Log-convexity and log-concavity for series in product ratios of rising factorials and gamma functions

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based on joint work with Sergei Sitnik and Segrei Kalmykov

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Log-concavity and log-convexity

Definition of log-concavity and log-convexity

A continuous function $f : (a, b) \to \mathbb{R}_+$ is log-concave on (a, b) if for any $\delta > 0$ and μ such that $[\mu - \delta, \mu + \delta] \subset (a, b)$

$$f(\mu)^2 \geq f(\mu + \delta) f(\mu - \delta). \quad (1)$$

If inequality (1) is reversed f is log-convex.

Some properties of log-convexity and log-concavity:

- Log-convexity is stronger then convexity
- Log-convexity is additive
- Log-convexity is not preserved by convolution
- Log-concavity is weaker than concavity
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Discrete and Wright log-concavity

Wright log-concavity

f is Wright log-concave if for any $\delta > 0$ and $\varepsilon > 0$:

$$f(\mu+\varepsilon)f(\mu+\delta) \geq f(\mu+\delta+\varepsilon)f(\mu) \quad (2)$$
$$\Updownarrow$$
$$\mu \to f(\mu+\delta)/f(\mu) \text{ is non-increasing}$$

For continuous functions Wright log-concativity=log-concavity

If (1) (or (2)) only holds for $\delta = 0, 1, 2, \ldots$ the function f will be called **discrete log-concave** (or **discrete Wright log-concave**). (2) $\Rightarrow$ (1)

Examples of discrete log-concavity: Newton's inequalities for elementary symmetric polynomials, Laguerre inequalities for derivatives of entire functions, Alexandrov-Fenchel inequalities for mixed volumes, log-concavity of combinatorial sequences, Turán inequalities for orthogonal polynomials (for latest development see Szwarc, Berg, Krasikov).

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General open problem

Under what conditions on the positive sequence $\{f_k\}$ and the numbers $a_1, \ldots, a_n, b_1, \ldots, b_m$ the functions:

$$\mu \to \sum_{k=0}^{\infty} f_k \frac{(a_1+\mu)_k \cdots (a_n+\mu)_k}{(b_1+\mu)_k \cdots (b_m+\mu)_k},$$

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is [discrete, Wright] log-concave or log-convex?

Instead of rising factorial we can consider another binomial sequence of polynomials or q-rising factorial, instead of Gamma function - another explicit function. . .

Instead of log-convexity we can consider convexity with respect to different means. . .

We can add (or subtract) μ to some parameters and ν to others and thus consider multidimensional analogues. . .

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Some history: Bessel functions

Lommel's (1870) formula

$$x^2[J_\nu^2(x) - J_{\nu+1}(x)J_{\nu-1}(x)] = \sum_{k=0}^{\infty}(2k+\nu+1)J_{2k+\nu+1}^2(x)$$

$$\Downarrow$$

$$\Delta_\nu := J_\nu^2(x) - J_{\nu+1}(x)J_{\nu-1}(x) \geq 0, \quad x \in \mathbb{R}, \quad \nu > -1.$$

Improvement by Szász (1950):

$$J_\nu^2(x) - J_{\nu+1}(x)J_{\nu-1}(x) > \frac{1}{\nu+1}J_\nu^2(x), \quad x \in \mathbb{R}, \quad \nu > 0.$$

Thiruvenkatachar and Nanjundiah (1951):

$$\Delta_\nu = \frac{1}{\nu+1}J_\nu^2(x) + \frac{2}{\nu+2}J_{\nu+1}^2(x) + 2\nu\sum_{k=2}^{\infty}\frac{J_{k+\nu}^2(x)}{(\nu+k-1)(\nu+k-1)}.$$

Log-concavity of $\nu \to J_\nu(x)$ on $(-1,\infty)$ and fixed $x > 0$ - Ismail and Muldoon (1978). Extensions to higher order inequalities - Skovgaard (1954), Karlin and Szegő (1960), Al-Salam (1961), Patrick (1973), Baricz and Pogány (2011, including a survey).

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Essentially equivalent inequality

$$xI_\nu'(x)/I_\nu(x) < \sqrt{x^2+\nu^2}$$

appeared in Gronwall (1932) for $\nu > 0$ and later in Phillips and Malin (1950) for integer ν.

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Some history: the Kummer function

Alzer (1990) inequality for exponential remainder:

${}_1F_1(1;n;x)^2 < {}_1F_1(1;n+\nu;x)\,{}_1F_1(1;n-\nu;x) \Leftrightarrow$ Gautschi (1982) inequality

Here n and $n-\nu$ are non-negative integers, $x>0$.

Sitnik (1993): $\mu \to {}_1F_1(1;\mu;x)$ is log-convex on$([0,\infty)$ and $\mu \to {}_1F_1(1;\mu;x)/\Gamma(\mu)$ is discrete log-concave:

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for all $a>0$, $c>a\geq n-1$ and $x\in\mathbb{R}$ or $a\geq n-1$, $c>-1$ $(c\neq 0)$, $x>0$, and positive integer n. If fact, they showed that the left hand side has positive Taylor coefficients.

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${}_1F_1(1;n;x)^2 < {}_1F_1(1;n+\nu;x){}_1F_1(1;n-\nu;x) \Leftrightarrow$ Gautschi (1982) inequality

Here n and $n-\nu$ are non-negative integers, $x>0$.

Sitnik (1993): $\mu \to {}_1F_1(1;\mu;x)$ is log-convex on$([0,\infty)$ and $\mu \to {}_1F_1(1;\mu;x)/\Gamma(\mu)$ is discrete log-concave:

$$ {}_1F_1(1;n+1;x)^2 > \frac{n}{n+1}{}_1F_1(1;n;x){}_1F_1(1;n+2;x) $$

Baricz (2008): $\mu \to {}_1F_1(a;c+\mu;x)$ is log-convex on $[0,\infty)$ for $a,c,x>0$ and $\mu \mapsto {}_1F_1(a+\mu;c+\mu;x)$ is log-convex on $[0,\infty)$ for $a>c>0$, $x>0$

Barnard-Gordy-Richards (2009):

$$ [{}_1F_1(a;c;x)]^2 - {}_1F_1(a+n;c;x){}_1F_1(a-n;c;x) \geq 0 $$

for all $a>0$, $c>a\geq n-1$ and $x\in\mathbb{R}$ or $a\geq n-1$, $c>-1$ $(c\neq 0)$, $x>0$, and positive integer n. If fact, they showed that the left hand side has positive Taylor coefficients.

Rising factorial series

Theorem 1 (K.-Sitnik, 2009)

Suppose $\{f_n\}_0^\infty$ is a positive log-concave (log-convex) sequence. Then the function

$$a \mapsto f(a,x) := \sum_{n=0}^{\infty} f_n \frac{(a)_n}{n!} x^n$$

is strictly log-concave (log-convex) on $(0,\infty)$ for each fixed $x>0$ and, moreover, given any positive a, b and δ the function

$$\varphi_{a,b,\delta}(x) := f(a+\delta,x)f(b,x) - f(b+\delta,x)f(a,x)$$

has positive (negative) power series coefficients so that the function $x \to \varphi_{a,b,\delta}(x)$ $(x \to -\varphi_{a,b,\delta}(x))$ is absolutely monotonic on $(0,\infty)$.

Corollaries and Conjectures

Corollary 1

Suppose $\{f_k\}_0^n$ is a log-concave sequence, $\alpha, \beta > 0$. Then the polynomial

$$P_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} f_k f_{n-k} \binom{n}{k} \left[(x+\alpha)_k (x+\beta)_{n-k} - (x+\alpha+\beta)_k (x)_{n-k}\right],$$

has no positive roots.

Conjecture 1

All coefficients of the polynomial $P_n^{\alpha,\beta}(x)$ are positive.

Conjecture 2

The polynomial $P_n^{\alpha,\beta}(x)$ is Hurwitz stable (all its roots have negative real parts).

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Theorem 2 - gamma function series (K.-Sitnik, 2009)

Suppose $\{g_n\}_0^\infty$ is a positive sequence. Then the function

$$a \to g(a,x) := \sum_{n=0}^{\infty} g_n \Gamma(a+n) x^n$$

is log-convex on $(0,\infty)$. Moreover, given any positive a, b and δ the function

$$\psi_{a,b,\delta}(x) := g(a+\delta,x)g(b,x) - g(b+\delta,x)g(a,x)$$

has negative power series coefficients so that $x \to -\psi_{a,b,\delta}(x)$ is absolutely monotonic on $(0,\infty)$.

Corollary 2

Let $f(a,x) = \sum_{n=0}^{\infty} f_n(a)_n x^n/n!$ with log-concave sequence $\{f_n\}$. Then

$$\frac{\Gamma(a+\delta)\Gamma(b)}{\Gamma(b+\delta)\Gamma(a)} < \frac{f(b+\delta,x)f(a,x)}{f(a+\delta,x)f(b,x)} < 1 \text{ for } b > a > 0 \text{ and } x > 0.$$

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Theorem 3 (K.-Sitnik, 2009)

Suppose $\{h_n\}_0^\infty$ is a positive sequence. Then the function

$$a \to h(a,x) := \sum_{n=0}^{\infty} \frac{h_n}{(a)_n} x^n$$

is log-convex on $(0,\infty)$. Moreover, given any positive a, b and δ the function

$$\lambda_{a,b,\delta}(x) := h(a+\delta,x)h(b,x) - h(b+\delta,x)h(a,x)$$

has negative power series coefficients so that $x \to -\lambda_{a,b,\delta}(x)$ is absolutely monotonic on $(0,\infty)$.

Theorem 4 (Kalmykov-K., 2011)

Suppose $\{q_n\}_0^\infty$ is a positive log-concave sequence. Then the function

$$a \mapsto q(a,x) := \sum_{n=0}^{\infty} \frac{q_n x^n}{n!\Gamma(a+n)}, \qquad (3)$$

is strictly log-concave on $(0,\infty)$ for each fixed $x>0$ and, moreover, given any positive a, b and δ the function

$$\eta_{a,b,\delta}(x) := q(a+\delta,x)q(b+\delta,x) - q(a+b+\delta,x)q(\delta,x)$$

has positive power series coefficients so that the function $x \to \eta_{a,b,\delta}(x)$ is absolutely monotonic on $(0,\infty)$.

Theorem 5 (Kalmykov-K., 2011)

Suppose $c > a > 0$ and $\{f_n\}_0^\infty$ is a positive log-concave sequence. Then the function

$$\mu \mapsto f(a+\mu, c+\mu; x) := \sum_{n=0}^{\infty} f_n \frac{(a+\mu)_n}{(c+\mu)_n} \frac{x^n}{n!},$$

is strictly discrete Wright log-concave on $(0, \infty)$ for each fixed $x > 0$. Moreover, given any $\mu > 0$ the function

$$\varphi_{a,c,\mu}(x) := f(a+1, c+1; x) f(a+\mu, c+\mu; x) - f(a, c; x) f(a+\mu+1, c+\mu+1; x)$$

has positive power series coefficients so that the function $x \to \varphi_{a,c,\mu}(x)$ is absolutely monotonic on $(0, \infty)$. If $a > c > 0$ and $\{f_n\}_0^\infty$ is any positive sequence, then $\mu \mapsto f(a+\mu, c+\mu; x)$ is strictly log-convex on $(0, \infty)$ for each fixed $x > 0$.

Series in ratios of gamma functions

Theorem 6 (Kalmykov-K., 2011)

Suppose $a > c > 0$ and $\{g_n\}_0^\infty$ is a positive log-concave sequence. Then the function

$$\mu \mapsto g(a+\mu, c+\mu; x) := \sum_{n=0}^{\infty} g_n \frac{\Gamma(a+\mu+n)}{\Gamma(c+\mu+n)} \frac{x^n}{n!},$$

is strictly discrete Wright log-concave on $(0, \infty)$ for each fixed $x > 0$. Moreover, given any $\mu > 0$ the function

$$\psi_{a,c,\mu}(x) :=$$
$$g(a+1, c+1; x)g(a+\mu, c+\mu; x) - g(a, c; x)g(a+\mu+1, c+\mu+1; x)$$

has positive power series coefficients so that the function $x \rightarrow \psi_{a,c,\mu}(x)$ is absolutely monotonic on $(0, \infty)$. If $c > a > 0$ and $\{g_n\}_0^\infty$ is any positive sequence, then $\mu \mapsto g(a+\mu, c+\mu; x)$ is strictly log-convex on $(0, \infty)$ for each fixed $x > 0$.

Conjecture

Conjecture 3

The word "discrete" may be removed from Theorems 5 and 6.

Lemma (Kalmykov-K., 2011)

The following identity holds for the Kummer function ${}_1F_1$:

$$
\begin{aligned}
&{}_1F_1(a+\mu;c+\mu;x)\,{}_1F_1(a+1;c+1;x)\\
&\qquad -{}_1F_1(a+\mu+1;c+\mu+1;x)\,{}_1F_1(a;c;x)\\
&\qquad = \frac{(c-a)x}{c(c+1)(c+\mu)(c+\mu+1)}\times\\
&\Big\{(c+\mu)(c+\mu+1)\,{}_1F_1(a+1;c+2;x)\,{}_1F_1(a+\mu+1;c+\mu+1;x)\\
&\qquad -c(c+1)\,{}_1F_1(a+1;c+1;x)\,{}_1F_1(a+\mu+1;c+\mu+2;x)\Big\}.
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\end{aligned}$$

Application to generalized hypergeometric function

Let $e_m(c_1, \ldots, c_q)$ denote m-th elementary symmetric polynomial,

$$e_m(c_1, \ldots, c_q) = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le q} c_{i_1} c_{i_2} \cdots c_{i_m}$$

Lemma (Heikkala, Vamanamurthy, Vuorinen, 2009), (K.-Sitnik, 2009)

Suppose $a_i, b_i > 0$, $i = 1, \ldots, q$. The sequence of hypergeometric terms

$$f_n = \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_q)_n} \quad \text{is log-concave if}$$

$$\frac{e_q(b_1, \ldots, b_q)}{e_q(a_1, \ldots, a_q)} \le \frac{e_{q-1}(b_1, \ldots, b_q)}{e_{q-1}(a_1, \ldots, a_q)} \le \cdots \le \frac{e_1(b_1, \ldots, b_q)}{e_1(a_1, \ldots, a_q)} \le 1. \quad (4)$$

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Some hypergeometric examples

- For $a > b > 0$, $c > 0$ and integer $m \geq 2$

$$ {}_4F_3\left(\begin{matrix} -m, a, 1-c-m, 1-am/(a+b) \\ c, 1-b-m, -am/(a+b) \end{matrix} \middle| -1\right) > 0, $$

  and for $b > a > 0$ the sign of inequality is reversed;
- The function $\alpha \mapsto {}_2F_1(\alpha, b; c; x)$ is log-concave, on $(0, \infty)$ if $0 < x < 1$, $b > c > 0$ or $x < 0$, $c > 0 > b$ and on $(-\infty, c]$ if $0 < x < 1$, $c > 0 > b$ or $x < 0$, $b > c > 0$;
- The function $\alpha \mapsto {}_3F_2\left(\alpha, a_1, a_2; b_1, b_2; x\right)$, $0 < x < 1$ is log-concave on $(0, \infty)$ if

$$ \frac{b_1 b_2}{a_1 a_2} \leq \frac{b_1 + b_2}{a_1 + a_2} \leq 1; $$

- The function $\alpha \mapsto {}_2F_2(\alpha, a_1; b_1, b_2; x)$, $0 < x < \infty$, is log-concave on $(0, \infty)$ if $a_1 \geq b_1 b_2/(b_1 + b_2)$;

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An application: directional statistics

Probability density function of multivariate Watson distribution:

$$\rho(\pm\mathbf{x};\boldsymbol{\mu},\kappa)=\frac{\Gamma(d/2)}{2\pi^{d/2}{}_1F_1(1/2;d/2;\kappa)}e^{\kappa(\boldsymbol{\mu},\mathbf{x})^2}.$$

The distribution is defined in projective hyperplane $\mathbb{P}^{d-1}$ = sphere $\mathbb{S}^{d-1}$ with opposite points identified. $\boldsymbol{\mu}$ and $\mathbf{x}$ are unit vectors in $\mathbb{R}^d$.

Maximum likelihood estimation for Watson distributions leads to a particular case of the equation

$$g(a,c,x):=\frac{{}_1F_1{}'(a,c;x)}{{}_1F_1(a,c;x)}=r,\quad r\in(0,1),\quad c>a>0. \qquad (6)$$

Theorem: uniqueness of solution (K.-Sra, 2010)

Let $c>a>0$. Then $g(a,c,x)$ is monotone decreasing on $\mathbb{R}$ mapping it onto $(0,1)$, so that for each $r\in(0,1)$ the solution of (6) exists and is unique.

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Introduce the notation:

$$L(r) = \frac{rc - a}{r(1-r)} \left(1 + \frac{1-r}{c-a}\right),$$

$$B(r) = \frac{rc - a}{2r(1-r)} \left(1 + \sqrt{1 + \frac{4(c+1)r(1-r)}{a(c-a)}}\right),$$

$$U(r) = \frac{rc - a}{r(1-r)} \left(1 + \frac{r}{a}\right).$$

Theorem: two-sided bounds (K.-Sra, 2010)

For $a/c < r < 1$ we have

$$L(r) < x(r) < B(r) < U(r). \tag{7}$$

For $0 < r < a/c$ we have

$$L(r) < B(r) < x(r) < U(r). \tag{8}$$

If $r = a/c$ we have $x = L(a/c) = B(a/c) = U(a/c) = 0$. All three bounds are also asymptotically precise at $r = 0$ and $r = 1$.

Curious open problems

Turán (1946) inequality for Legendre polynomials:

$$ {}_2F_1(-\mu, 1+\mu; 1; y)^2 > {}_2F_1(-\mu-\delta, 1+\mu+\delta; 1; y)\,{}_2F_1(-\mu+\delta, 1+\mu-\delta; 1; y), $$

for $y \in (0,1)$, $\mu = 1, 2, \ldots$ and $\delta = 1$.

Conjecture 4

Turán inequality is true for all $\mu > 0$ and $0 < \delta < 1$.

Prékopa-Ninh while studying some convex optimization problems conjectured that

$$ \left[(1+k)\,{}_2F_2\left(\begin{matrix} k/2+1, k/2+3/2 \\ 3/2, 2 \end{matrix}\middle| x\right)\right]^2 \geq $$

$$ k\,{}_2F_2\left(\begin{matrix} k/2+1/2, k/2+1 \\ 3/2, 2 \end{matrix}\middle| x\right)(k+2)\,{}_2F_2\left(\begin{matrix} k/2+2, k/2+3/2 \\ 3/2, 2 \end{matrix}\middle| x\right) $$

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THANK YOU FOR ATTENTION!