Discrete and Continuous Painlevé Equations

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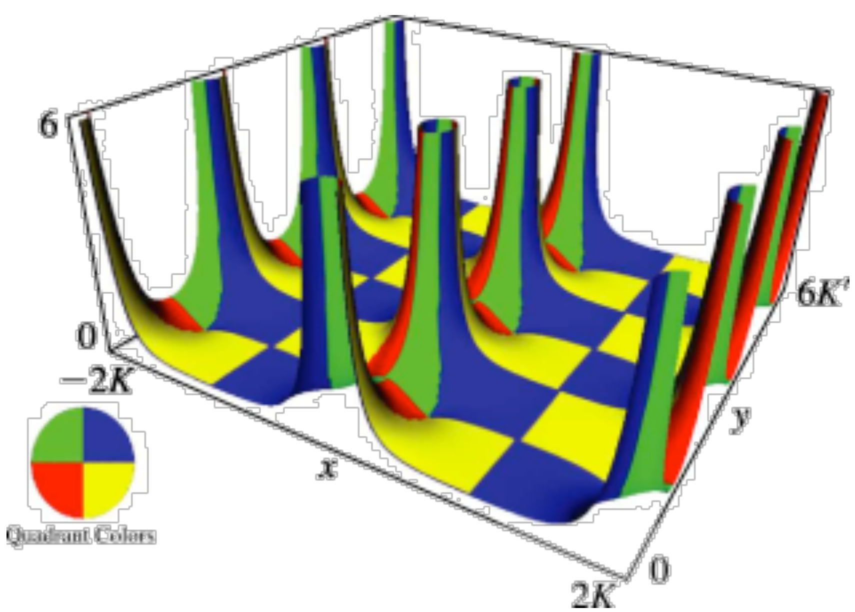
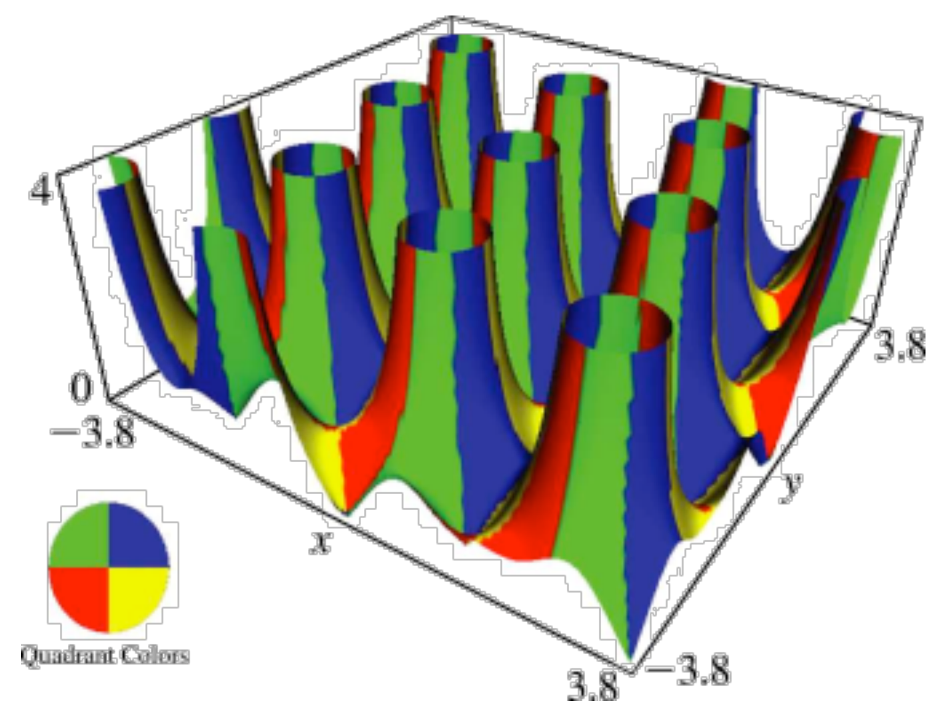
DLMF

- Weierstrass elliptic functions $\wp(z)$





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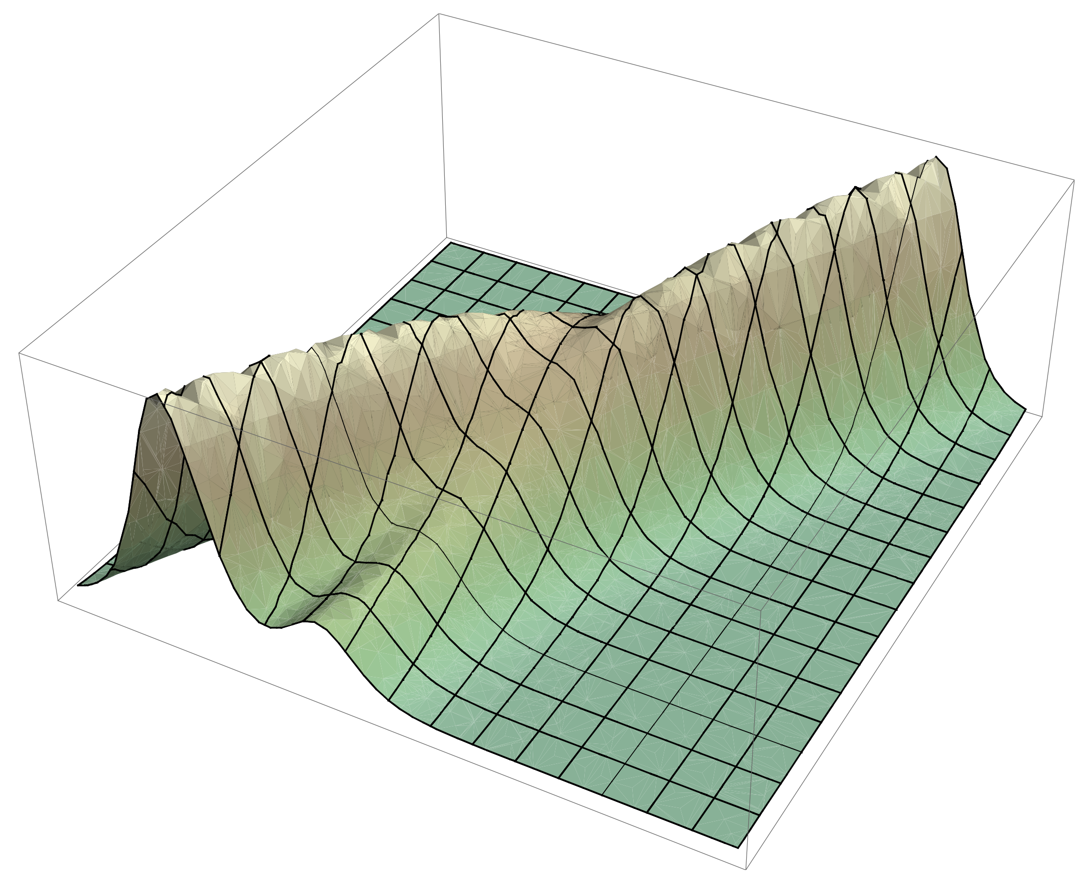
Painlevé Equations

The Painlevé Equations

$$
\begin{aligned}
\mathrm{P_I}:\ y'' &= 6y^2 + x \\
\mathrm{P_{II}}:\ y'' &= 2y^3 + xy + \alpha \\
\mathrm{P_{III}}:\ y'' &= \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y} \\
\mathrm{P_{IV}}:\ y'' &= \frac{y'^2}{2y} + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y} \\
\mathrm{P_V}:\ y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2 y}(\alpha y^2 + \beta) \\
&\quad + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1} \\
\mathrm{P_{VI}}:\ y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y' \\
&\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right)
\end{aligned}
$$

Order and Chaos

- In 1953, around the same time as Lorenz discovered chaos, Fermi, Pasta and Ulam found astonishingly well-ordered and predictable behaviour in models used to describe thermal properties of metals, leading to the theory of soliton equations.



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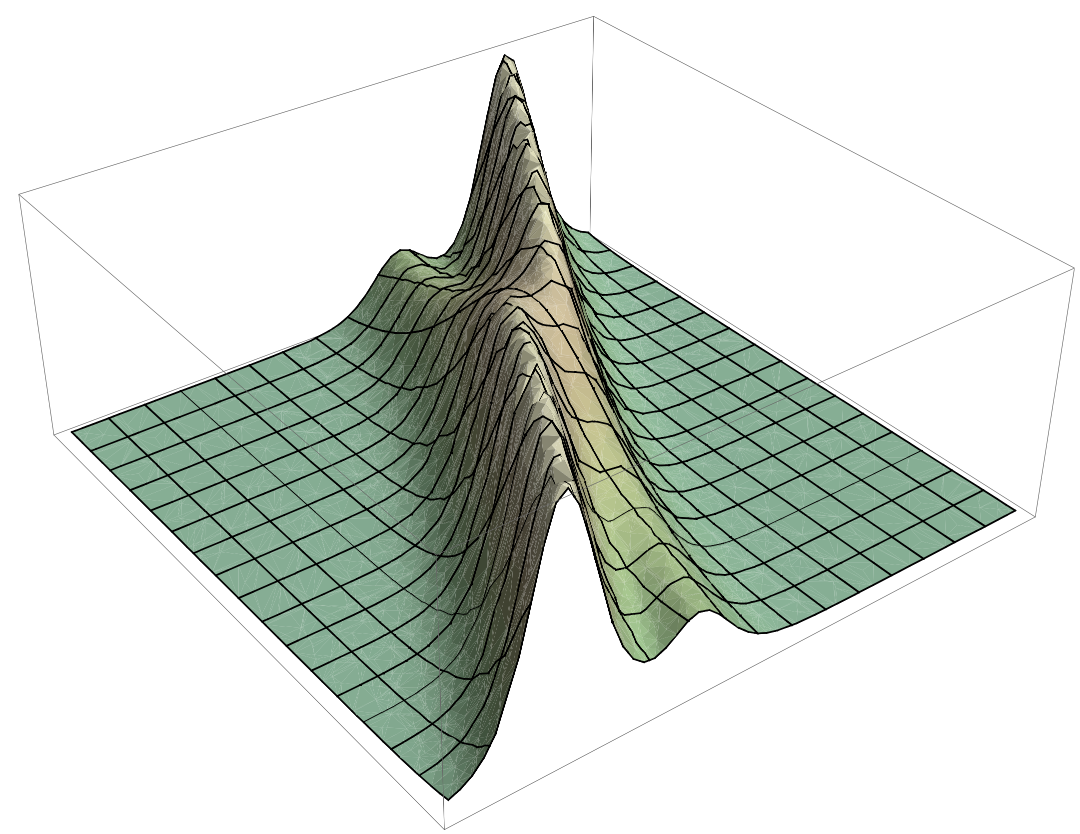

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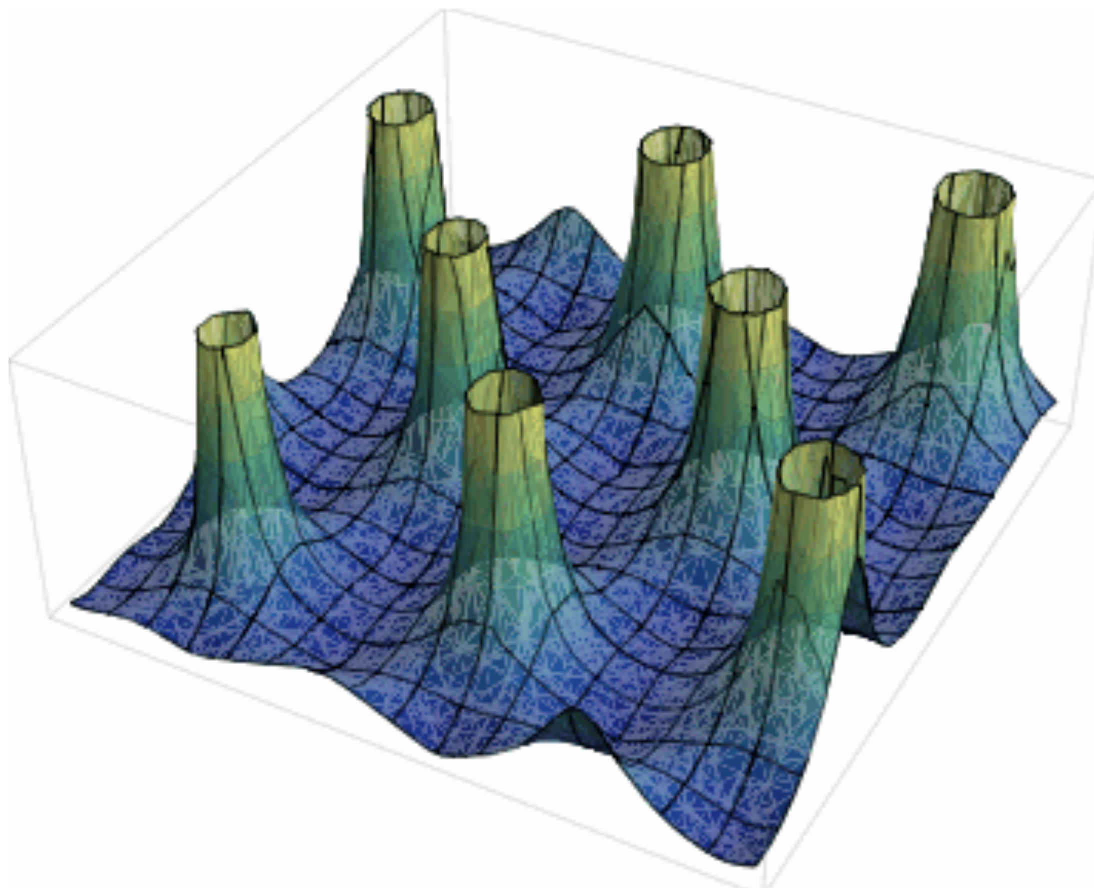
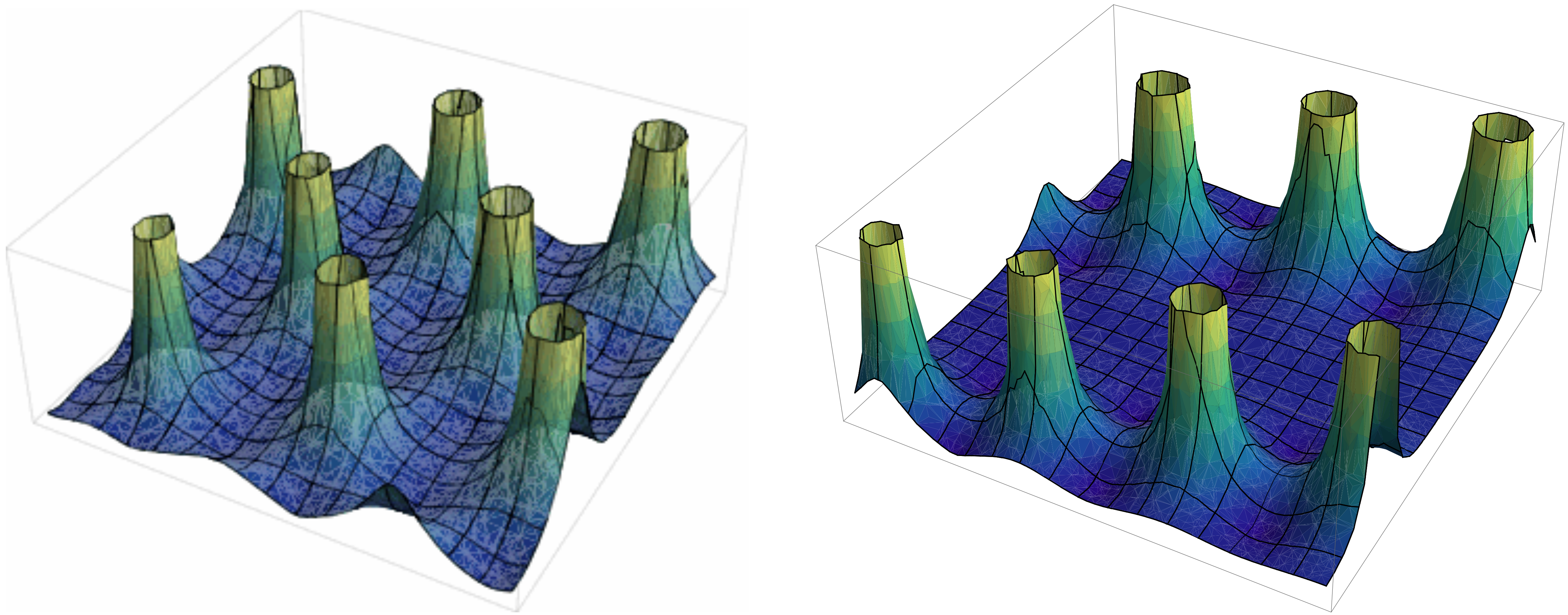
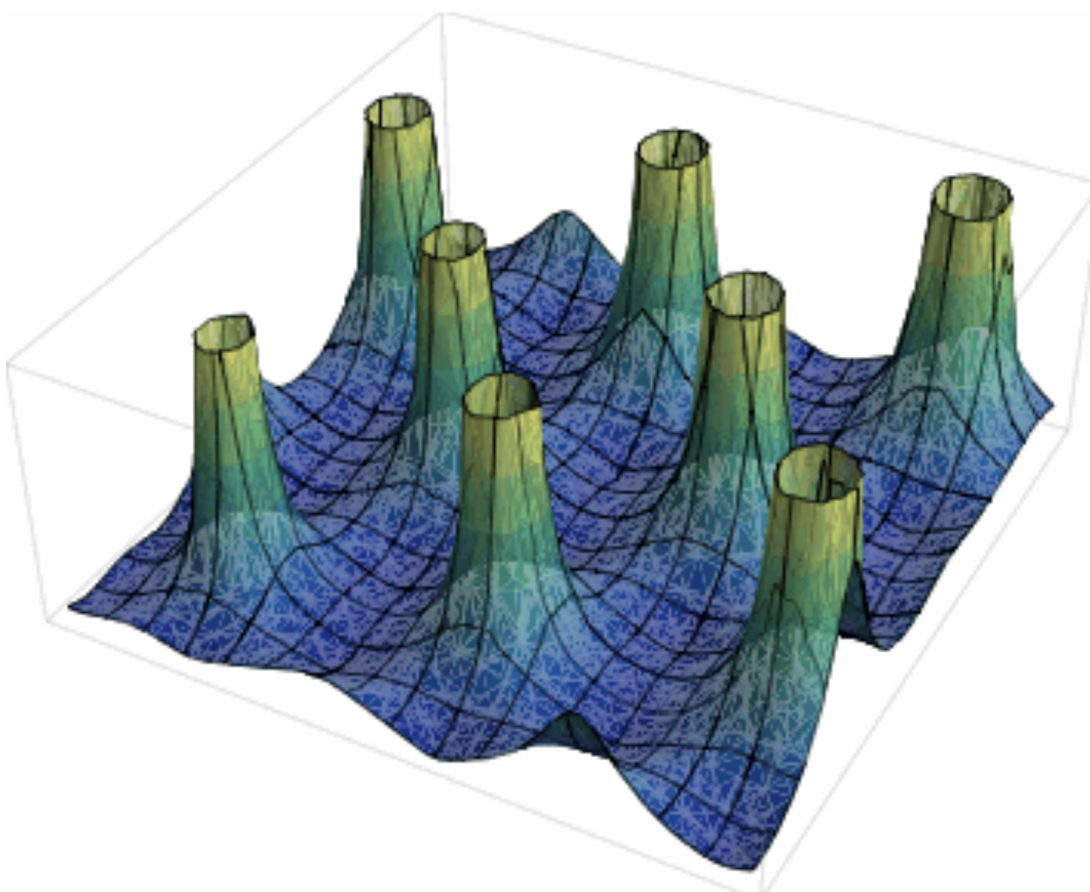
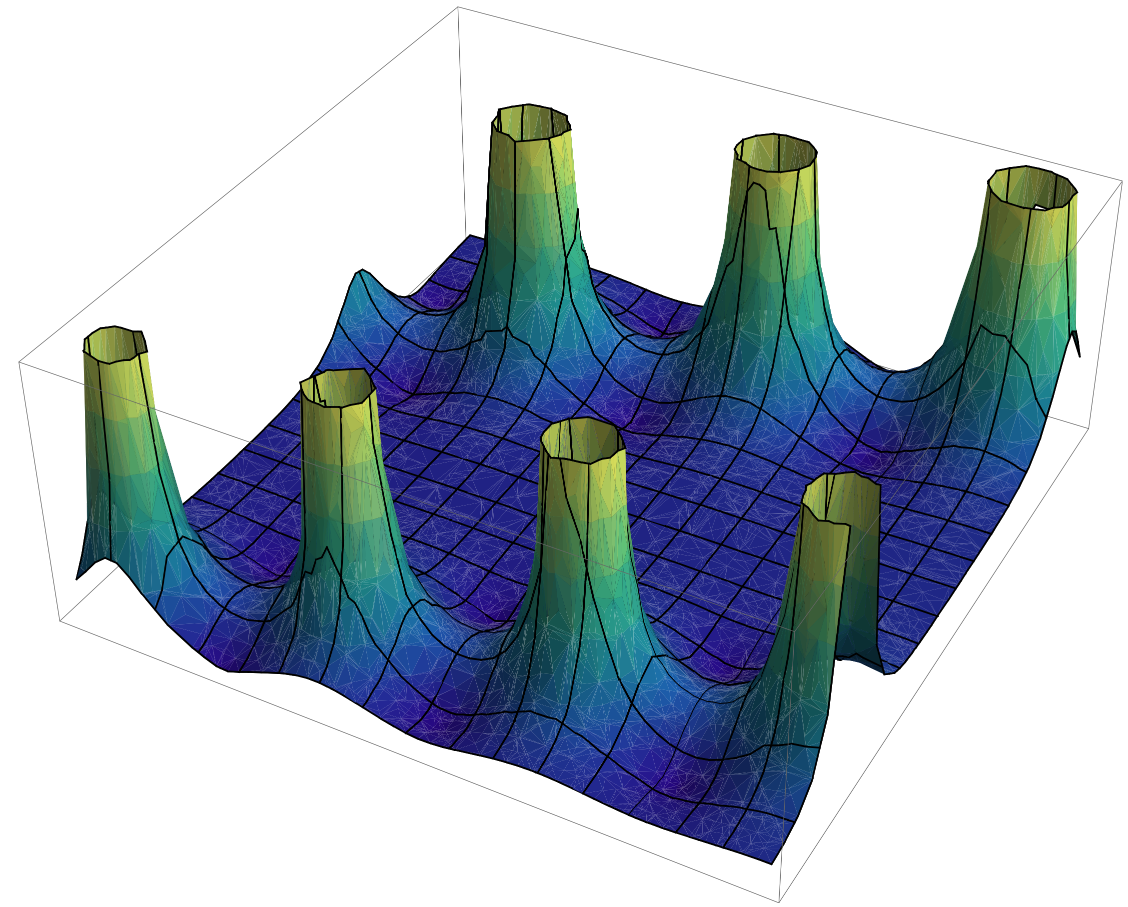




- General solutions are new transcendental functions, with movable poles.





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- P_{II}-P_{VI} have special solutions for certain parameter values (⇒ rational or hypergeometric-type solutions).

Linear problems

- Each Painlevé equation is a compatibility condition for pairs of linear problems:

$$\left.\begin{aligned} \frac{\partial Y}{\partial \lambda} &= A(\lambda, t)\, Y \\ \frac{\partial Y}{\partial t} &= B(\lambda, t)\, Y \end{aligned}\right\} \Rightarrow \quad \frac{\partial A}{\partial t} - \frac{\partial B}{\partial \lambda} + [A, B] = 0$$

called Lax pairs or iso-monodromy problems.

- These provide information about the solutions of Painlevé equations.

P_{II} Lax pair

$$A = -4\,i\,\lambda^2\sigma_3 + 4\,y\,\lambda\,\sigma_1 - 2\,y'\sigma_2 - i\,(2y^2 + t)\,\sigma_3 - \frac{\alpha}{\lambda}\sigma_1$$

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where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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➡ $$\mathrm{P}'_{\mathrm{II}} : \; y''' \;=\; 6y^2y' + t\,y' + y$$

Information near 0, ∞

- The Stokes multipliers near ∞
- The monodromy matrix around 0
- The connection matrix between 0 & ∞

remain unchanged as *t* varies. When $\lambda \gg 1$

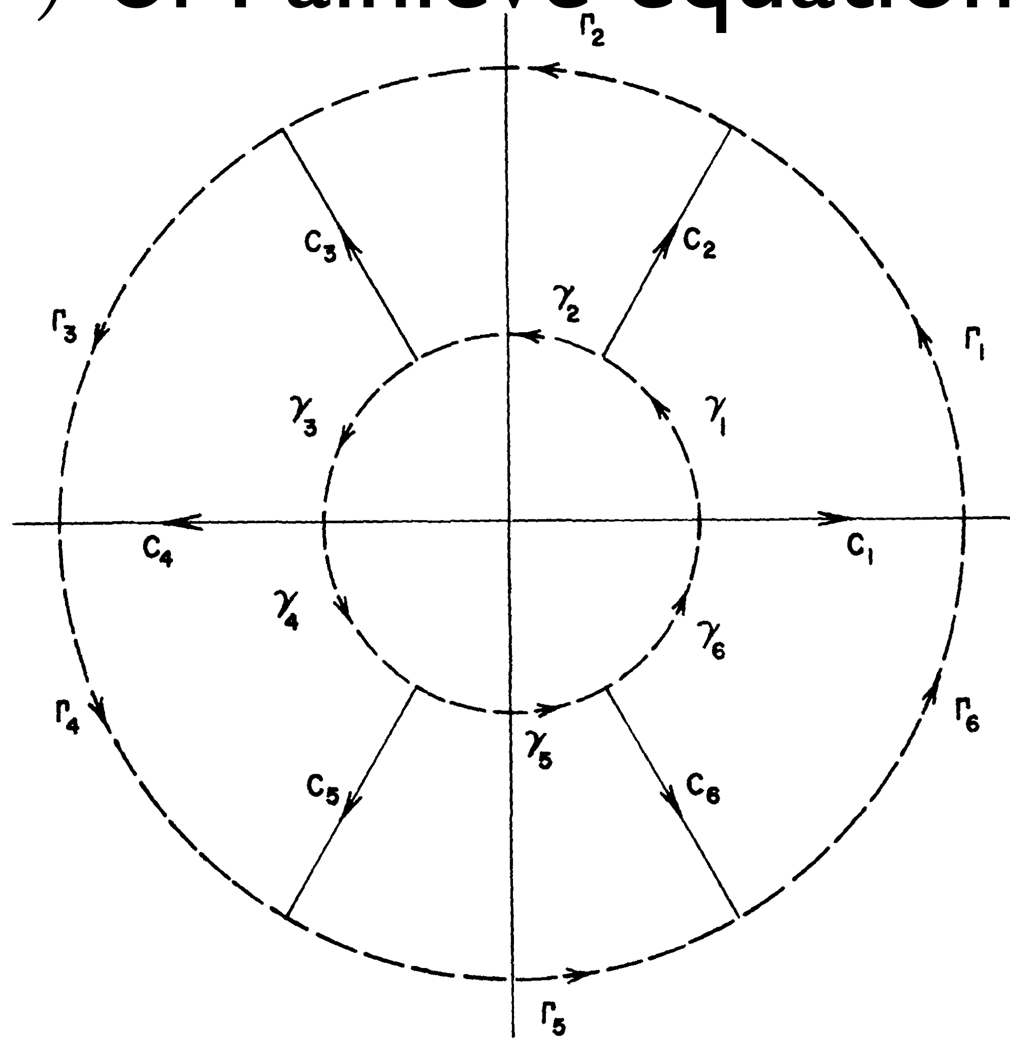
$$Y = a_1 \exp^{-i(4\lambda^3/3+\lambda\, t)}\ Y_1 + a_2 \exp^{i(4\lambda^3/3+\lambda\, t)}\ Y_2$$

provides global information for $t \gg 1$, through a WKB approach.

Riemann-Hilbert Approach

- This inverts monodromy data characterising solutions $Y(\lambda, t)$ of the linear system to describe solutions $y(t)$ of Painlevé equations.

Most effective in limits such as $t \to \infty$



and for special solns, e.g., $y(t) \equiv 0$ for $\alpha = 0$

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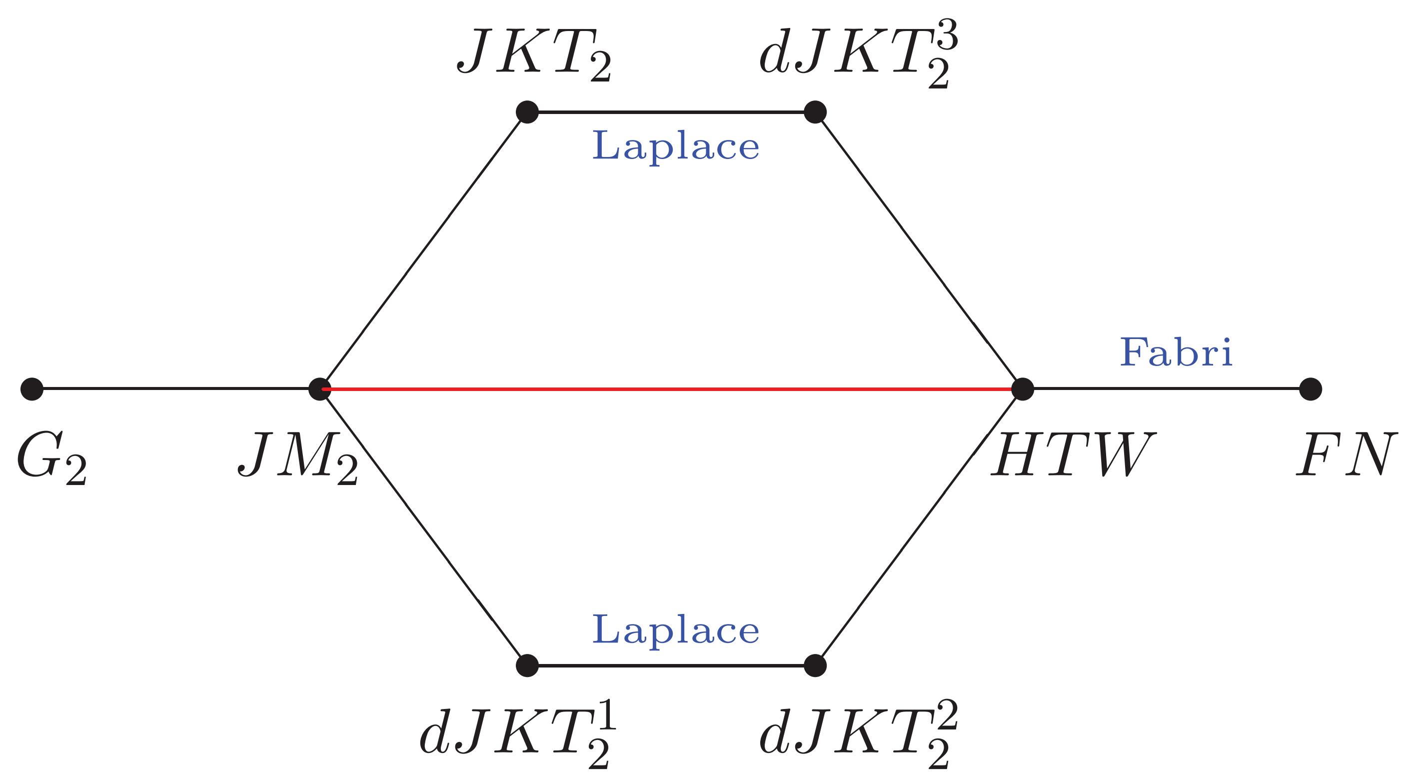
$$\mathrm{P}_{\mathrm{II}} : \; y'' \quad = \quad 2y^3 + t\,y + \frac{1}{2} - \theta$$

Relating Lax pairs

- Recently, we found that the Lax pairs for PII can be mapped invertibly to each other



J, Kitaev and Treharne *JPhysA* 42 (2009) 055208

Water Waves

- Dubrovin, Grava and Klein *J. Nonlin. Sci* (2009) analysed critical behaviour of non-linear water waves under Hamiltonian perturbations

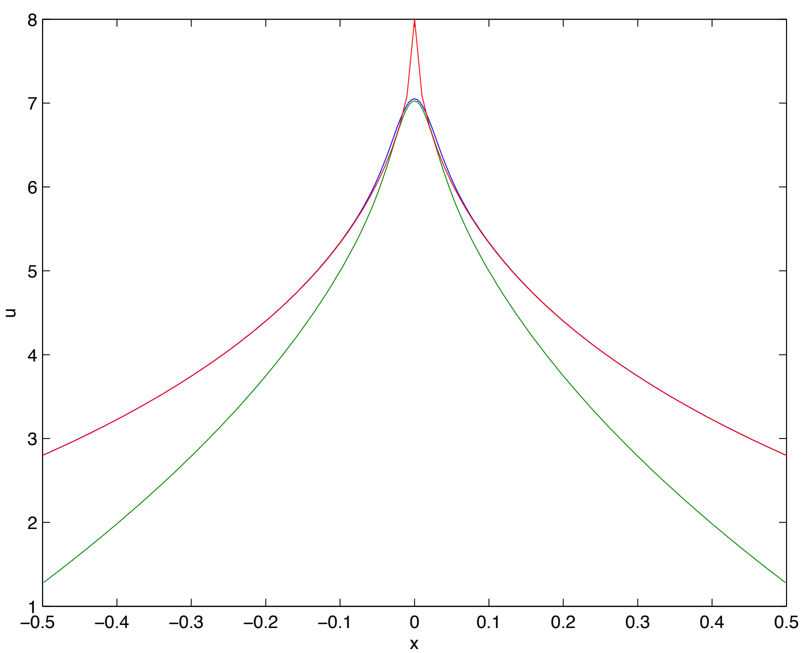


Figure 8: The blue line is the function u of the solution to the focusing NLS equation for the initial data $u(x,0) = 2\,\mathrm{sech}\,x$ and $\epsilon = 0.04$ at the critical time, and the red line is the corresponding semiclassical solution given by formulas (2.4). The green line gives the multiscales solution via the tritronquée solution of the Painlevé I equation.

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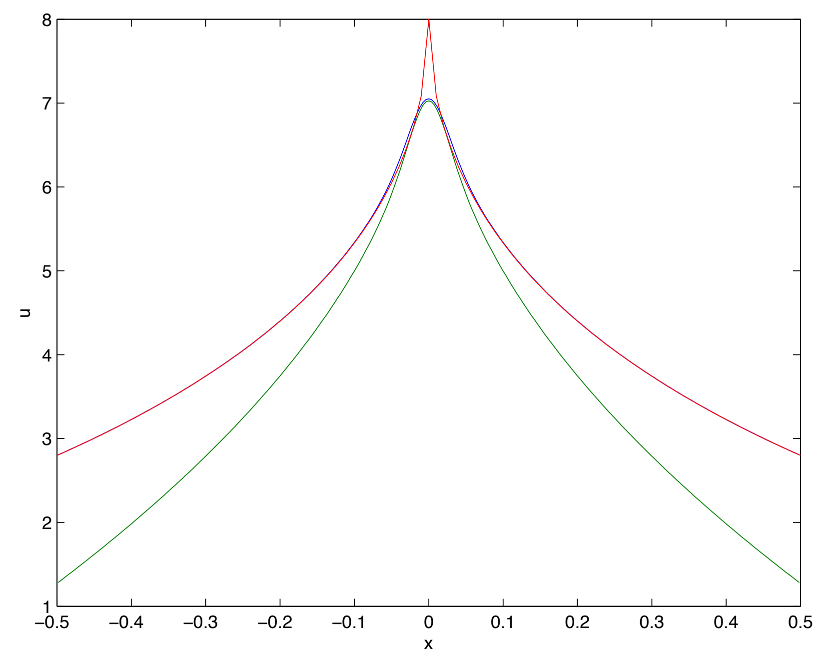

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Tritronquée Solutions

- These are asymptotic to an algebraic expansion y_f in sectors of width $4\pi/5$ in $\mathbb{C}$.

From Dubrovin *et al arXiv:0704.0501*

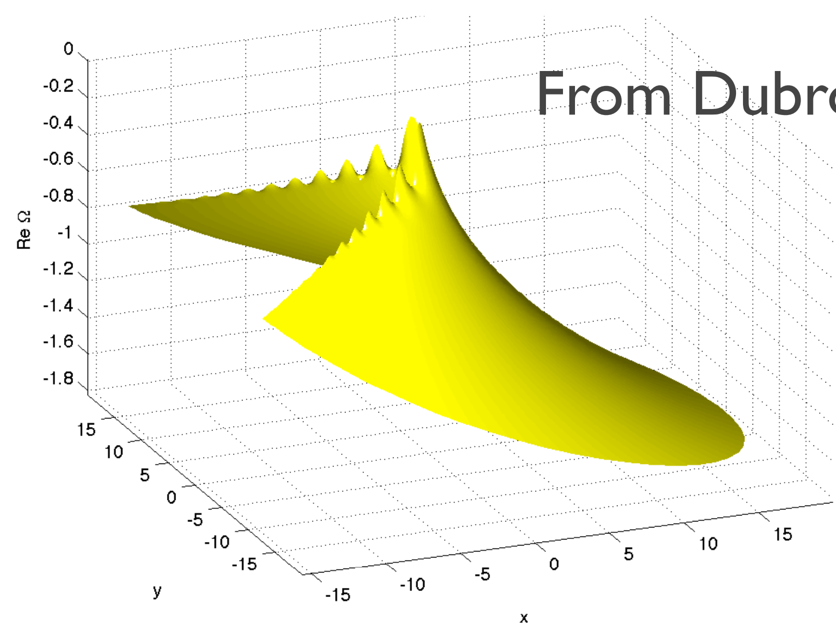
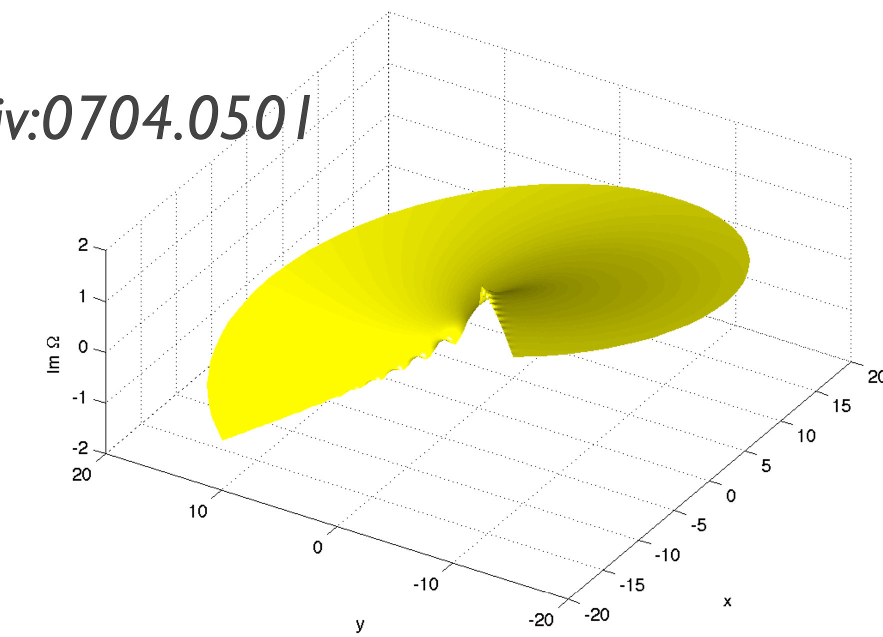


Figure 5: Real part of the *tritronquée* solution in the sector $r < 20$ and $|\phi| < 4\pi/5 - 0.05$.



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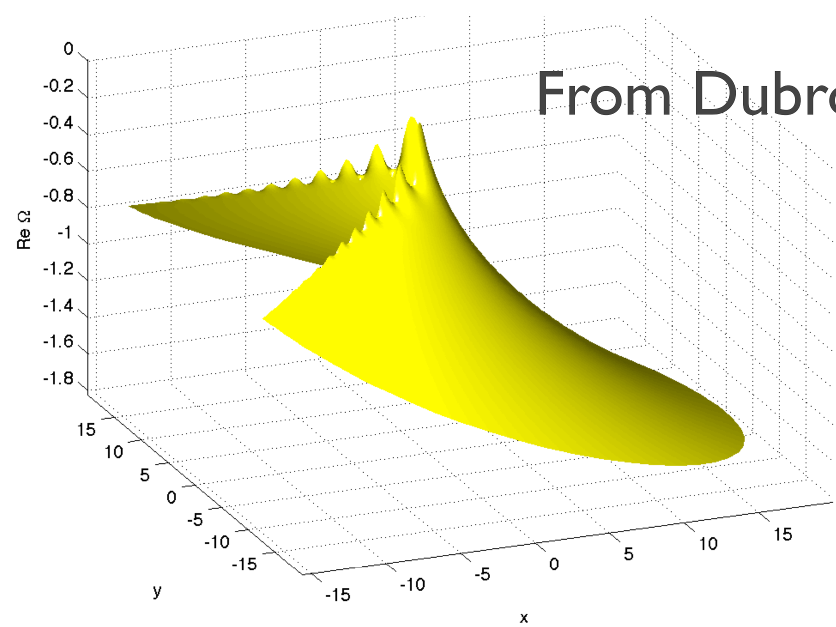

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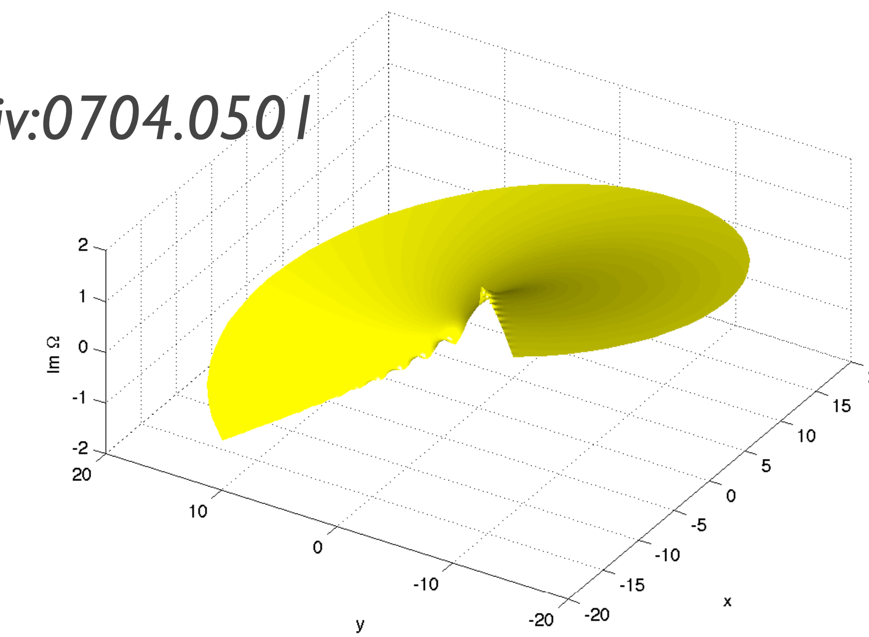

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Dubrovin's conjecture in $|\arg(x)| < 4\pi/5$

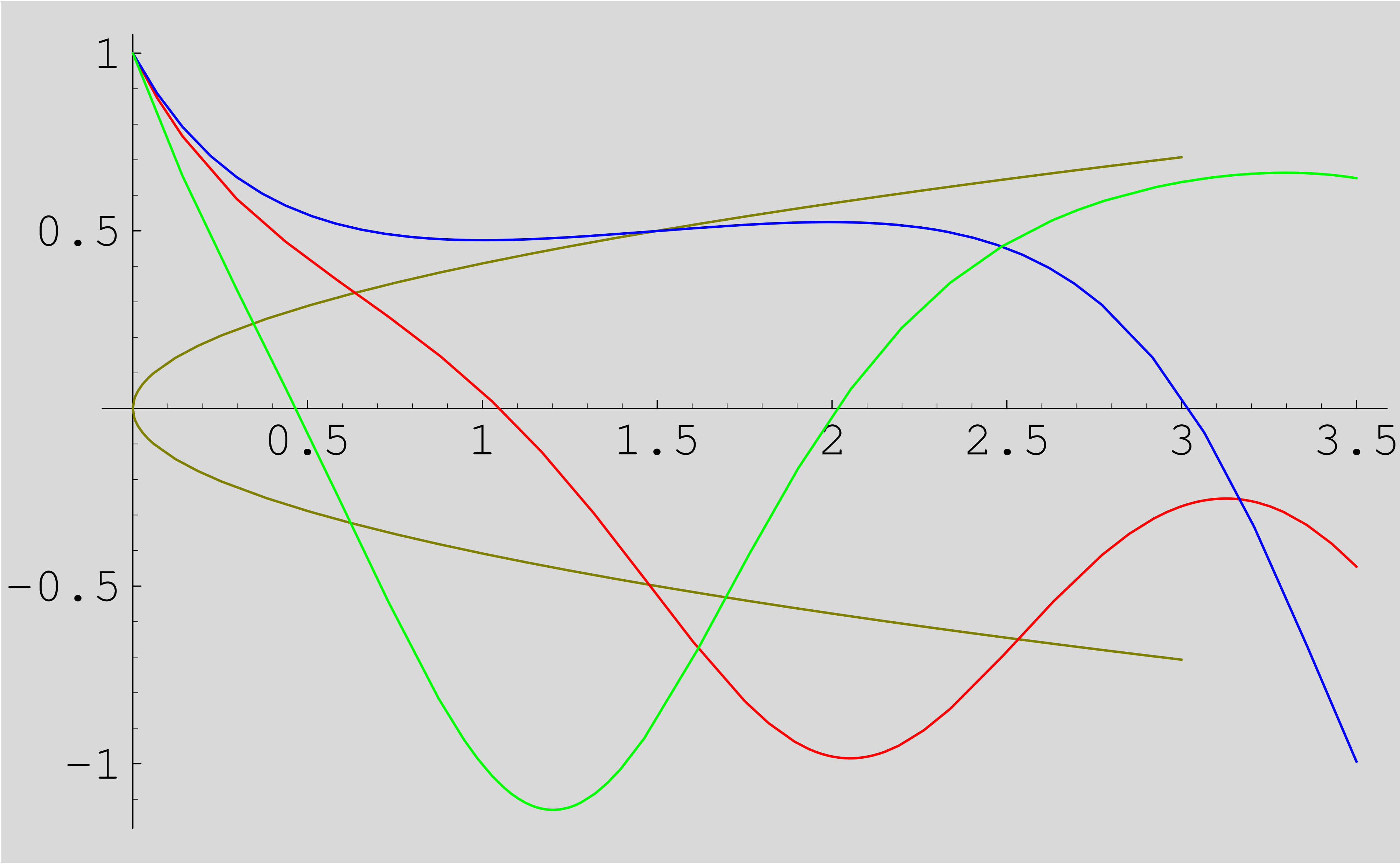
In the Finite Plane

- While asymptotic behaviours of solutions are now well known, finite behaviours remain open.
- We started a study of Painlevé transcendents by starting with initial value problems at the origin.
- This approach provided us with the first proof that the real *tritronquée* solution has no poles on the positive real line, for

$$y'' = 6\,y^2 - x$$

Real Solutions

- Consider P_I $y'' = 6\,y^2 - x$ for *y(x)*, *x*$\in\mathbb{R}$



The Real Tritronquée

- Theorem: ∃ unique solution *Y(x)* of PI which has asymptotic expansion

$$y_f = -\sqrt{\frac{x}{6}}\sum_{k=0}^{\infty}\frac{a_k}{(x^{1/2})^{5k}}, \text{ in } |\arg(x)| \le 4\pi/5$$

and

- *Y(x) is real for real x*
- *Its interval of existence I contains* $\mathbb{R}$
- *Y(x) lies below* Π_-
- *It is monotonically decaying in I.*

From *J.& Kitaev Studies in Appl Math (2001).*

Poles & Zeroes

- From the proof, we found

$$Y(0) = -0.18755430\ldots \quad Y'(0) = -0.3049055\ldots$$

- Let x_p be its first real pole, ζ be its first zero, define c by

$$y(x) = \frac{1}{(x-x_p)^2} + \frac{x_p}{10}(x-x_p)^2 + \frac{1}{6}(x-x_p)^3 + c\,(x-x_p)^4 + \ldots$$

- Then

$$\zeta = -0.49991255\ldots \quad Y'(\zeta) = -0.46886551\ldots$$

$$x_p = -2.3841687\ldots \quad c = -0.06213573$$

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- How does $Y(x)$ behave in $\mathbb{C}$?
- How can we describe solutions in finite plane?

Near ∞

- Consider (Duistermaat & J: *arXiv 1010:5563)*
  $y'' = 6y^2 + x$ in Boutroux's coordinates

$$y(x) = x^{1/2}\, u(z), \quad z = \frac{4\, x^{5/4}}{5}$$

$$\Rightarrow \qquad \ddot{u} = 6\, u^2 + 1 - \frac{\dot{u}}{z} + \frac{4\, u}{25\, z^2}$$

$$\Rightarrow \qquad \begin{cases} \dot{u}_1 = u_2 - \dfrac{2\, u_1}{5\, z} \\ \dot{u}_2 = 6\, u_1^2 + 1 - \dfrac{3\, u_2}{5\, z} \end{cases}$$

The Space of Initial Values

- Okamoto showed how to compactify and regularizing the space of initial values *(Japan J. Math 5 (1979)).*

- To compactify, we first embed into the projective plane

$$\overbrace{\left[1:\frac{u_{011}}{u_{010}}:\frac{u_{012}}{u_{010}}\right]}^{Affine\ coordinates} \quad \Leftrightarrow \quad \overbrace{\left[u_{010}:u_{011}:u_{012}\right]}^{Homogeneous\ coordinates}$$

$$u_{010}=0 \quad \Leftrightarrow \quad \mathcal{L}_0$$

The Projective Plane $\mathbb{CP}_2$

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First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021}$$

$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1}u_{022}$$

The Projective Plane $\mathbb{CP}_2$

First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021}$$

$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1}u_{022}$$

Second chart: $[u_2^{-1} : u_1\, u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$

$$\dot{u}_{031} = -u_{031}^2 - 6u_{032}^2 + 3(5z)^{-1}u_{031}$$

$$\dot{u}_{032} = -u_{031}u_{032} - 6u_{031}^{-1}u_{032}^3 + 1 + (5z)^{-1}u_{032}$$

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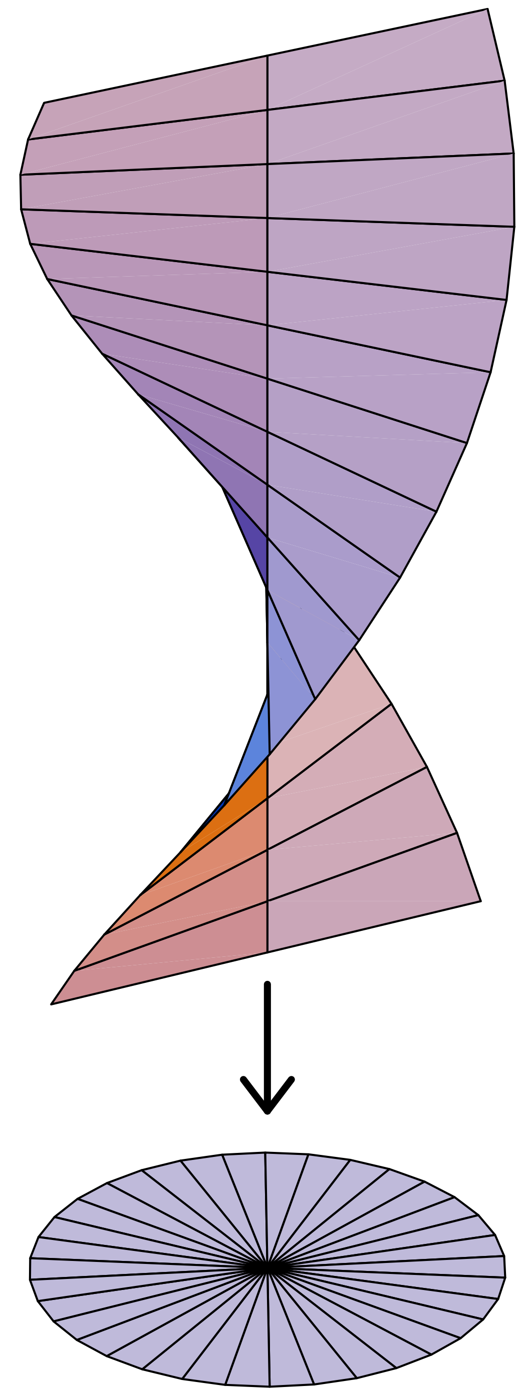
Second chart: $[u_2^{-1} : u_1 u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$

$$\dot{u}_{031} = -u_{031}^2 - 6u_{032}^2 + 3(5z)^{-1} u_{031}$$

$$\dot{u}_{032} = -u_{031} u_{032} - 6u_{031}^{-1} u_{032}^3 + 1 + (5z)^{-1} u_{032}$$

→ base pt $b_0 : \; u_{031} = 0, u_{032} = 0$

Blowing up at a base pt



From JJ Duistermaat, QRT Maps and Elliptic Surfaces, Springer Verlag, 2010

First Blow-up

First Blow-up

- Chart (1,1): $[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$

$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$

$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

First Blow-up

- Chart (1,1): $[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$

$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$

$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

- Chart (1,2): $[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$

$$\dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\,(5z)^{-1}\,u_{121}$$

$$\dot{u}_{122} = u_{121}^{-1} - 2\,(5z)^{-1}\,u_{122}$$

First Blow-up

- **Chart (1,1):** $[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$

$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$

$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

- **Chart (1,2):** $[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$

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First Blow-up

- **Chart (1,1):** $[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$

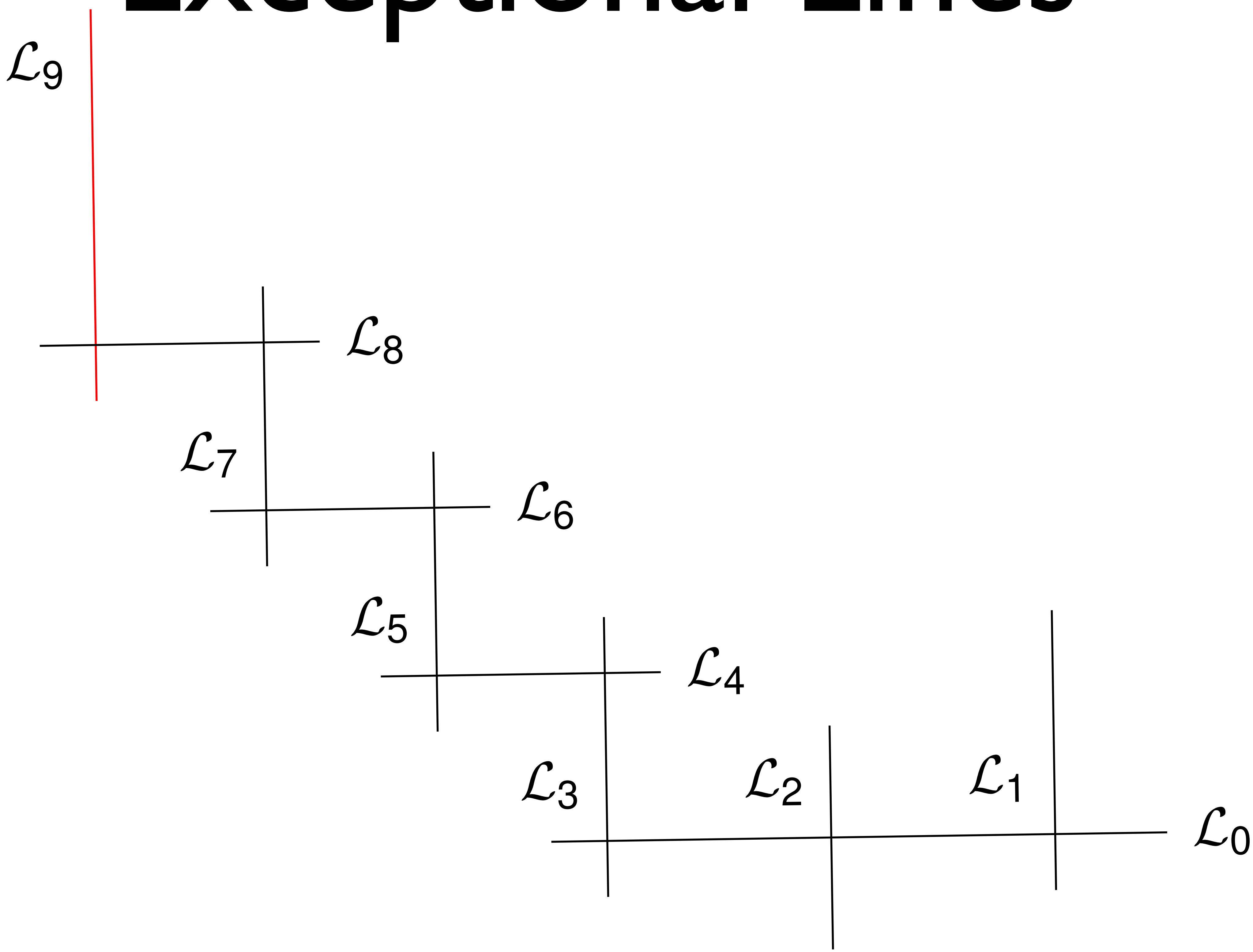
$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$
$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

- **Chart (1,2):** $[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$

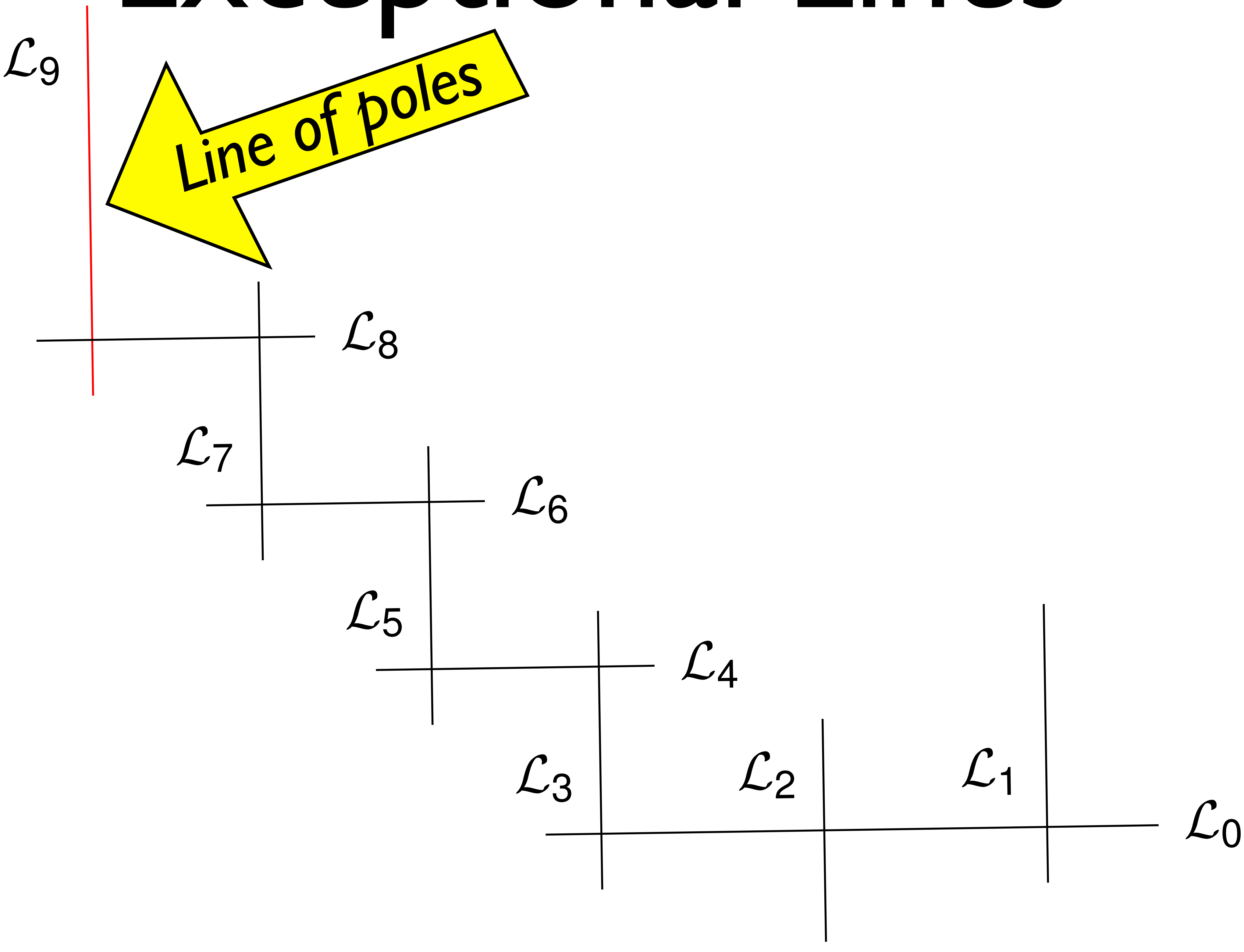
$$\dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\,(5z)^{-1}\,u_{121}$$
$$\dot{u}_{122} = u_{121}^{-1} - 2\,(5z)^{-1}\,u_{122}$$

→ **base pt** $b_1 : \; u_{111} = 0, u_{112} = 0$

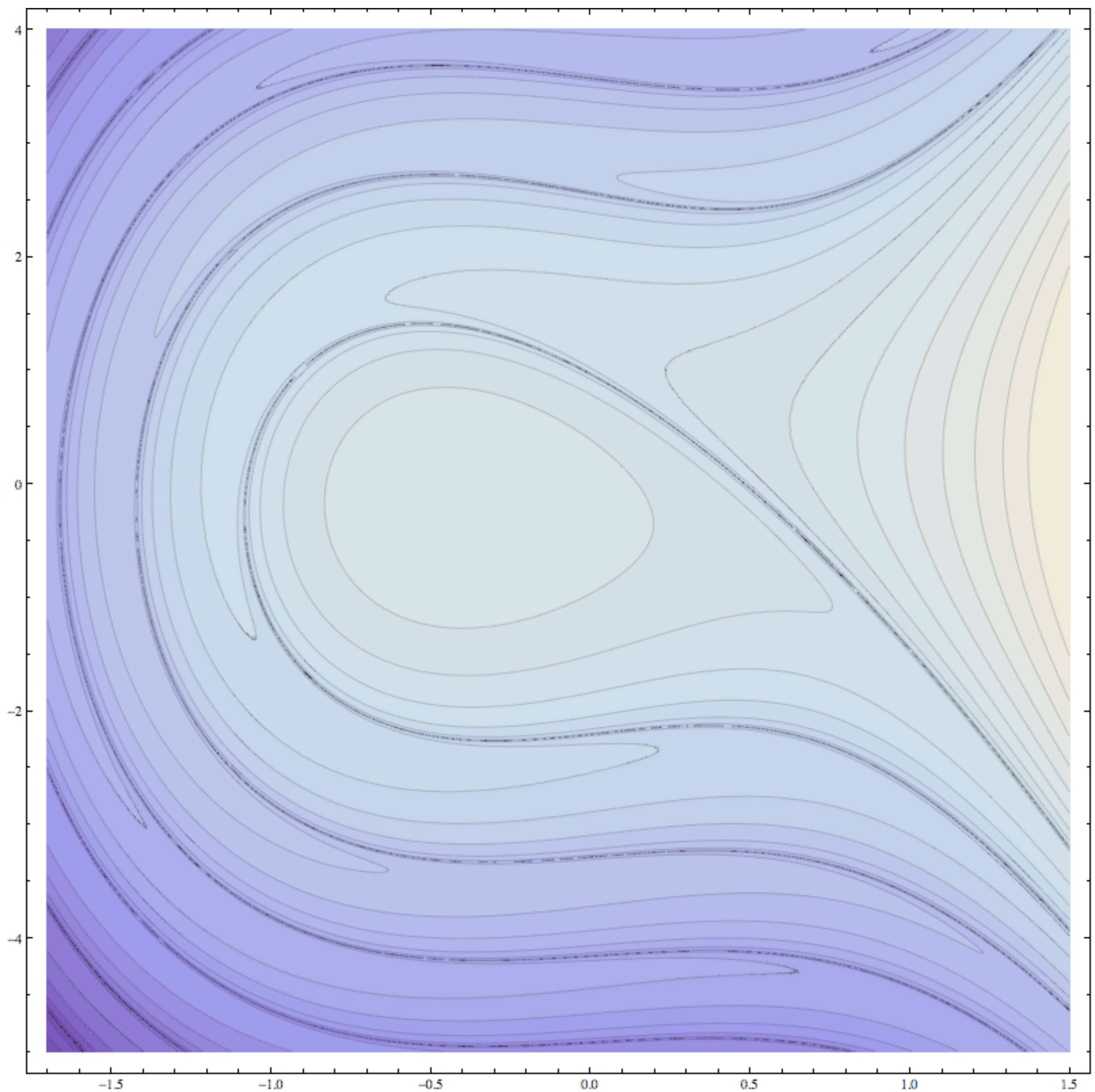
Exceptional Lines



Exceptional Lines



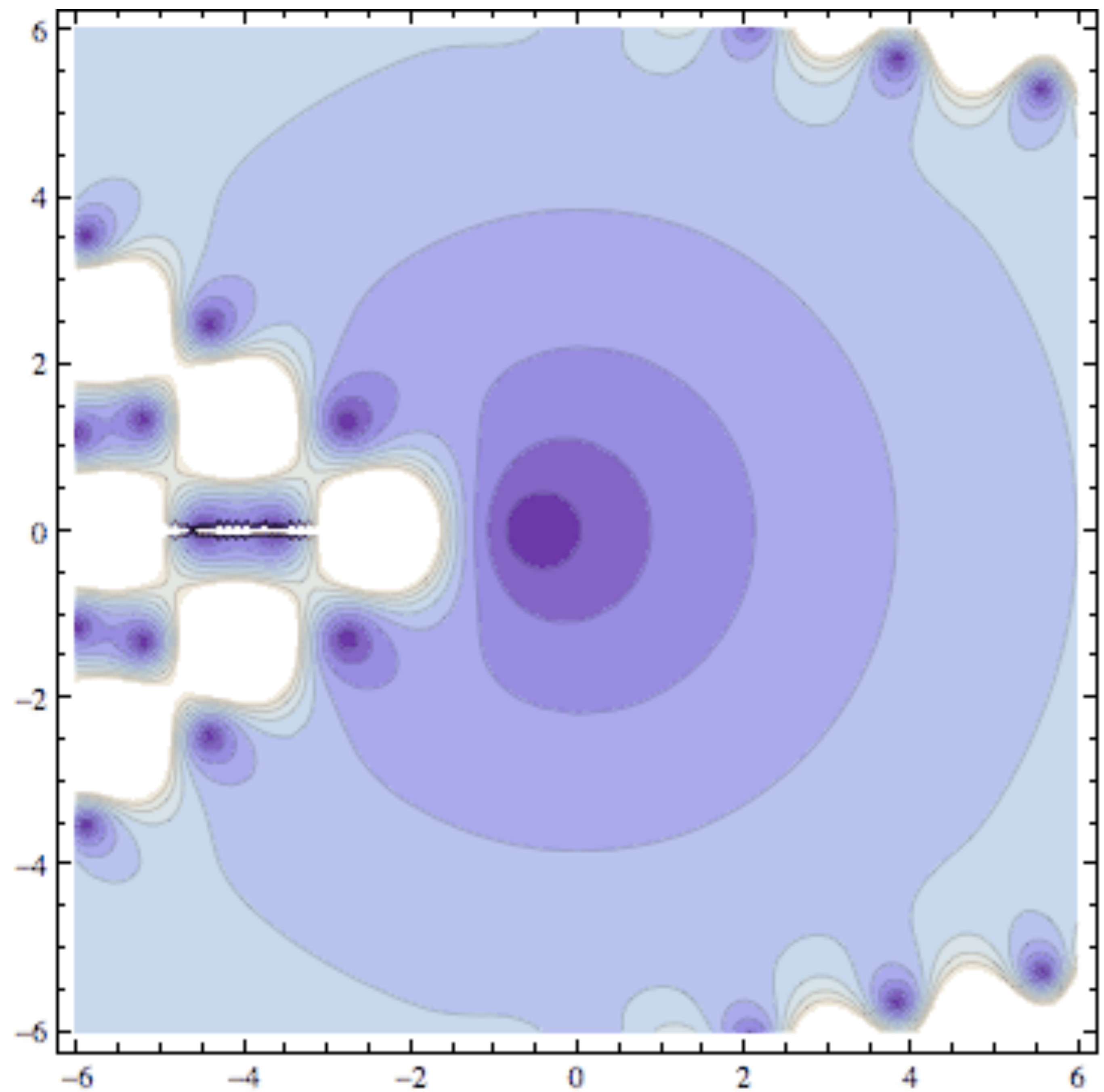
The Phase Plane



Main results

- The union of exceptional lines is a *repellor* for the flow.
- The limit set is non-empty, connected and compact subset of Okamoto's space.
- ∃ unique solutions near equilibria of the flow. They are bounded sufficiently far from *0*.

Approaching *Y*



Discrete Integrable Systems

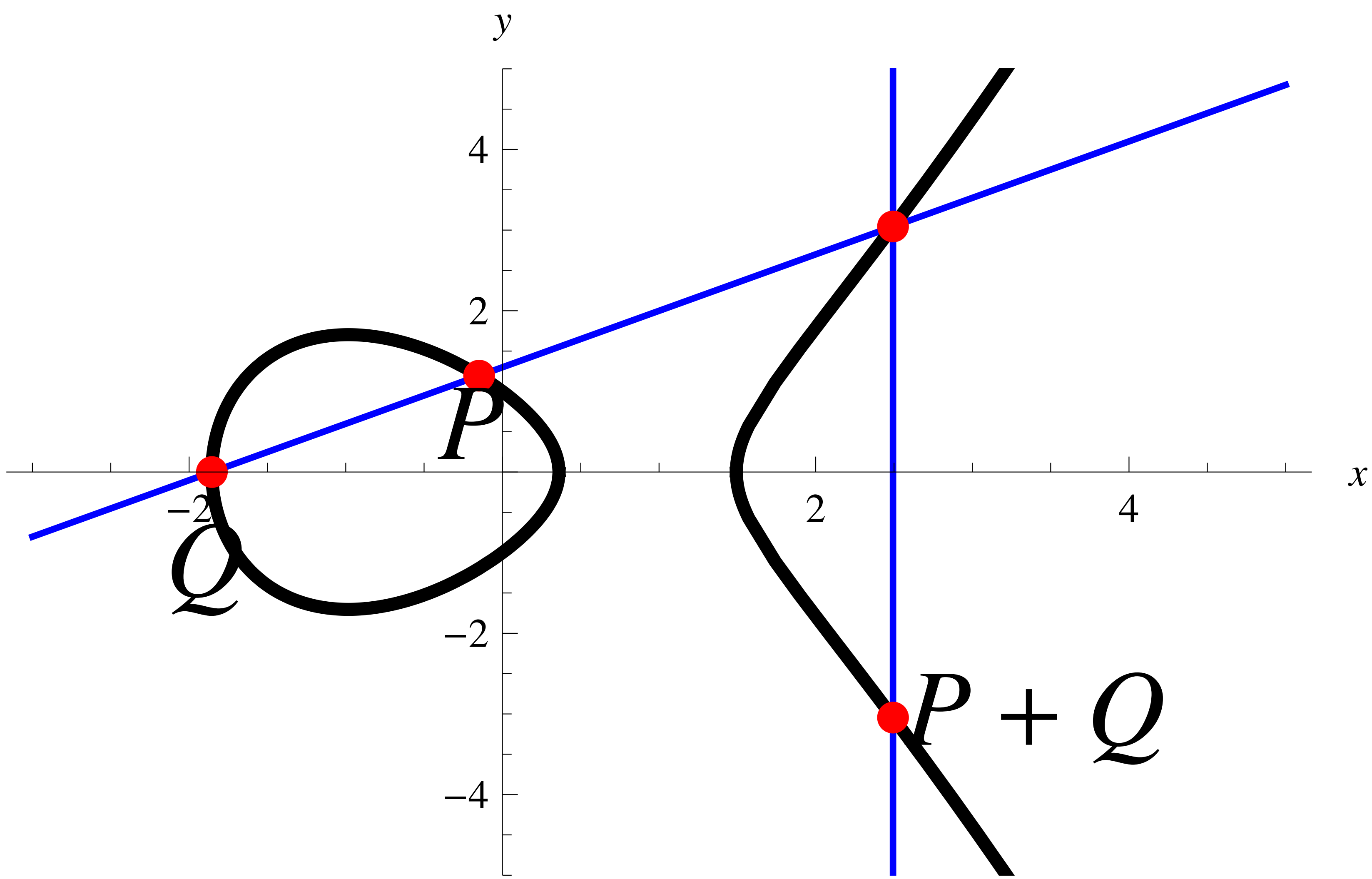
The surprising properties of the Painlevé equations also extend to certain discrete (or difference) equations.

These are of three types:

- $$x_{n+1} + x_{n-1} = F\big(x_n, n\big)$$
- $$x_{n+1}\, x_{n-1} = F\big(x_n, q^n\big)$$
- $$x_{n+1}\, x_{n-1} = F\big(x_n, \theta(n)\big)$$

where $\theta(n)$ is a theta function.

Discrete Elliptic Equations



Addition formula for elliptic functions provide iterations.

Recurrence Relations

- Bäcklund transformations of the Painlevé equations give rise to discrete Painlevé equations. E.g., solutions $w_n(t)$ of $P_{IV}(n)$ corresponding to

$$\alpha_n = -\tfrac{n}{2} + c_0 + c_1(-1)^n,\ \beta_n = n - 2c_0 + \tfrac{2c_1}{3}(-1)^n$$

transform with

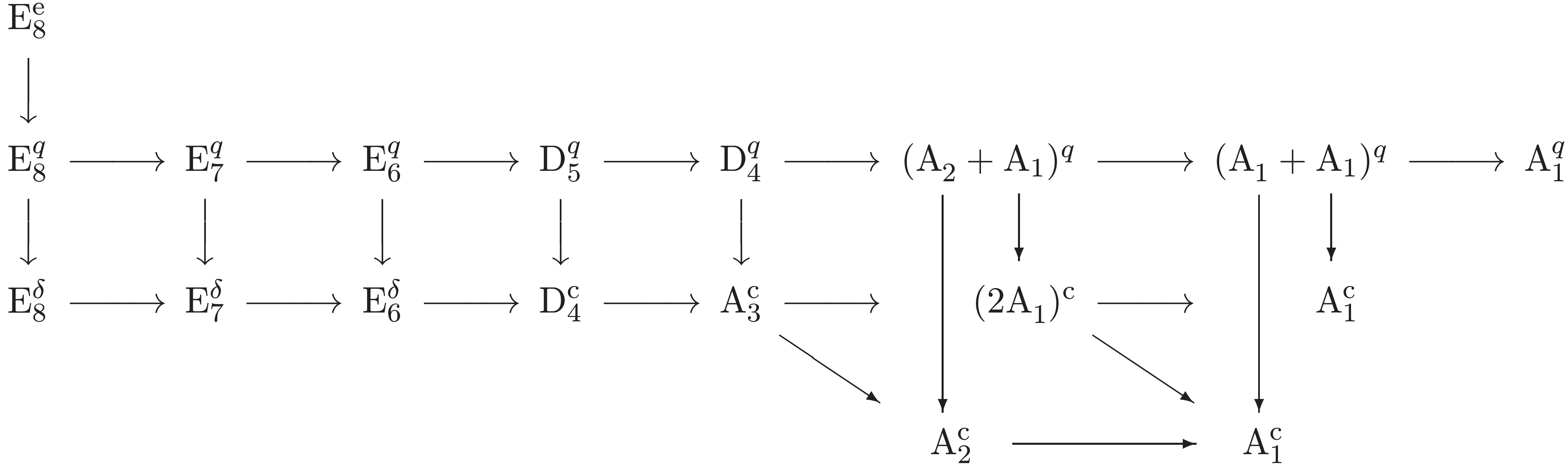
$$\begin{aligned} 2\,w_n\,w_{n+1} &= -\,w_n{}' - w_n^2 - 2t\,w_n + \beta_n \\ 2\,w_n\,w_{n-1} &= \quad\ w_n{}' - w_n^2 - 2t\,w_n + \beta_n, \end{aligned}$$

eliminating $w'_n(t)$ gives the discrete P_I :

$$w_n\left(w_{n+1} + w_n + w_{n-1}\right) = \beta_n - 2t\,w_n$$

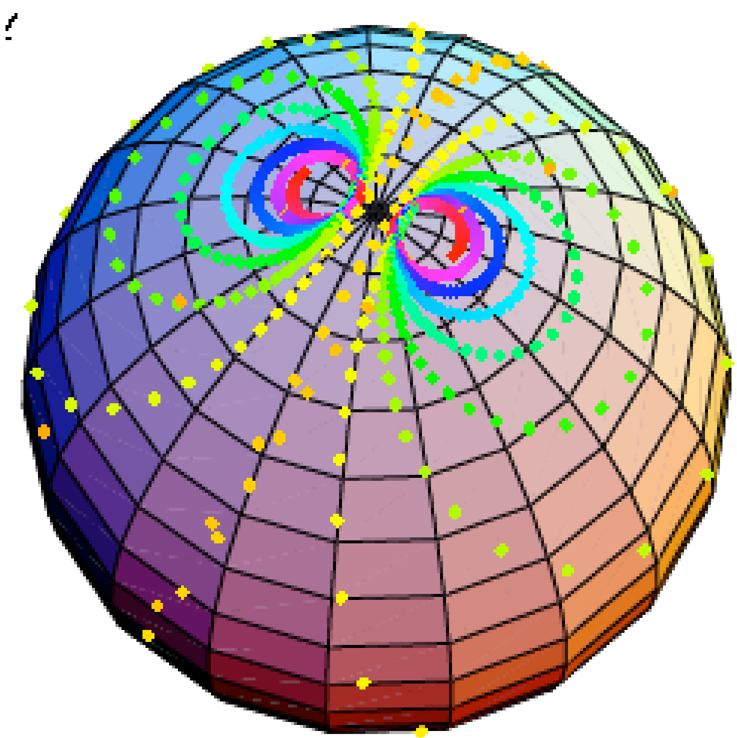
Geometric Origin of Discrete Painlevé Equations

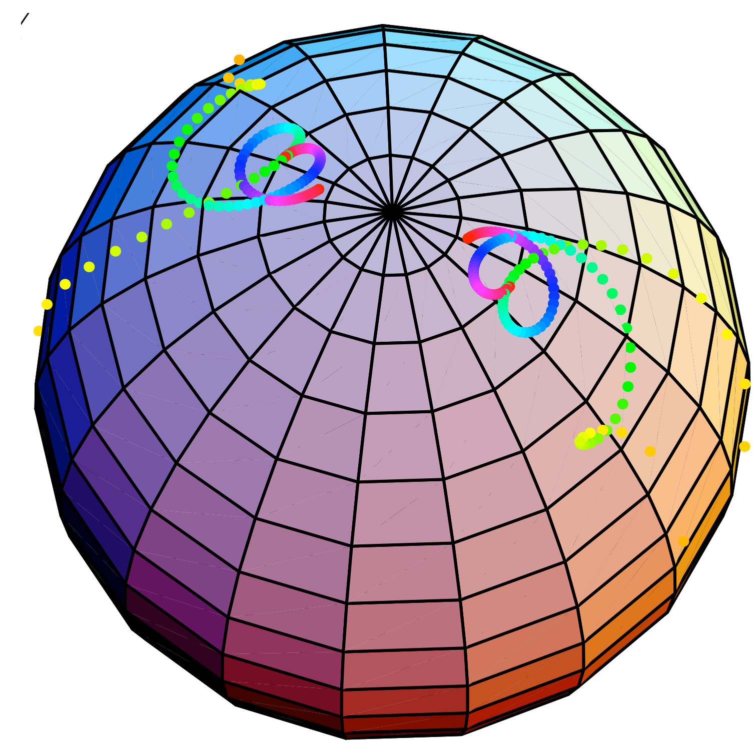
- Sakai *CMP 2001* classified all possible equations whose initial value space is regularized by a 9-point blow-up of $\mathbb{P}_1$ x $\mathbb{P}_1$.



from Grammaticos and Ramani, Regul Chaotic Dyn. 2000

Solutions?





Very little is known about their transcendental solutions.

Summary

- The solutions of the Painlevé equations are non-linear special functions of fundamental importance in modern science.
- Very little is known about the transcendental solutions in finite regions. Major conjectures remain open.
- Even less is known about the solutions of discrete Painlevé equations.