Zeros of Special Functions: uses of continuity and analyticity with respect to parameters

Martin E. Muldoon
York University, Canada

Special Functions in the 21st Century:
Theory & Applications - Washington, DC
Dedicated to Frank W. J. Olver

April 6–8, 2011

Around 1964, I was a graduate student struggling with a major part of my dissertation involving an approximation in a turning point problem. Since I had to integrate the approximation, I had to be particularly careful about the error. There was no shortage of discussion of this problem in the literature. But I was not completely comfortable with the O-symbols and uniformity statements in these articles. I was very fortunate to find Olver's [12]:

J. Soc. Indust. Appl. Math.
Vol. 11, No. 3, September, 1963
Printed in U.S.A.

ERROR BOUNDS FOR FIRST APPROXIMATIONS IN TURNING-POINT PROBLEMS*

F. W. J. OLVER†

1. Introduction and summary. In this paper we consider approximate solutions of the differential equation

$$\frac{d^2w}{dx^2} = \{u^2 p(u, x) + q(u, x)\}w, \tag{1.01}$$

in which u is a large parameter. We suppose that $p(u, x)$ and $q(u, x)$ are free from singularities in the x-region considered, and that $p(u, x)$ has there a simple zero, a so-called *turning-point* or *transition-point* of the differential equation. It is well known that in these circumstances approximate solutions can be expressed in terms of Airy functions. The asymptotic nature of these approximations as $|u| \to \infty$ has been investigated by many writers, particularly Langer [1, 2], Cherry [3], Jeffreys [4], Erdélyi [5, 6] and

Error Bounds

differential equation

$$\frac{d^2w}{dx^2} = \{x + f(x)\}w \tag{3.01}$$

has solutions $w_1(x)$, $w_2(x)$, *such that*

$$w_1(x) = \mathrm{Ai}(x) + \epsilon_1(x), \qquad w_1{}'(x) = \mathrm{Ai}'(x) + \eta_1(x), \tag{3.02}$$

$$w_2(x) = \mathrm{Bi}(x) + \epsilon_2(x), \qquad w_2{}'(x) = \mathrm{Bi}'(x) + \eta_2(x), \tag{3.03}$$

where

$$\begin{aligned} |\,\epsilon_1(x)\,| &\leqq \lambda_1^{-1}\{e^{\lambda_1 F_1(x)} - 1\}E^{-1}(x)M(x), \\ |\,\eta_1(x)\,| &\leqq \lambda_1^{-1}\{e^{\lambda_1 F_1(x)} - 1\}E^{-1}(x)N(x), \end{aligned} \tag{3.04}$$

$$\begin{aligned} |\,\epsilon_2(x)\,| &\leqq (\lambda_2/\lambda_1)\{e^{\lambda_1 F_2(x)} - 1\}E(x)M(x), \\ |\,\eta_2(x)\,| &\leqq (\lambda_2/\lambda_1)\{e^{\lambda_1 F_2(x)} - 1\}E(x)N(x); \end{aligned} \tag{3.05}$$

Error bounds

What was remarkable about Olver's article was the clarity of the statements about error bounds [12, (3.04), (3.05)]. The properties of functions F_1, F_2, M and E were treated in detail. They could be thought of as "special functions" in their own right. This was exactly what I needed and I was able to complete the related parts of my dissertation.

SInce then, I have had many opportunities to enjoy and benefit from Frank's written work. I have also benefitted from his tremendous support for some of my own work. Thanks, Frank!

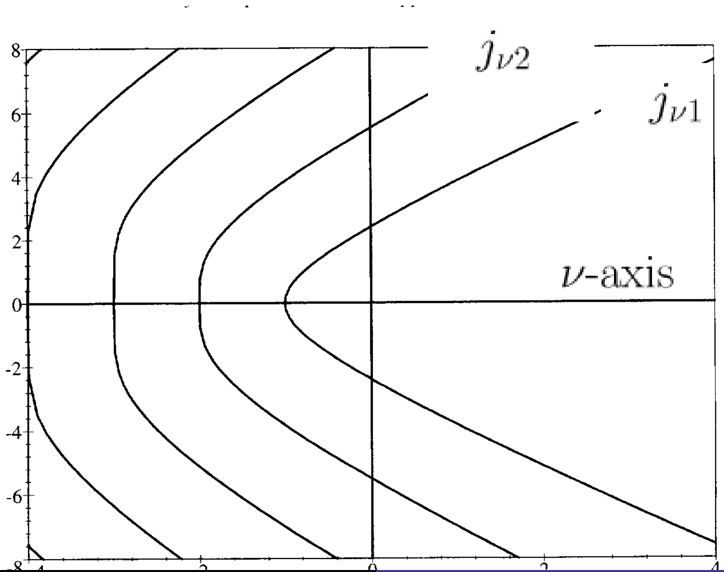
Bessel function $J_\nu(x)$

In case $\nu > -1$,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$$

has an infinite set of positive zeros. $j_{\nu k},\ k = 1, 2, \ldots$.
$x^{-\nu} J_\nu(x)$ is even. For individual values of ν, the form of the graph of $J_\nu(x)$ is well-known [14, Fig. 10.3.1].
There is a diagram in [18, p. 510] that gives a good idea of the behaviour of the zeros *as functions of* ν note that $j_{\nu k} \to 0$ as $\nu \to -k$:

"Watson" diagram



Cylinder functions

We also consider the Bessel function $Y_\nu(x)$ with positive zeros $y_{\nu k}$, $k = 1, 2, \ldots$. **Cylinder functions** are linear combinations of $J_\nu(x)$ and $Y_\nu(x)$:

$$C_\nu(x, \alpha) = \cos\alpha J_\nu(x) - \sin\alpha Y_\nu(x)$$

We use the notation $c(\nu, \alpha, k)$ for a zero of $C_\nu(x, \alpha)$, k representing the rank (first, second, etc.)
How does $c(\nu, \alpha, k)$ vary with ν, k, α?

- $dc/d\nu$ – Watson 1922 [18, p. 508]
- $dc/d\alpha$ – Olver 1950 [10, (2.12)]
- dc/dk – Elbert 2001 [2, (1.4)]

Variation of $c(\nu, \alpha, k)$ with ν

G. N. Watson (1922)

$$\frac{dc}{d\nu} = 2c \int_0^\infty K_0(2c\,\mathrm{sinh} t) e^{-2\nu\, t}\, dt$$

where

$$K_0(x) = \int_0^\infty e^{-x \mathrm{cosh} t}\, dt.$$

Á. Elbert (1977) used this formula to show that $j_{\nu k}$ is a concave increasing function of ν on $-k < \nu < \infty$.
Á. Elbert and A. Laforgia use this formula very effectively during the 1980s and 1990s to get inequalities and other properties for the zeros of Bessel functions. See [2] for references.

Olver [10], variation with α:

A NEW METHOD FOR THE EVALUATION OF ZEROS OF BESSEL FUNCTIONS AND OF OTHER SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

By F. W. J. OLVER

Communicated by E. T. Goodwin

Received 10 March 1950

1. *Introduction*. Methods for the evaluation of zeros of the Bessel functions J, Y ıve been described in detail in a series of Notes by Bickley, Miller and Jones (1). In e course of work for the British Association Mathematical Tables Committee, they ıcounted difficulties in computing these zeros for functions of orders other than zero ıd unity. The two simplest methods of computation are inverse interpolation where e functions are tabulated at a small interval, and the McMahon expansion for the rger zeros. To the accuracy required, the ranges of applicability of these methods ·erlap for the functions of zero and unit order, but after these a gap appears and idens steadily with increasing order.

Attempts to close this gap were made in the series of Notes referred to above, in

Variation of $c(\nu, \alpha, k)$ with α

F. W. J. Olver (1950): linearly independent solutions w_1, w_2 of

$$\frac{d^2 w}{dz^2} + q(z) w = 0,$$

with $q(z)$ analytic.

$$C(z, \alpha) = \cos\alpha \; w_1(z) - \sin\alpha \; w_2(z)$$

with a zero $\rho(\alpha)$. Then with $' = d/d\alpha$, $\rho(\alpha)$ satisfies

$$2\rho'\rho''' - 3\rho''^2 + 4q(\rho)\rho'^4 - 4\rho'^2 = 0.$$

Can be approximated by $q(\rho)\rho'^4 - \rho'^2 = 0$, when $\rho(\alpha)$ varies slowly with α.
This observation was the basis for a successful method of finding the zeros of Bessel functions in the "gap region" where neither methods based on series nor those based on asymptotic expansions were effective.

Reprinted from the *Proceedings of the Cambridge Philosophical Society*,
Volume 47, Part 4, pp. 699–712, 1951.

A FURTHER METHOD FOR THE EVALUATION OF ZEROS OF BESSEL FUNCTIONS AND SOME NEW ASYMPTOTIC EXPANSIONS FOR ZEROS OF FUNCTIONS OF LARGE ORDER

By F. W. J. OLVER

Communicated by E. T. Goodwin

Received 4 December 1950

1. *Introduction.* In a recent paper (1) I described a method for the numerical evaluation of zeros of the Bessel functions $J_n(z)$ and $Y_n(z)$, which was independent of computed values of these functions. The essence of the method was to regard the zeros ρ of the cylinder function

$$\mathscr{C}_n(z) \equiv J_n(z)\cos\pi t - Y_n(z)\sin\pi t, \qquad (1\cdot 1)$$

as a function of t and to solve numerically the third-order non-linear differential equation satisfied by $\rho(t)$. It has since been successfully used to compute ten-decimal values of $j_{n,s}$, $y_{n,s}$, the sth positive zeros* of $J_n(z)$, $Y_n(z)$ respectively, in the ranges

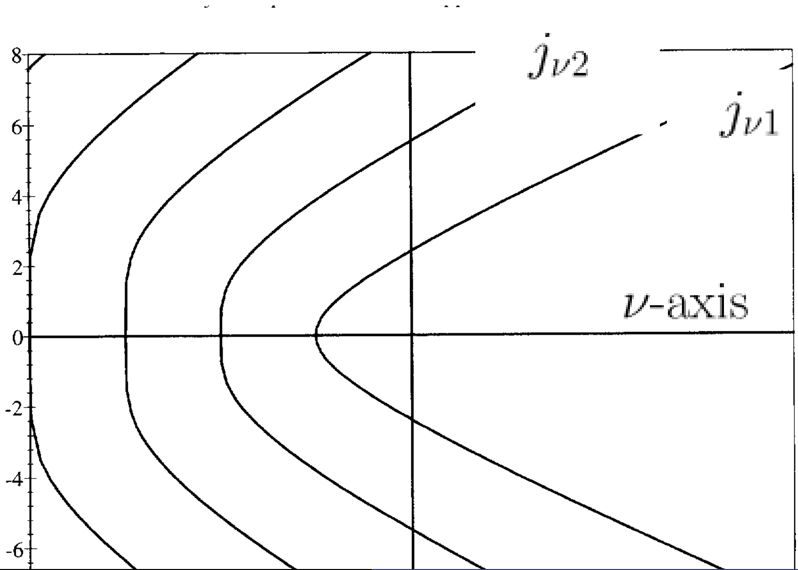
Olver 1951 (cont.)

values of $j_{n,s}, y_{n,s}$, the sth positive zeros* of $J_n(z)$, $Y_n(z)$ respectively, in the ranges $n = 10\,(1)\,20$, $s = 1\,(1)\,20$. During the course of this work it was realized that the least satisfactory feature of the new method was the time taken for the evaluation of the first three or four zeros in comparison with that required for the higher zeros; the direct numerical technique for integrating the differential equation satisfied by $\rho(t)$ becomes unwieldy for the small zeros and a different technique (described in the same paper) must be employed. It was also apparent that no mere refinement of the existing methods would remove this defect and that a new approach was required if it was to be eliminated. The outcome has been the development of the method to which the first part (§§ 2–6) of this paper is devoted.

The basis of the new method is the integration of an integro-differential equation. Previously ρ was regarded as a function of t for fixed n. This situation is now reversed; keeping t fixed, we consider the integration of the equation satisfied by ρ as a function of n, taken as a continuous positive variable. This equation is given by Watson (2) in another context but not in a form suitable for immediate numerical application. In order to use his result it is necessary to derive an asymptotic expansion of the equation.

The method has been extended to include the determination of the values of the derivative $\mathscr{C}_n'(z)$ at the zeros of $\mathscr{C}_n(z)$, and also to the determination of the zeros of $\mathscr{C}_n'(z)$ and the corresponding stationary values of $\mathscr{C}_n(z)$. One advantage of these methods is their production of zeros for half-odd-integer orders at the same time as for integer

Variation of $c(\nu, \alpha, k)$ with α: continuous rank



Continuous rank (cont.)

$$y(x,\alpha) = \cos\alpha\, y_1(x) - \sin\alpha\, y_2(x)$$

The change of variables $y(x) = [p(x)]^{1/2}u(t),\ x'(t) = p(x)$ transforms (1) into the trigonometric equation $u''(t) + u(t) = 0$ (see, e.g., [8, Lemma 2.3]) with general solution $u(t) = A\sin(t + B)$. Hence the general solution of (1) is given by

$$y(x) = A[p(x)]^{1/2}\sin\left(\int_a^x \frac{dt}{p(t)} + B\right).$$

We may redefine

$$y_1(x) = \sqrt{p(x)}\sin\left(\int_a^x \frac{du}{p(u)}\right),$$

$$y_2(x) = -\sqrt{p(x)}\cos\left(\int_a^x \frac{du}{p(u)}\right).$$

Continuous rank (cont.)

The zeros of $y_1(x)$ on (a, b) are the (finitely or infinitely many) numbers x_k for which

$$\int_a^{x_k} \frac{dt}{p(t)} = k\pi, \quad k = 1, 2, \ldots .$$

We define a function $x(\kappa)$ of the continuous variable κ by

$$\int_a^{x(\kappa)} \frac{dt}{p(t)} = \kappa\pi, \; 0 \leq \kappa < \infty.$$

This idea is due to J. Vosmanský [17].

Continuous rank (cont.)

For positive integer values of κ, $x(\kappa)$ is a zero of y_1. For each nonintegral value of κ, $x(\kappa)$ is a zero of some solution of (1) other than y_1. In fact, for $0 < \alpha < \pi$, the solution

$$y(x, \alpha) = \cos \alpha y_1(x) - \sin \alpha y_2(x) = [p(x)]^{1/2} \sin \left(\int_a^x \frac{dt}{p(t)} + \alpha \right)$$

has its zeros where

$$\int_a^x \frac{dt}{p(t)} = (k - \alpha/\pi)\pi,$$

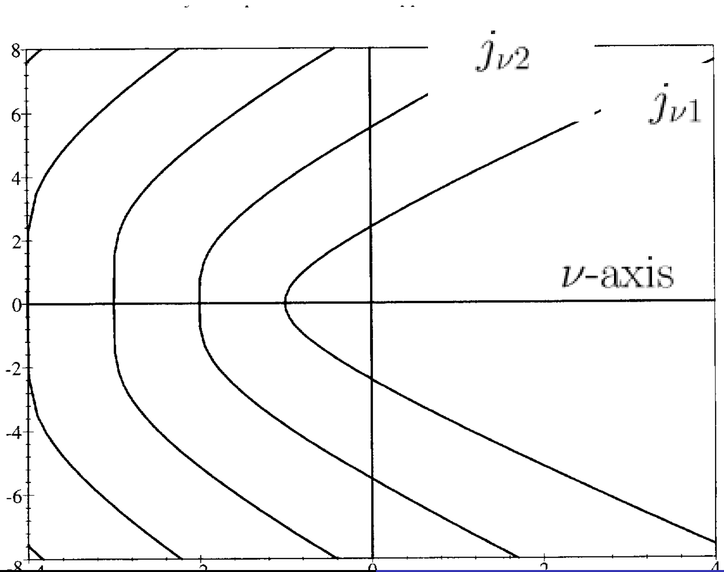
i.e., at the points $x(k - \alpha/\pi)$, $k = 0, 1, 2, \ldots$.

Continuous rank (cont.)

α and k are not really independent; they may be subsumed in a single variable $\kappa = k - \alpha/\pi$. Elbert and Laforgia [3] explained this idea in the case of Bessel functions. Thus each zero of $y(x, \alpha)$ increases from one zero x_k of $y_1(x)$ to the next larger one x_{k+1}. At the same time a new smallest zero appears and increases from a to x_1. Thus it makes sense to define $x(\kappa)$ for any real $\kappa \geq 0$, by $x(0) = a$ and $x(\kappa) = x_k(\alpha)$ where $k = \lceil \kappa \rceil$ is the largest integer *less than* $\kappa + 1$ and $\alpha = \pi(k - \kappa)$. Thus $x(\kappa)$ is a continuous increasing function of κ on $[0, \infty)$. The positive zeros of $y_1(x)$ correspond to $x(k)$, $k = 1, 2, \ldots$ and those of $y_2(x)$ correspond to $x(k - 1/2)$, $k = 1, 2, \ldots$.

The graphs of the zeros of cylinder functions thus fill in the gaps in the diagram:

Zeros of cylinder functions



Variation of $c(\nu, \alpha, \kappa)$ with κ

Á. Elbert [2]

$$\frac{d}{d\kappa} j_{\nu\kappa} = \frac{\pi^2}{2} \left[J_\nu^2(j_{\nu\kappa}) + Y_\nu^2(j_{\nu\kappa}) \right]$$

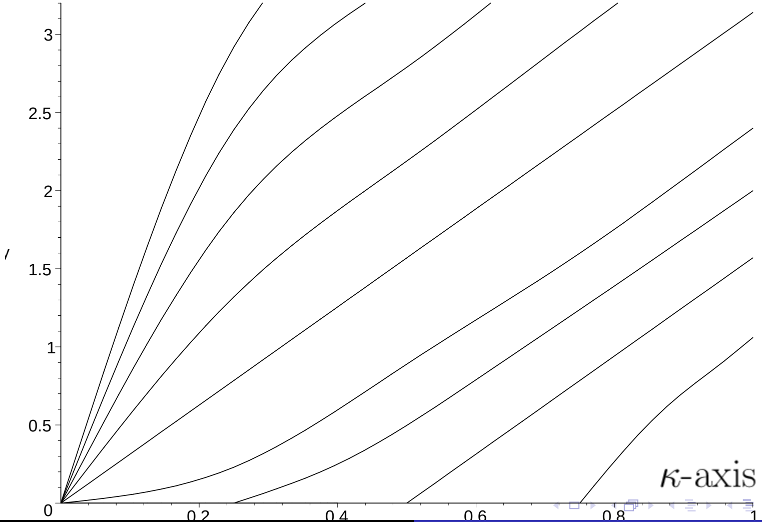
$$\frac{d}{d\kappa} j_{\nu\kappa} = 4 j_{\nu\kappa} \int_0^\infty K_0(2 j_{\nu\kappa} \sinh t) \cosh 2\nu t \, dt$$

Compare (Watson)

$$\frac{d}{d\nu} j_{\nu\kappa} = 2 j_{\nu\kappa} \int_0^\infty K_0(2 j_{\nu\kappa} \sinh t) \exp(-2\nu t) \, dt$$

The followings graphs are from [9].

Variation of $c(\nu, \alpha, \kappa)$ with κ



Variation of $c(\nu, \alpha, \kappa)$ with κ (cont.)

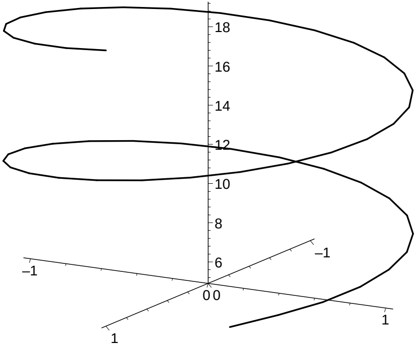


FIGURE 2. Approximate graph of the curve
$x = \cos\kappa, \quad y = \sin\kappa, \quad z = j_{5\kappa}, \ 0.2 \leq \kappa \leq 4.$

Variation of $c(\nu, \alpha, \kappa)$ with κ (cont.)

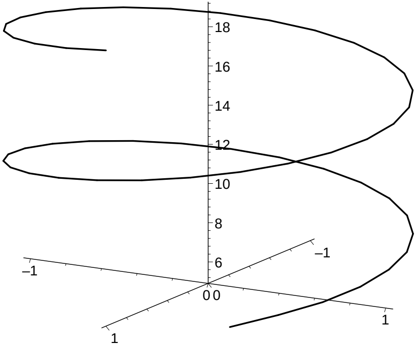
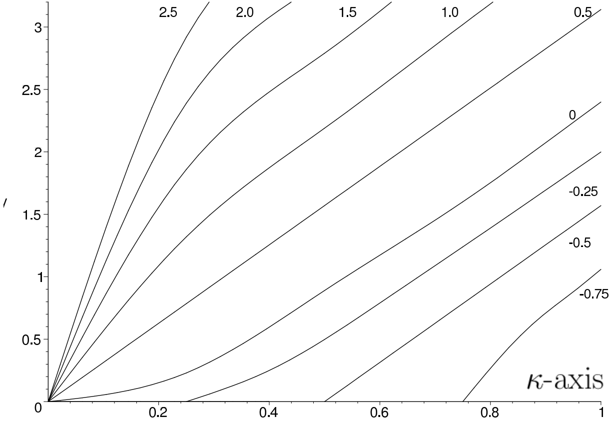


FIGURE 2. Approximate graph of the curve
$x = \cos\kappa, \quad y = \sin\kappa, \quad z = j_{5\kappa}, \ 0.2 \le \kappa \le 4.$

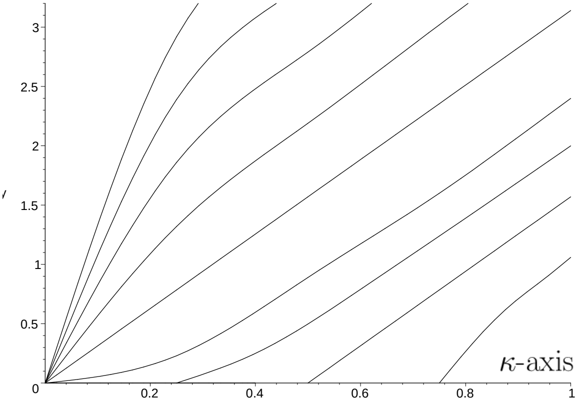
Variation of $c(\nu, \alpha, \kappa)$ with κ

$j_{\nu\kappa}$ as a function of κ



$$j_{1/2,\kappa} = \kappa\pi, \;\; j_{-1/2,\kappa} = (\kappa - 1/2)\pi$$

$j_{\nu\kappa}$ is convex for $|\nu| < 1/2$, concave for $|\nu| > 1/2$



We have

$$(-1)^{n+1}\frac{d^n}{d\kappa^n}j_{\nu\kappa} > 0, \quad |\nu| > 1/2, \; n = 1, 2, \ldots$$

This generalizes the result of L. Lorch and P. Szego [8]

$$(-1)^{n+1}\Delta_k^n c_{\nu k} > 0, \quad |\nu| > 1/2, \; n = 1, 2, \ldots$$

Hermite functions

The differential equation

$$y'' - 2ty' + 2\lambda y = 0.$$

has a solution

$$H_\lambda(t) = -\frac{\sin \pi\lambda \;\; \Gamma(1+\lambda)}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left((n-\lambda)/2\right)}{\Gamma(n+1)} (-2t)^n$$

which reduces to the Hermite polynomials for $\lambda = 0, 1, 2, \ldots$. In terms of confluent hypergeometric functions,

$$H_\lambda(t) = \frac{2^\lambda}{\sqrt{\pi}} \left[\cos \frac{\lambda\pi}{2} \Gamma(\frac{\lambda}{2} + \frac{1}{2}) {}_1F_1(-\frac{\lambda}{2}, \frac{1}{2}; t^2) \right.$$

$$\left. + 2t \sin \frac{\lambda\pi}{2} \Gamma(\frac{\lambda}{2} + 1) {}_1F_1(-\frac{\lambda}{2} + \frac{1}{2}, \frac{3}{2}; t^2) \right].$$

Hermite functions (cont.)

We can define a solution $G_\lambda(t)$, so that $e^{-t^2/2}H_\lambda(t)$ and $e^{-t^2/2}G_\lambda(t)$ are linearly independent solutions of the modified Hermite equation

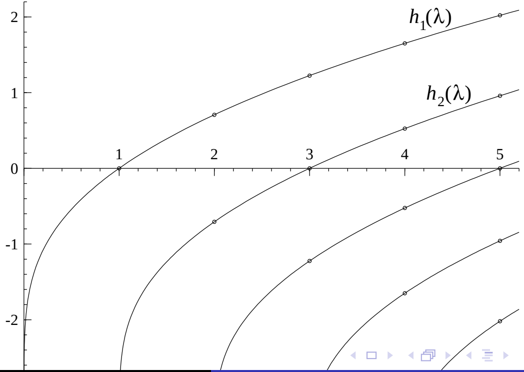
$$y'' + (2\lambda + 1 - t^2)y = 0.$$

The Wronskian of $e^{-t^2/2}H_\lambda(t)$ and $e^{-t^2/2}G_\lambda(t)$ is given by

$$W = \pi^{-1/2}2^{\lambda+1}\Gamma(\lambda + 1).$$

The zeros of $H_\lambda(t)$ as functions of λ were considered in [4].

Zeros of $H_\lambda(t)$ as functions of λ, $\lambda > -1$



Hermite functions (cont.)

Á. Elbert and MEM [4] proved a formula for the derivative with respect to λ of a zero $h(\lambda)$ of a solution of a linear combination of H_λ and G_λ.

$$\frac{dh}{d\lambda} = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-(2\lambda+1)\tau} \phi(h\sqrt{\tanh \tau}) \frac{d\tau}{\sqrt{\sinh \tau \, \cosh \tau}}, \qquad (2)$$

for $\lambda > -1$, where

$$\phi(x) = e^{x^2} \mathrm{erfc}(x),$$

Hermite functions (cont.)

This formula (2) for $dh/d\lambda$ was useful [5] in finding an asymptotic expansion for the zeros (as $\lambda \to +\infty$)

The function

$$y(t) = e^{-t^2/2}[\cos\alpha\ H_\lambda(t) - \sin\alpha\ G_\lambda(t)]$$

satisfies the differential equation

$$y'' + (2\lambda + 1 - t^2)y = 0,$$

and hence, if we write

$$\mu = (2\Lambda)^{-1/3},\ \ \Lambda = \sqrt{2\lambda + 1},\ \ C_\lambda = \pi^{-1/3} 2^{-\lambda/2 - 1/4} \lambda^{1/2},$$

we find, after some simplification, that

$$Y(\lambda, t) = C_\lambda y\left(\frac{1}{2\mu^3} - \mu t\right)$$

satisfies the differential equation

$$\frac{d^2 Y}{dt^2} + (t - \mu^4 t^2)Y = 0. \qquad (3)$$

Asymptotic information shows that $Y(\lambda, t)$ satisfies the initial conditions

$$Y(0) = \cos\alpha \; Ai(0) - \sin\alpha \; Bi(0), \;\; Y'(0) = \cos\alpha \; Ai'(0) - \sin\alpha \; Bi'(0). \tag{4}$$

The initial conditions (4) are independent of μ and the coefficient term $t - \mu^4 t^2$ in (3) is an entire function of μ for each fixed t. Hence, for fixed t, the solutions of (3), (4) are entire functions of μ.

Hermite functions (cont.)

Thus a zero $z(\mu)$ of a solution of a nontrivial solution of (3, 4) is analytic in μ in a neighbourhood of $\mu = 0$:

$$z(\mu) = \sum_{k=1}^{\infty} c_k \mu^{k-1}, \ \ |\mu| < R,$$

for some $R > 0$, where c_1 is the corresponding zero of $\cos\alpha\, Ai(-x) - \sin\alpha\, Bi(-x)$. In other words, if $h(\lambda)$ is a zero of a solution of (28), then

$$h(\lambda) = \Lambda + \Lambda^{-1/3} \sum_{k=1}^{\infty} a_k \Lambda^{-4(k-1)/3},$$

where the series converges for $\Lambda > M$, for some $M > 0$.
This is also an asymptotic series

$$h(\lambda) \sim \Lambda + \Lambda^{-1/3} \sum_{k=1}^{\infty} a_k \Lambda^{-4(k-1)/3}, \ \Lambda \to \infty,$$

in the usual sense.

Hermite functions (cont.)

For $h(k, \alpha)$, the kth zero, in decreasing order, of a Hermite function

$$\cos\alpha H_\lambda(x) - \sin\alpha G_\lambda(x),$$

the expansion will involve $a - a(k, \alpha)$ where $2^{-1/3}a$ is the kth positive zero of

$$\cos\alpha Ai(-x) - \sin\alpha Bi(-x).$$

Using (2), the first five terms are given by [5]

$$\begin{aligned} h(k, \alpha) = \Lambda - a\Lambda^{-1/3} - \frac{1}{10}a^2\Lambda^{-5/3} + \left[\frac{9}{280} - \frac{11}{350}a^3\right]\Lambda^{-3} \\ + \left[\frac{277}{12600}a - \frac{823}{63000}a^4\right]\Lambda^{-13/3} + \cdots, \end{aligned}$$

Tricomi's asymptotic formula

G. N. Watson [18] provides some large ν asymptotic approximations for zeros of the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ and the first of these was generalized by F. G. Tricomi [16]. This is expansion is also convergent.
As pointed out by L. Gatteschi (1974), if $\mu = 2^{2/3}\nu^{-2/3}$, a constant multiple of the function $J_\nu[\nu \exp(-\mu x)]$ satisfies

$$\frac{d^2y}{dx^2} - xy = c(\mu, x)y, \tag{5}$$

$$y(0) = Ai(0), \;\; y'(0) = Ai'(0),$$

where

$$c(\mu, x) = \begin{cases} \mu^{-1}[1 - \mu x - e^{-\mu x}], & \mu \neq 0, \\ 0, & \mu = 0 \end{cases} \tag{6}$$

Tricomi's asymptotic formula (cont.)

For each fixed x, $c(\mu, x)$ is analytic at $\mu = 0$. Thus the zeros of solutions of (5) are analytic functions of μ and may be expressed in the form

$$\sum_{k=0}^{\infty} a_n \mu^n, \quad |\mu| < R,$$

for some $R > 0$.

Now $J_\nu[\nu \exp(-\mu x)]$ has its zeros x where $j = \nu \exp(-\mu x)$. Hence

$$j = \nu \exp\left[-\sum_{k=0}^{\infty} a_n \mu^n\right] = \nu(1 - a_0\mu + b\mu^2 + \dots),\ \mu = (2/\nu)^{2/3}.$$

Analyticity of $j_{\nu 1}^2/(4(\nu+1))$

R. Piessens (1984) showed that

$$j_{\nu 1} = 2(\nu+1)^{1/2}\left[1 + \frac{\nu+1}{4} - \frac{7(\nu+1)^2}{96} + \frac{49(\nu+1)^3}{1152} - \cdots\right]$$

for $-1 < \nu < 0$. As ν decreases through -1, the zeros $\pm j_{\nu 1}$ become purely imaginary and move away from the origin returning there as ν approaches -2. This suggests considering $j_{\nu 1}^2,\ \nu > -2$:

Graph of $j_{\nu 1}^2,\ \nu > -2$ [7]

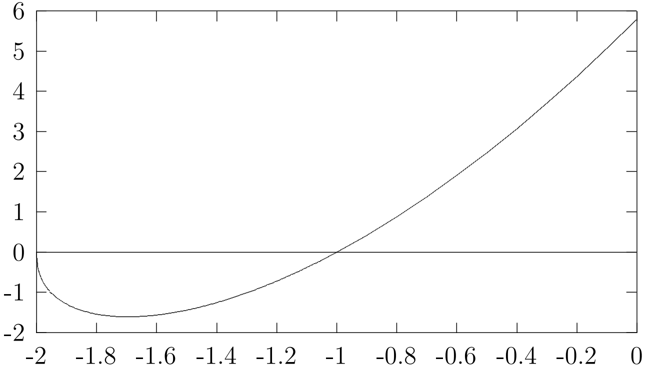


Figure 1: $j_{\nu 1}^2$ *vs.* ν

Analyticity of $j_{\nu 1}^2/(4(\nu+1))$ (cont.)

It is an easy step from this to

$$j_{\nu 1}^2 = 4(\nu{+}1)\left[1 + \frac{\nu+1}{2} - \frac{(\nu+1)^2}{12} + \frac{7(\nu+1)^3}{144} - \frac{293(\nu+1)^4}{8640} + \cdots\right]$$

which is valid for $-2 < \nu < 0$ once we note that $j_{\nu 1}^2/(4(\nu+1))$ is analytic at -1.
This follows since $z^{1/2} J_\nu(2\sqrt{(\nu+1)z})$ satisfies

$$\frac{d^2y}{dz^2} + \left[\frac{1-\nu^2}{4z^2} + \frac{\nu+1}{z}\right] y = 0,$$

the coefficient function being analytic in ν at $\nu = -1$.

References I

[1] Á. Elbert, Concavity of the zeros of Bessel functions, *J. Comp. Appl. Math.* **12** (1977), 81—88.

[2] Á. Elbert, Some recent results on the zeros of Bessel functions and orthogonal polynomials, *J. Comp. Appl. Math.* **133** (2001), 66–83.

[3] Á. Elbert, and A. Laforgia, On the square of the zeros of Bessel functions, *SIAM J. Math. Anal.* **15** (1984), 206–212.

[4] Á. Elbert and M. E. Muldoon, Inequalities and monotonicity properties for zeros of Hermite functions, *Proc. Roy. Soc. Edinburgh, Section A* **129** (1999), 57–75.

References II

[5] Á. Elbert and M. E. Muldoon, Approximations for zeros of Hermite functions, pp. 117–126 in D. Dominici and R. S. Maier, eds., "Special Functions and Orthogonal Polynomials", *Contemporary Mathematics* **471** (2008).

[6] L. Gatteschi, *Funzioni Speciali*, UTET, Torino, 1973.

[7] C. G. Kokologiannaki, M. E. Muldoon and P. D. Siafarikas, A unimodal property of purely imaginary zeros of Bessel and related functions, *Canad. Math. Bull.* **37** (1994), 365–373.

[8] L. Lorch and P. Szego, Higher monotonicity properties of certain Sturm-Liouville functions, *Acta Math.* **109** (1963), 55–73.

[9] M. E. Muldoon, Continuous ranking of zeros of special functions, *J. Math. Anal. Appl.* **343** (2008), 436–445.

[10] F. W. J. Olver, A new method for the evaluation of zeros of Bessel functions and of other solutions of second-order differential equations, *Proc. Cambridge Philos. Soc.* **46** (1950), 570–580.

[11] F. W. J. Olver, A further method for the evaluation of zeros of Bessel functions and some new asymptotic expansions for zeros of functions of large order, *Proc. Cambridge Philos. Soc.* **47** (1951), 699-712.

[12] F. W. J. Olver, Error bounds for first approximations in turning-point problems, *J. Soc. Indust. Appl. Math.* **11** (1963), 748–772.

[13] F. W. J. Olver, Selected Papers, Parts I, II. Edited by Roderick Wong. World Scientific Series in 20th Century Mathematics, 7. 2000.

References IV

[14] F. W. J. Olver, et al., eds, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010. `dlmf.nist.gov`.

[15] R. Piessens, A series expansion for the first positive zero of the Bessel functions, *Math. Comp.* **42** (1984), 195–197.

[16] F. G. Tricomi, Sulle funzioni di Bessel di ordine e argomento pressoché uguali, *Atti Accad. Sci. Torino* **88** (1949), 8–20.

[17] J. Vosmanský, Zeros of solutions of linear differential equations as continuous functions of the parameter k, in J. Wiener and J. K. Hale (eds.), Partial differential equations. Proceedings of conference held at Edinburg, Texas, 1991. *Pitman Res. Notes Math.* Ser. **273** (1992), 253–257.

References V

[18] G. N. Watson, A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge University Press, 1944.