Olver's asymptotic theory, Green's functions and fixed point theorems

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Second order linear differential equation:

$$\ddot{w} - [\Lambda^2 f(t) + h(t)]w = 0, \qquad \Lambda \text{ large.}$$

Double change of variable:

$$\begin{cases} t \to x \\ w \to y = \dot{x}^{1/2} w \end{cases} \implies y'' = \left[\Lambda^2 \left(\frac{dt}{dx}\right)^2 f(x) + g(x)\right] y.$$

$t(x)$ fixed by the conditions:

- t and x are analytic functions of each other at the transition point (if any),
- For $g(x) = 0$, solutions which are functions of a single variable.

$$\text{Case I.} \quad \left(\frac{dt}{dx}\right)^2 f(x) = 1, \quad \text{which means} \quad x = \int^t f^{1/2}(s)ds.$$

In case I the DE reduces to

$$y'' = [\Lambda^2 + g(x)]y.$$

When $\Lambda \to \infty$ we seek for a formal solution of the form

$$y_1(x) \sim e^{\Lambda x} \sum_{n=0}^{\infty} \frac{A_n(x)}{\Lambda^n}.$$

- $A_0(x) =$ constant (we may take $A_0(x) = 1$ without loss of generality).
- $$A_{n+1}(x) = -\frac{1}{2}A_n'(x) + \frac{1}{2}\int^x g(t)A_n(t)dt, \qquad n = 0, 1, 2, \ldots,$$

A second formal solution:

$$y_2(x) \sim e^{-\Lambda x} \sum_{n=0}^{\infty} (-1)^n \frac{A_n(x)}{\Lambda^n}.$$

In general, these expansions are divergent. Olver's theory:

- Proof of the asymptotic character of these expansions.
- Error bounds for the remainders of the expansions:

$$R_{n,1}(x) := y_1(x) - e^{\Lambda x} \sum_{k=0}^{n-1} \frac{A_k(x)}{\Lambda^k}; \quad R_{n,2}(x) := y_2(x) - e^{-\Lambda x} \sum_{k=0}^{n-1} (-1)^k \frac{A_k(x)}{\Lambda^k}.$$

- Behavior of the coefficients $A_k(x)$ at the singularities of the DE (if any),
- Uniformity properties.
- Discussions about the regions of validity of the expansions.

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Consider the following initial value problem:

$$\begin{cases} y'' - \Lambda^2 y - g(x)y = 0 & \text{in } [0, X], \\ y(0) = y^0, \quad y'(0) = y'^0, \end{cases}$$

$X > 0, \quad y^0, y'^0, \Lambda \in \mathbb{C}, \quad \Re\Lambda \geq 0, \quad g : [0, X] \to \mathbb{C}$ continuous.

Consider the auxiliary initial value problem

$$\begin{cases} \phi'' - \Lambda^2 \phi = 0 & \text{in } [0, X], \\ y(0) = y^0, \quad y'(0) = y'^0, \end{cases}$$

Unique solution:

$$\phi(x) := y^0 \cosh(\Lambda x) + \frac{y'^0}{\Lambda} \sinh(\Lambda x).$$

Change of unknown $y(x) \to u(x) = y(x) - \phi(x) \Rightarrow$ homogeneous IC:

$$\begin{cases} u'' - \Lambda^2 u = (u+\phi)g \quad \text{in } [0,X], \\ u(0) = 0, \quad u'(0) = 0, \end{cases}$$

We seek for solutions of $\mathbf{L}[u] := u'' - \Lambda^2 u - (u+\phi)g = 0$ in

$$\mathcal{B}_0 = \{u : [0,X] \to \mathbb{C}, u'' \in \mathcal{C}[0,X]; u(0) = u'(0) = 0\}$$

equipped with the norm

$$||u||_\infty = \text{Sup}_{x\in[0,X]}|u(x)|.$$

Key point, write:

$$\mathbf{L}[u] = \underbrace{u'' - \Lambda^2 u}_{\mathbf{M}[u]} - (u+\phi)g$$

For large Λ:

$$(u+\phi)g \quad \text{is negligible} \Longrightarrow \mathbf{L}[u] \sim \mathbf{M}[u].$$

Then solve the equation $\mathbf{L}[u] = 0$ in the form

$$u = \mathbf{M}^{-1}[(u+\phi)g],$$

where

$$\mathbf{M}^{-1}(v) = \int_0^X G(x,t)v(t)dt,$$

$G(x,t)$ is the Green function of the problem $\mathbf{M}[u]=0$:

$$\begin{cases} G_{xx}(x,t)-\Lambda^2 G(x,t)=\delta(x-t) & \text{in } [0,X], \\ G(0,t)=G_x(0,t)=0, & t\in[0,X], \end{cases}$$

$$G(x,t)=\frac{1}{\Lambda}\sinh[\Lambda(x-t)]\chi_{[0,x]}(t).$$

Then, any solution $u(x)$ of the IVP is a solution of the integral equation

$$u(x)=\mathbf{M}^{-1}[(u+\phi)g]=\frac{1}{\Lambda}\int_0^x \sinh[\Lambda(x-t)]g(t)[u(t)+\phi(t)]dt.$$

Equivalently, defining

$$\tilde{u}(x) := e^{-\Lambda x} u(x) \qquad \text{and} \qquad \tilde{\phi}(x) := e^{-\Lambda x} \phi(x),$$

any solution $u(x) = e^{\Lambda x}\tilde{u}(x)$ of the IVP is a solution of

$$\tilde{u}(x) = [\mathbf{T}\tilde{u}](x),$$

$$[\mathbf{T}\tilde{u}](x) := \frac{1}{2\Lambda}\int_0^x \left[1 - e^{2\Lambda(t-x)}\right] g(t)[\tilde{u}(t) + \tilde{\phi}(t)]dt.$$

From the fixed point theorem, if $\mathbf{T}^n$ is contractive in $\mathcal{B}_0 \Rightarrow$

- $\tilde{u}(x) = [\mathbf{T}\tilde{u}](x)$ has a unique solution $\tilde{u}(x)$ and
- $\tilde{u}_{n+1} = \mathbf{T}(\tilde{u}_n)$, $\tilde{u}_0 = 0$, converges to $\tilde{u}(x)$.

We show this by using the bound

$$\left| \frac{1 - e^{2\Lambda(t-x)}}{2\Lambda} \right| \leq x - t, \qquad \text{for } t \leq x.$$

in

$$[\mathbf{T}\tilde{u}](x) := \frac{1}{2\Lambda} \int_0^x \left[1 - e^{2\Lambda(t-x)}\right] g(t)[\tilde{u}(t) + \tilde{\phi}(t)]dt.$$

By means of induction over n, for $n = 1, 2, 3, ...$,

$$||\mathbf{T}^n z - \mathbf{T}^n w||_\infty \leq \frac{||g||_\infty^n X^{2n}}{(2n)!} ||z - w||_\infty.$$

Therefore $\mathbf{T}^n$ is contractive for large enough $n \Rightarrow$ the sequence

$$y_n(x) = e^{\Lambda x}[\tilde{u}_n(x) + \tilde{\phi}(x)]$$

converges uniformly in $x \in [0, X]$ to the unique solution of the IVP.

Moreover:

$$|\tilde{u}(x) - \tilde{u}_n(x)| \leq \frac{||g||_\infty^n x^{2n}}{(2n)!} ||\tilde{u}||_\infty.$$

Using $$y(x) = e^{\Lambda x}\tilde{u}(x) + \phi(x)$$

and $$y_n(x) = e^{\Lambda x}\tilde{u}_n(x) + \phi(x)$$

we find

$$|R_n(x)| \leq \frac{||g||_\infty^n x^{2n}}{(2n)!} ||e^{-\Lambda x}(y - \phi)||_\infty.$$

Theorem 1. *Let* $g : [0, X] \to \mathbb{C}$ *be continuous. Then,*

$$\begin{cases} y'' - \Lambda^2 y - g(x) y = 0 & \text{in } [0, X], \\ y(0) = y^0, \quad y'(0) = y'^0, \end{cases}$$

has a unique solution $y(x)$*. Moreover, for* $n = 0, 1, 2, \ldots,$

$$y_{n+1}(x) = \phi(x) + \frac{1}{\Lambda} \int_0^x \sinh[\Lambda(x - t)] g(t) y_n(t) dt,$$

$$y_0(x) = \phi(x) := y^0 \cosh(\Lambda x) + \frac{y'^0}{\Lambda} \sinh(\Lambda x)$$

converges to $y(x)$ *uniformly in* $x \in [0, X]$.
The remainder $R_n(x) := e^{-\Lambda x}[y(x) - y_n(x)]$ *is bounded by*

$$|R_n(x)| \leq \frac{||g||_\infty^n x^{2n}}{(2n)!} ||e^{-\Lambda x}(y - \phi)||_\infty.$$

Observations:

- $\exists$ and uniqueness $\to$ direct consequence of the Picard-Lindelöf's Th.

- Different election of the "main operator" $\mathbf{M}[u]$:
  $\mathbf{M}[u] = u''$ in the standard Picard-Lindelöf's theorem.
  Here $\mathbf{M}[u] = u'' + \Lambda^2 u$.

- For large Λ, $\mathbf{M}[u] = u'' + \Lambda^2 u$ "closer" to $\mathbf{L}[u]$ than $\mathbf{M}[u] = u''$:
  Similar error bound in the Picard-Lindelöf's iteration, but replacing

$$||g||_\infty \quad \to \quad ||g||_\infty + \Lambda^2.$$

  When $\Lambda >> ||g||_\infty$, we have a faster convergence.

- Moreover, the recurrence $y_n(x)$ is an asymptotic expansion of $y(x)$ for large Λ.

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We have

$$y(x) = \lim_{n\to\infty} y_n(x) \qquad \text{uniformly in } [0, X].$$

In other words, $y(x)$ admits the series expansion

$$y(x) = \phi(x) + \sum_{k=0}^{\infty}[y_{k+1}(x) - y_k(x)] = \phi(x) + e^{\Lambda x}\sum_{k=0}^{\infty}[\tilde{u}_{k+1}(x) - \tilde{u}_k(x)],$$

with

$$\tilde{u}_n(x) := e^{-\Lambda x}[y_n(x) - \phi(x)], \qquad n = 0, 1, 2, \ldots$$

We define the remainder of this expansion in the form

$$R_n(x) := e^{-\Lambda x}[y(x) - y_n(x)], \qquad n = 0, 1, 2, \ldots$$

Then we may write the series expansion in the form

$$y(x) = \phi(x) + \sum_{k=0}^{n-1} [y_{k+1}(x) - y_k(x)] + e^{\Lambda x} R_n(x) =$$

$$\phi(x) + e^{\Lambda x} \left[\sum_{k=0}^{n-1} [\tilde{u}_{k+1}(x) - \tilde{u}_k(x)] + R_n(x) \right].$$

To show the convergence of the recurrence $y_n(x)$ we used the bound

$$\left| \frac{1 - e^{2\Lambda(t-x)}}{2\Lambda} \right| \leq x - t, \qquad \text{for } \; t \leq x.$$

Bad bound for large Λ.
To show the asymptotic character we need:

$$\left| \frac{1 - e^{2\Lambda(t-x)}}{2\Lambda} \right| \leq \frac{1}{|\Lambda|}, \qquad \Re\Lambda \geq 0, \qquad x \geq t.$$

in

$$[\mathbf{T}\tilde{u}](x) := \frac{1}{2\Lambda} \int_0^x \left[1 - e^{2\Lambda(t-x)} \right] g(t)[\tilde{u}(t) + \tilde{\phi}(t)]dt.$$

We obtain:

$$||u_{n+1} - u_n||_\infty \leq \frac{X}{|\Lambda|} ||g||_\infty ||u_n - u_{n-1}||_\infty.$$

We also have:

$$||\tilde{u} - \tilde{u}_n||_\infty \leq \frac{||g||_\infty^n X^n}{|\Lambda|^n} ||\tilde{u}||_\infty$$

and $\tilde{u} = \lim_{n\to\infty} \tilde{u}_n = \sum_{k=0}^{\infty} [\tilde{u}_{k+1} - \tilde{u}_k] = \sum_{k=0}^{\infty} \mathcal{O}(\Lambda^{-k-1}) = \mathcal{O}(\Lambda^{-1})$.

Theorem 2. *Let $g : [0, X] \to \mathbb{C}$ be continuous in $[0, X]$. Then,*

$$y(x) = \phi(x) + e^{\Lambda x}\left[\sum_{k=0}^{n-1}[\tilde{u}_{k+1}(x) - \tilde{u}_k(x)] + R_n(x)\right].$$

is an asymptotic expansion for large Λ of the unique solution of

$$\begin{cases} y'' - \Lambda^2 y - g(x)y = 0 \quad \text{in} \ [0, X], \\ y(0) = y^0, \quad y'(0) = y'^0, \end{cases}$$

uniformly in $x \in [0, X]$. More precisely, for $n = 1, 2, 3, \ldots,$

$$\tilde{u}_n(x) - \tilde{u}_{n-1}(x) = \mathcal{O}(\Lambda^{-n}) \qquad \text{and} \qquad R_n(x) = \mathcal{O}(\Lambda^{-n-1})$$

uniformly for $x \in [0, X]$.

Observations:

- This expansion is not of Poincaré type.
- Compare the construction of Olver's and this expansion:

$$y_1(x) \sim e^{\Lambda x} \sum_{n=0}^{\infty} \frac{A_n(x)}{\Lambda^n}, \qquad A_{n+1}(x) = -\frac{1}{2}A_n'(x) + \frac{1}{2}\int^x g(t)A_n(t)dt,$$

and

$$\tilde{u}_{n+1}(x) = \frac{1}{2\Lambda}\int_0^x \left[1 - e^{2\Lambda(t-x)}\right] g(t)[\tilde{u}_n(t) + \tilde{\phi}(t)]dt,$$

The integrand in the RHS of Olver's recursion is independent of Λ
The integrand in the RHS of the recurrence $\tilde{u}_n(x)$:
$\mathcal{O}(1)$ as $\Lambda \to \infty$,
contains an exponentially small dependence on Λ.

Example. For any $\Lambda \in \mathbb{C}$ and $X > 0$, the unique solution of the IVP

$$\begin{cases} y'' - \left(\Lambda^2 + \frac{x^2}{4}\right) y = 0 \quad \text{in } [0, X], \\ y(0) = U_{\Lambda^2}(0), \quad y'(0) = U'_{\Lambda^2}(0), \end{cases}$$

is the Parabolic Cylinder function $U_{\Lambda^2}(x)$. For this problem

$$\phi(x) = \frac{\sqrt{\pi}}{2^{\Lambda^2/2+1/4}} \left[\frac{\cosh[\Lambda x]}{\Gamma(\Lambda^2/2+3/4)} - \frac{\sqrt{2}\sinh[\Lambda x]}{\Lambda\Gamma(\Lambda^2/2+1/4)} \right],$$

$y_0(x) = \phi(x)$ and, for $n = 0, 1, 2, \ldots$,

$$y_{n+1}(x) = \phi(x) + \frac{1}{4\Lambda} \int_0^x t^2 \sinh[\Lambda(x-t)] y_n(t) dt.$$

$y_n(x)$ converges absolutely and uniformly in $[0, X]$ to $U_{\Lambda^2}(x)$.
The sequence $y_n(x)$ is also an asymptotic expansion of $U_{\Lambda^2}(x)$.

Numerical experiments:

n	3	5	7	10
Olver's method	0.029931	0.711066	2.397264	34.189849
$y_{n+1} = \phi + \mathbf{T}y_n$	2.9242086E-7	2.117221E-13	0E-19	0E-11

Table: Parameter values: $x = 1,\ \Lambda = 0.5$.

n	3	5	7	10
Olver's method	0.035784	1.046597	1.215351	20.55906
$y_{n+1} = \phi + \mathbf{T}y_n$	2.401301E-7	1.701563E-13	0E-19	0E-12

Table: Parameter values: $x = 1,\ \Lambda = 0.5i$.

n	3	5	7	10
Olver's method	0.103680	0.025607	0.626	1.015444
$y_{n+1} = \phi + \mathbf{T} y_n$	0.117456	0.00313190E-2	0.229839E-4	2.373983E-9

Table: Parameter values: $x = -4,\ \Lambda = 1$.

n	3	5	7	10
Olver's method	0.178926E-3	1.888427E-6	4.157060E-9	1.105044E-10
$y_{n+1} = \phi + \mathbf{T} y_n$	0.203539E-2	4.885380E-6	4.823895E-9	4.315667E-14

Table: Parameter values: $x = -4,\ \Lambda = 10$.

n	3	5	7	10
Olver's method	2.386986E-5	1.437328E-6	4.800630E-8	6.697113E-10
$y_{n+1} = \phi + \mathbf{T}y_n$	9.812156E-3	2.960339E-4	2.435708E-8	9.491949E-13

Table: Parameter values: $x = -4, \ \Lambda = 10i$.

n	3	5	7	10
Olver's method	2.975095E-8	3.729794E-12	1.568155E-16	1.023331E-20
$y_{n+1} = \phi + \mathbf{T}y_n$	1.49924E-5	5.268923E-10	8.74661E-15	1.108888E-22

Table: Parameter values: $x = 4, \ \Lambda = 100i$.

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Consider

$$\begin{cases} y'' - \Lambda^2 y - f(x,y) = 0 & \text{in } [0,X], \\ y(0) = y^0, \quad y'(0) = y'^0, \end{cases}$$

where $f : [0,X] \times \mathbb{C} \to \mathbb{C}$ is continuous in its two variables.

Linear case: $f(x,y) = g(x)y$, $g(x)$ continuous.

Nonlinear case, we require the Lipschitz condition:

$$|f(x,y) - f(x,z)| \leq K|y - z| \qquad \forall\ y,z \in \mathbb{C} \text{ and } x \in [0,X], \qquad K > 0.$$

This condition replaces

$$|g(t)|\,|y(x) - z(x)| \leq ||g||_\infty\,||y - z||_\infty$$

used in the linear case.

Repeating the arguments of the linear case, but replacing the bound

$$|g(t)|\,|y(x)-z(x)| \leq ||g||_\infty \,||y-z||_\infty$$

by the bound

$$|f(x,y)-f(x,z)| \leq K|y-z| \qquad \forall \;\; y,z \in \mathbb{C} \;\text{ and }\; x \in [0,X], \quad K>0,$$

Theorem 3. *Let* $f : [0, X] \times \mathbb{C} \to \mathbb{C}$ *continuous and satisfy the Lipschitz condition. Then, the problem*

$$\begin{cases} y'' - \Lambda^2 y - f(x, y) = 0 & in \;\; [0, X], \\ y(0) = y^0, \qquad y'(0) = y'^0, \end{cases}$$

has a unique solution $y(x)$*. Moreover:*

- *For* $n = 0, 1, 2, \ldots$ *and* $y_0(x) = \phi(x)$*, the sequence*

$$y_{n+1}(x) = \phi(x) + \frac{1}{\Lambda} \int_0^x \sinh[\Lambda(x - t)] f(t, y_n(t)) dt,$$

$$\phi(x) := y^0 \cosh(\Lambda x) + \frac{y'^0}{\Lambda} \sinh(\Lambda x)$$

  converges to $y(x)$ *uniformly in* $x \in [0, X]$.
- *The remainder* $R_n(x) := e^{-\Lambda x}[y(x) - y_n(x)]$ *is bounded by*

$$|R_n(x)| \leq \frac{K^n x^{2n}}{(2n)!} ||e^{-\Lambda x}(y - \phi)||_\infty.$$

Observations:

- Existence and uniqueness of the solution well known from the Picard-Lindelöf's theorem.
- Similar error bound for the standard Picard-Lindelöf's iteration replacing K by $K + \Lambda^2$ in

$$|R_n(x)| \leq \frac{K^n x^{2n}}{(2n)!} ||e^{-\Lambda x}(y - \phi)||_\infty.$$

- When Λ is large compared with K, we have that $y_n(x)$ converges faster than the standard Picard-Lindelöf's iteration.
- Moreover, $y(x)$ is also an asymptotic expansion of $y(x)$ for large Λ:

Theorem 4. *Let* $f : [0, X] \times \mathbb{C} \to \mathbb{C}$ *be continuous and Lipschitz's continuous. Then, the expansion*

$$y(x) = \phi(x) + \sum_{k=0}^{n-1} [y_{k+1}(x) - y_k(x)] + e^{\Lambda x} R_n(x) =$$

$$\phi(x) + e^{\Lambda x} \left[\sum_{k=0}^{n-1} [\tilde{u}_{k+1}(x) - \tilde{u}_k(x)] + R_n(x) \right].$$

is an asymptotic expansion for large Λ *of* $y(x)$*, uniformly in* $x \in [0, X]$*. More precisely, for* $n = 1, 2, 3, ...,$

$$\tilde{u}_n(x) - \tilde{u}_{n-1}(x) = \mathcal{O}(\Lambda^{-n}) \qquad \text{and} \qquad R_n(x) = \mathcal{O}(\Lambda^{-n-1})$$

uniformly for $x \in [0, X]$.

Example. Consider, for $b, c \in \mathbb{C}$, $\Re\Lambda \geq 0$, the Mathieu-Duffing equation

$$y'' - (\Lambda^2 + b\cos x)y - cy^3 = 0,$$

and the corresponding initial value problem

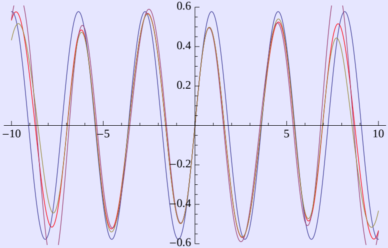
$$\begin{cases} y'' - \Lambda^2 y = f(x,y) := b\,y\,\cos x + c\,y^3 & \text{in } [0,X], \\ y(0) = 0, \quad y'(0) = 1, \end{cases}$$

$$|f(x,y) - f(x,z)| \leq [|b| + |c|\,|y^2 + yz + z^2|]|y - z|.$$

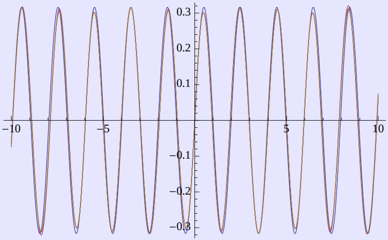
Lipschitz's continuous for $y, z \in D \subset \mathbb{C}$, D compact.
When all the $y_n(x)$ are uniformly bounded in $x \in [0,X]$,

$$y_{n+1}(x) = \frac{\sinh(\Lambda x)}{\Lambda} + \frac{1}{\Lambda}\int_0^x \sinh[\Lambda(x-t)][b\,y_n(t)\,\cos t + c\,y_n^3(t)]dt.$$

$$b = c = -1$$



$\Lambda^2 = -3$



$\Lambda^2 = -10$

Exact solution (red), $y_1(x)$ (blue), $y_2(x)$ (pink) and $y_3(x)$ (gold).

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$$\begin{cases} y'' - \Lambda^3 xy - g(x)y = 0 \quad \text{in} \;\; [0, X], \\ y(0) = y^0, \qquad y'(0) = y'^0, \end{cases}$$

For $n = 0, 1, 2, \ldots$ and $y_0(x) = 0$, the sequence

$$y_{n+1}(x) = \phi(x) + \frac{\pi}{\Lambda}\left[Bi(\Lambda x)\int_0^x Ai(\Lambda t)g(t)y_n(t)dt - \right.$$

$$\left. Ai(\Lambda x)\int_0^x Bi(\Lambda t)g(t)y_n(t)dt\right],$$

$$\phi(x) := \pi\left\{\left[y^0 Bi'(0) - \frac{y'^0}{\Lambda}Bi(0)\right]Ai(\Lambda x) - \left[y^0 Ai'(0) - \frac{y'^0}{\Lambda}Ai(0)\right]Bi(\Lambda x)\right\}.$$

converges to $y(x)$ uniformly in $[0, X]$.

For negative X, the expansion

$$y(x) = \phi(x) + \sum_{k=0}^{n-1} [y_{k+1}(x) - y_k(x)] + R_n(x)$$

is an asymptotic expansion for large Λ of $y(x)$, uniformly in $x \in [0, X]$.

$$\begin{cases} xy'' - \Lambda^2 y - xg(x)y = 0 & \text{in } [0, X], \\ y'(0) = y'^0, \end{cases}$$

For $n = 0, 1, 2, \ldots$ and $y_0(x) = 0$, the sequence

$$y_{n+1}(x) = \phi(x) + \pi\sqrt{x}\left[Y_1(2\Lambda\sqrt{x})\int_0^x \sqrt{t}J_1(2\Lambda\sqrt{t})g(t)y_n(t)dt - \right.$$

$$\left. J_1(2\Lambda\sqrt{x})\int_0^x \sqrt{t}Y_1(2\Lambda\sqrt{t})g(t)y_n(t)dt\right],$$

$$\phi(x) := \frac{y_0'}{\Lambda}\sqrt{x}J_1(2\Lambda\sqrt{x})$$

converges to $y(x)$ uniformly in $[0, X]$.

For positive X, the expansion

$$y(x) = \phi(x) + \sum_{k=0}^{n-1} [y_{k+1}(x) - y_k(x)] + R_n(x)$$

is an asymptotic expansion for large Λ of $y(x)$, uniformly for $x \in [0, X]$.


THANKS FOR YOUR ATTENTION