in a domain bounded away from cuts $[\alpha_n, \beta_n]$, $[iA, +i\infty)$ and $(-i\infty, -iA]$, where
$$D(z) = -2i \operatorname{Im} \sqrt{(-iA - \beta_n)(iA - \alpha_n)}\sigma(z).$$

The mapping $\phi_n(z)$

- The auxiliary function $\phi_n(z)$ is defined by

$$\phi_n(z) = \int_{\beta_n}^{z} \nu_n(s)ds,$$

which is analytic on $\mathbb{C} \setminus ((-\infty, \beta_n] \cup [iA, i\infty) \cup (-i\infty, -iA])$.

- Symmetrically, the function $\tilde{\phi}_n(z)$ is

$$\tilde{\phi}_n(z) = \int_{\alpha_n}^{z} \nu_n(s)ds,$$

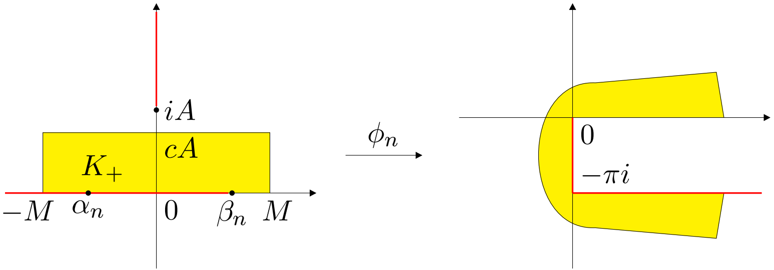
$$z \in \mathbb{C} \setminus ([\alpha_n, \infty) \cup [iA, i\infty) \cup (-i\infty, -iA]).$$

The mapping $\phi_n(z)$

- Given $0 < c < 1$ and $M > \max\{|\alpha_n|, |\beta_n|\}$, define the rectangle $K = K(c, M) = \{z \in \mathbb{C} : |\operatorname{Re} z| < M, |\operatorname{Im} z| < cA\}$, and $K_\pm$ the upper and lower half of K.
- The mapping properties of $\phi_n(z)$ on the real axis:
  - If $x \in [\beta_n, \infty)$, then $\phi_n(x) \in [0, \infty)$, and when x moves from ∞ to β_n, $\phi_n(x)$ moves from ∞ to 0 decreasingly.
  - If $x \in [\alpha_n, \beta_n]$, then $\phi_{n,+}(x) \in [-i\pi, 0]$, and when x moves from β_n to α_n, $\phi_{n,+}(x)$ moves from 0 to $-i\pi$ monotonically.
  - If $x \in (-\infty, \alpha_n]$, then $\phi_{n,+}(x) \in [-i\pi, \infty - i\pi)$, and when x moves from α_n to $-\infty$, $\phi_{n,+}(x)$ moves from $-i\pi$ to $\infty - i\pi$ increasingly.

The mapping $\phi_n(z)$

- There is $0 < c < 1$, for any $M > \max\{|\alpha_n|, |\beta_n|\}$, $\phi_n(z)$ is a one-to-one mapping from the upper-half rectangle $K_+ = K_+(c, M)$ to a region in $\mathbb{C} \setminus \{z : \mathrm{Re}\, z \geq 0, -\pi \leq \mathrm{Im}\, z \leq 0\}$.



- Symmetrically on the lower half rectangle K_-.

Asymptotics of $U(z)$ outside of K

Now we can follow from the standard arguments of the Riemann-Hilbert approach by a series transformations.

- $U \to T$: the normalization of $U(z)$ at infinity by using the logarithm potential of equilibrium measure,
- $T \to S$: the matrix decomposition and the contour deformation,
- S has an approximation S_∞ which satisfies a solvable RHP.

Solving this limit RHP, we can get the asymptotic behavior of $U(z)$ outside of a neighborhood of $[\alpha_n, \beta_n]$ (e.g. outside of the rectangle K).

Asymptotics of $U(z)$ outside of K

- The result asymptotic behavior of $U(z)$ outside of K is given by

$$U(z) \sim e^{\frac{1}{2}\tau \ell_n \sigma_3} \tilde{V}_{out}(z) w_n(z)^{-\frac{1}{2}\sigma_3}, \quad z \in \mathbb{C} \setminus (K \cup \mathbb{R}),$$

- where

$$\tilde{V}_{out}(z) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ \frac{-i(2z-\alpha_n-\beta_n)}{\beta_n-\alpha_n} & -2i \end{pmatrix} b_n(z)^{-\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} e^{-\tau\phi_n(z)\sigma_3},$$

  - $b_n(z) = [(z-\alpha_n)(z-\beta_n)]^{1/4}/\sqrt{\beta_n - \alpha_n}$ for $z \in \mathbb{C} \setminus (-\infty, \beta_n]$,
  - the constants $\ell_n \sim 2A \log \tau$
  - and σ_3 is Pauli's matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Asymptotics of $U(z)$ inside in K

To obtain the asymptotic behavior of $U(z)$ inside in K, we need to construct a parametrix $V(z) = \tilde{V}_{in}(z)$ such that

- it satisfies the jump condition $V_+(x) = V_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$,
- and it has asymptotic behavior like $\tilde{V}_{out}$ on the boundary of K (matching condition).

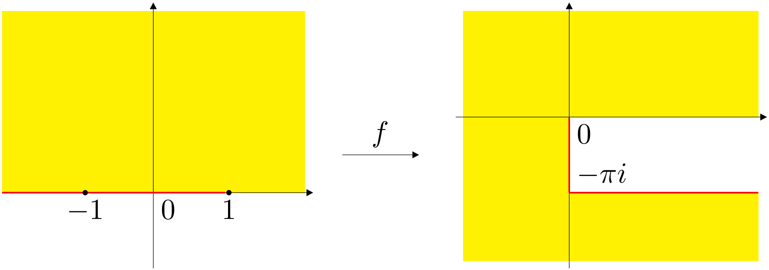
The mapping properties of $\phi_n(z)$ invokes us to construct our approximate solution by using *the parabolic cylinder function*.

Asymptotics of $U(z)$ inside in K

From F. Olver's significant work on the asymptotics of the parabolic cylinder functions $U(-\tau, 2\sqrt{\tau}\xi)$ as $\tau \to \infty$, we introduce the function

$$f(\xi) = \xi\sqrt{\xi^2 - 1} - \log(\xi + \sqrt{\xi^2 - 1}), \quad \xi \in \mathbb{C} \setminus (-\infty, 1].$$

This is a one-to-one mapping from upper half plane $\mathbb{C}^+$ to the region $\mathbb{C} \setminus \{z : \operatorname{Re} z \geq 0, -\pi \leq \operatorname{Im} z \leq 0\}$.



Asymptotics of $U(z)$ inside in K

- Combining $f(\xi)$ with the mapping properties of the auxiliary function $\phi_n(z)$, we establish a one-to-one mapping between $\xi \leftrightarrow z$ defined by

$$f(\xi(z)) = \phi_n(z), \quad \text{or equiv.} \quad \xi(z) = f^{-1} \circ \phi_n(z),$$

  for $z \in K$.
- This mapping maps the rectangle K to a neighborhood of $[-1, 1]$, and $\xi(\alpha_n) = -1, \xi(\beta_n) = 1$.

Asymptotics of $U(z)$ inside in K

To construct the parametrix satisfying the jump condition, we use the connection formula for the parabolic cylinder functions

$$\sqrt{2\pi}U(a,\pm x)$$
$$=\Gamma(\frac{1}{2}-a)\{e^{-i\pi(\frac{1}{2}a+\frac{1}{4})}U(-a,\pm ix)+e^{i\pi(\frac{1}{2}a+\frac{1}{4})}U(-a,\mp ix)\}.$$

This yields the matrix equation (jump relation)

$$\begin{pmatrix} U(-\tau,2\sqrt{\tau}\xi) & \frac{n!}{\sqrt{2\pi}i^n}U(\tau,-2i\sqrt{\tau}\xi) \\ \frac{1}{\sqrt{\tau}}U'(-\tau,2\sqrt{(\tau)}\xi) & \frac{n!}{\sqrt{2\pi\tau}i^{n+1}}U'(\tau,-2i\sqrt{\tau}\xi) \end{pmatrix}=$$
$$\begin{pmatrix} U(-\tau,2\sqrt{\tau}\xi) & -\frac{n!i^n}{\sqrt{2\pi}}U(\tau,2i\sqrt{\tau}\xi) \\ \frac{1}{\sqrt{\tau}}U'(-\tau,2\sqrt{(\tau)}\xi) & -\frac{n!i^{n+1}}{\sqrt{2\pi\tau}}U'(\tau,2i\sqrt{\tau}\xi) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Asymptotics of $U(z)$ inside in K

- To match the behavior of $\tilde{V}_{out}$ on ∂K, we use Olver's uniform asymptotic approximation of the parabolic cylinder functions:

$$U(-\tau, 2\sqrt{\tau}\xi) \sim \frac{1}{\sqrt{2}}\tau^{\frac{\tau}{2}-\frac{1}{4}}e^{-\frac{\tau}{2}}\frac{1}{[\xi^2(z)-1]^{1/4}}e^{-\tau\phi_n(z)},$$
$$U(\tau, -2i\sqrt{\tau}\xi) \sim \frac{i^{n+1}}{\sqrt{2}}\tau^{-\frac{\tau}{2}-\frac{1}{4}}e^{\frac{\tau}{2}}\frac{1}{[\xi^2(z)-1]^{1/4}}e^{\tau\phi_n(z)},$$

  uniformly for $\xi \in \mathbb{C} \setminus (-\infty, 1]$ ($z \in \mathbb{C} \setminus (-\infty, \beta_n]$).
- $U'(-\tau, 2\sqrt{\tau}\xi)$ and $U'(\tau, -2i\sqrt{\tau}\xi)$ have the corresponding approximations.

Asymptotics of $U(z)$ inside in K

- Insert above asymptotic approximations into the matrix equation for jump relation, and comparing it with $\tilde{V}_{out}(z)$, we construct the parametrix for $z \in K_+$

$$\tilde{V}_{in}(z) = \frac{1}{\sqrt{2}} \tau^{-\frac{\tau}{2}+\frac{1}{4}} e^{\frac{\tau}{2}} \begin{pmatrix} 1 & 0 \\ \frac{-i(2z-\alpha_n-\beta_n)}{\beta_n-\alpha_n} & -2i \end{pmatrix} \left(\frac{(\xi^2-1)^{1/4}}{b_n(z)} \right)^{\sigma_3} \cdot \begin{pmatrix} U(-\tau, 2\sqrt{\tau}\xi) & \frac{n!}{\sqrt{2\pi} i^n} U(\tau, -2i\sqrt{\tau}\xi) \\ \frac{1}{\sqrt{\tau}} U'(-\tau, 2\sqrt{(\tau)}\xi) & \frac{n!}{\sqrt{2\pi\tau} i^{n+1}} U'(\tau, -2i\sqrt{\tau}\xi) \end{pmatrix}.$$

- Similar construction of the parametrix can be given for $z \in K_-$.

Asymptotic expansions of $U(z)$

- Define

$$\tilde{U}(z) = \begin{cases} e^{\frac{1}{2}\tau \ell_n \sigma_3} \tilde{V}_{in}(z) w_n(z)^{-\frac{1}{2}\sigma_3}, & z \in K \setminus \mathbb{R}, \\ e^{\frac{1}{2}\tau \ell_n \sigma_3} \tilde{V}_{out}(z) w_n(z)^{-\frac{1}{2}\sigma_3}, & z \in \mathbb{C} \setminus (K \cup \mathbb{R}). \end{cases}$$

  We have formally that $U(z) \sim \tilde{U}(z)$.

- To give a rigorous prove, and to obtain the asymptotic expansion, we define the matrix

$$S(z) = e^{-\frac{1}{2}\tau \ell_n \sigma_3} U(z) \tilde{U}^{-1}(z) e^{\frac{1}{2}\tau \ell_n \sigma_3}.$$

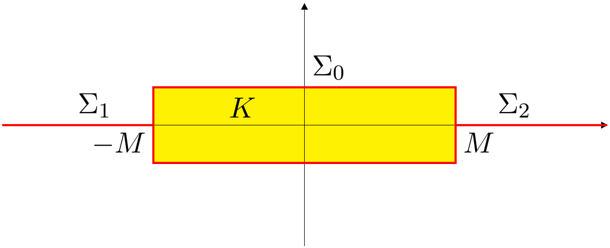
Asymptotic expansions of $U(z)$

It is easy to verify that $S(z)$ is the solution of the following RHP:

(S_a) $S(z)$ is analytic in $\mathbb{C} \setminus \Sigma$, where $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \partial K$, $\Sigma_1 = (-\infty, -M]$ and $\Sigma_2 = [M, \infty)$;

(S_b) $S_+(\zeta) = S_-(\zeta) J_S(\zeta)$ for $\zeta \in \Sigma$;

(S_c) $S(z) \sim I + O(1/z)$ as $z \in \mathbb{C} \setminus \Sigma$ and $z \to \infty$.



Asymptotic expansions of $U(z)$

- Applying the uniform asymptotic expansions of the parabolic cylinder functions $U(-\tau, 2\sqrt{\tau}\xi)$, $U(\tau, -2i\sqrt{\tau}\xi)$, etc., the jump matrix $J_S(\zeta)$ has an asymptotic expansion on the contour Σ:

$$J_S(\zeta) \sim I + \sum_{m=1}^{\infty} \frac{J_S^{(m)}(\zeta)}{(2\tau)^m}, \quad \zeta \in \Sigma_0,$$

$$J_S(x) \sim I + O(e^{-cn^{1/4}}), \quad x \in \Sigma_1 \cup \Sigma_2.$$

- The coefficients $J_S^{(m)}(\zeta)$ can be determined by the coefficients of expansions of the parabolic cylinder functions.

Asymptotic expansions of $U(z)$

- From the expansion of J_S on Σ, we can prove that the solution $S(z)$ of RHP $(S_a)-(S_c)$ also has a uniform asymptotic expansion:

$$S(z) \sim I + \sum_{m=1}^{\infty} \frac{S^{(m)}(z)}{(2\tau)^m},$$

  where the coefficients $S^{(m)}(z)$ can be determined recursively.

- Then we obtain

$$U(z) \sim e^{\frac{1}{2}\tau\ell_n\sigma_3}\left[I + \sum_{m=1}^{\infty} \frac{S^{(m)}(z)}{(2\tau)^m}\right] e^{-\frac{1}{2}\tau\ell_n\sigma_3}\tilde{U}(z).$$

- Take the (1,1)-entry, we get the uniform asymptotic expansion of $\pi_n(\tau z)$.

Asymptotic expansions of $\pi_n(\tau z)$

- In the rectangle K, we have the uniform asymptotic expansion

$$\pi_n(\tau z) = \frac{1}{\sqrt{2}}\, e^{\frac{\tau}{2}(\ell_n+1)}\tau^{\frac{n}{2}} w_n(z)^{-\frac{1}{2}}\big[\, U(-\tau, 2\sqrt{\tau}\xi(z))\, A(z,n) + U'(-\tau, 2\sqrt{\tau}\, \xi(z))\, B(z,n)\,\big]$$

where $A(z,n)$ and $B(z,n)$ are analytic functions of z, and

$$A(z,n) \sim \frac{(\xi^2-1)^{\frac{1}{4}}}{b_n(z)}\left[1 + \sum_{k=1}^{\infty} \frac{A_k(z)}{\tau^k}\right],$$

$$B(z,n) \sim \frac{b_n(z)}{(\xi^2-1)^{\frac{1}{4}}} \sum_{k=1}^{\infty} \frac{B_k(z)}{\tau^{k+\frac{1}{2}}}.$$

Asymptotic expansions of $\pi_n(\tau z)$

- Outside of K, we have the uniform asymptotic expansion

$$\pi_n(\tau z) \sim \frac{1}{2}\tau^n e^{\frac{\tau}{2}\ell_n} b_n(z)^{-1} w_n(z)^{-\frac{1}{2}} e^{-\tau\phi_n(z)} \left[1 + \sum_{k=1}^{\infty} \frac{C_k(z)}{\tau^k}\right]$$

Asymptotic expansions of $\pi_n(\tau z)$

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Thank you for your attention.