Error bounds for Cherry's asymptotic expansions for turning point problems.

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1. Introduction.

We consider differential equations of the form

$$\frac{d^2w}{dz^2} = \left\{u^2 f(z) + g(z)\right\} w \,, \tag{1.1}$$

where u is large and z lies in a real or complex region **D** containing at most one transition point z_0 (pole or zero of f). Many of the special functions of mathematical physics satisfy ODEs of this form, as well as various one-dimensional quantum mechanical problems.

Olver (1974/1997) identifies 3 main cases:

Case I: **D** is free from transition points (Liouville-Green/WKBJ approximations).

Case II: z_0 is a simple zero of f and g is analytic at z_0 .

Case III: z_0 is a simple pole of f and $\left(z-z_0\right)^2 g\left(z\right)$ is analytic at z_0 .

In the 60s and 70s Olver provided explicit error bounds for asymptotic solutions in Cases I, II, III and gave conditions for uniform validity when **D** is unbounded.

For Case I the standard expansions are of the form

$$w \sim f^{-1/4} \exp\left\{\pm u\xi\right\} \sum_{s=0}^{\infty} \frac{A_s\left(\xi\right)}{u^s} , \tag{1.2}$$

where

$$\xi = \int f^{1/2}\left(z\right) dz , \tag{1.3}$$

$A_0\left(\xi\right) = 1$, with the other coefficients satisfying the linear recursion relation

$$A_{s+1}\left(\xi\right) = -\tfrac{1}{2} A_s'\left(\xi\right) + \tfrac{1}{2} \int \psi\left(\xi\right) A_s\left(\xi\right) d\xi \quad \left(s \ge 0\right) . \tag{1.4}$$

In quantum mechanics alternative expansions of the form

$$w \sim f^{-1/4} \exp\left\{ \pm u\xi + \sum_{s=0}^{\infty} (\pm 1)^s \frac{E_s(\xi)}{u^s} \right\}, \tag{1.5}$$

are often more suitable in finding certain eigenvalues. Here $E_1(\xi) = \frac{1}{2}\int \psi(\xi)\,d\xi$, and the other coefficients satisfy the *nonlinear* recursion relation

$$E_{s+1}(\xi) = -\tfrac{1}{2}E_s'(\xi) - \tfrac{1}{2}\sum_{j=1}^{s-1}\int E_j'(\xi)E_{s-j}'(\xi)\,d\zeta \quad (s \geq 1). \tag{1.6}$$

Error bounds, using Olver's technique, were given by Dunster (1998).

For Case II we start with the Liouville transformation

$$\tfrac{2}{3}\zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t)\,dt, \quad W = \left(\frac{f(z)}{\zeta}\right)^{1/4} w, \tag{1.7}$$

to give

$$\frac{d^2W}{d\zeta^2} = \left\{u^2\zeta + \psi(\zeta)\right\}W, \tag{1.8}$$

where $\psi(\zeta)$ is analytic at $\zeta = 0$ ($z = z_0$).

The comparison equation $W'' = u^2 \zeta W$ has a solution $W(u,\zeta) = \mathrm{Ai}\left(u^{2/3}\zeta\right)$, and Olver's expansions are of the form

$$W_{2n+1,2}(u,\zeta) = \mathrm{Ai}\left(u^{2/3}\zeta\right)\sum_{s=0}^{n}\frac{A_s(\zeta)}{u^{2s}} + \frac{\mathrm{Ai}'\left(u^{2/3}\zeta\right)}{u^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\zeta)}{u^{2s}} + \varepsilon_{2n+1,2}(u,\zeta). \tag{1.9}$$

Olver obtained explicit bounds for $\varepsilon_{2n+1,2}(u,\zeta)$ by expressing this error term as a solution of an integral equation and then using successive approximations. This is nicely packaged in his following theorem.

Theorem 1. *Let* $h(\zeta)$ *satisfy*

$$h(\zeta) = \int_\alpha^\zeta \mathsf{K}(\zeta,t)\phi(t)\left\{J(t) + h(t)\right\}dt\ . \tag{1.10}$$

Assume

$$\mathsf{K}(\zeta,\zeta) = 0,\ \left|\mathsf{K}(\zeta,t)\right| \le P_0(\zeta)Q(t)\ \ (\alpha < t \le \zeta < \beta), \tag{1.11}$$

where $\phi(t)$, $J(t)$, $\psi_0(t)$, $P_0(\zeta)$, $Q(t)$ *are continuous on* (α,β). *Then*

$$\left|h(\zeta)\right| \le (\kappa/\kappa_0)P_0(\zeta)\left[\exp\left\{\kappa_0\int_\alpha^\zeta\left|\phi(t)\right|dt\right\} - 1\right],\ \ \left|h'(\zeta)\right| \le \cdots, \tag{1.12}$$

where

$$\kappa = \sup\left\{Q(\zeta)\left|J(\zeta)\right|\right\},\ \ \kappa_0 = \sup\left\{P_0(\zeta)Q(\zeta)\right\}. \tag{1.13}$$

In his 1950 paper T. M. Cherry (a.k.a. Professor Sir Thomas MacFarland Cherry, Kt., Sc.D., F.A.A., F.R.S.) approached the problem of approximating solutions to the same equation

$$\frac{d^2W}{d\zeta^2} = \left\{u^2\zeta + \psi(\zeta)\right\}W \ , \tag{1.14}$$

by defining a new independent variable

$$\hat{\zeta} = \zeta + \mathscr{A}_n(u,\zeta) \, , \tag{1.15}$$

where

$$\mathscr{A}_n(u,\zeta) = \sum_{s=1}^{n} \frac{a_s(\zeta)}{u^{2s}} \, . \tag{1.16}$$

Cherry then obtained a new ODE with $\hat{\zeta}$ as the independent variable. This makes it a little harder to obtain error bounds, as well as describe regions of validity in the complex plane in terms of ζ. Hence our approach, as follows.

For brevity we only consider real variables (complex variables done similarly). We assume u is sufficiently large so that $d\hat{\zeta} / d\zeta > 0$, i.e.

$$1 + \sum_{s=1}^{n} \frac{a_s'(\zeta)}{u^{2s}} > 0 \, . \tag{1.17}$$

Now $w = \mathrm{Ai}_n(u,\zeta) = \left\{ d\hat{\zeta} / d\zeta \right\}^{-1/2} \mathrm{Ai}\left(u^{2/3}\hat{\zeta}\right)$ satisfies

$$\frac{d^2 w}{d\zeta^2} = \left\{ u^2\zeta + \hat{\psi}_n(u,\zeta) \right\} w \,, \tag{1.18}$$

where

$$\hat{\psi}_n(u,\zeta) = \sum_{s=0}^{\infty} \frac{\hat{\psi}_{n,s}(\zeta)}{u^{2s}} \,. \tag{1.19}$$

We choose $a_s(\zeta)$ ($s = 1,2,3,\cdots n$) so that $\hat{\psi}_n = \psi + O\left(u^{-2n}\right)$. Thus

$$\hat{\psi}_{n,0}(\zeta) = \psi(\zeta), \; \hat{\psi}_{n,1}(\zeta) = \hat{\psi}_{n,2}(\zeta) = \cdots = \hat{\psi}_{n,n-1}(\zeta) = 0 \,. \tag{1.20}$$

This gives

$$a_1(\zeta) = \frac{1}{2\zeta^{1/2}} \int_0^{\zeta} \frac{\psi(t)}{t^{1/2}} dt \,, \tag{1.21}$$

and for $1 \le s \le n-1$

$$a_{s+1}(\zeta) = \frac{1}{2\zeta^{1/2}} \int_0^{\zeta} \frac{F_s(t)}{t^{1/2}} dt \,, \tag{1.22}$$

where

$$F_s(\zeta) = \frac{1}{2\pi i} \oint_{|w|=\delta} \left[\frac{\{2\mathscr{B}_s + (\zeta + \mathscr{B}_s)\mathscr{B}_s'\}\mathscr{B}_s'}{w^{s+2}} + \frac{2(1+\mathscr{B}_s')\mathscr{B}_s''' - 3\mathscr{B}_s''^2}{4(1+\mathscr{B}_s')^2 w^{s+1}} \right] dw, \tag{1.23}$$

$$\mathscr{B}_s = \sum_{j=1}^{s} a_j(\zeta) w^j, \quad \mathscr{B}_s' = \sum_{j=1}^{s} a_j'(\zeta) w^j, \text{ etc.} \tag{1.24}$$

From this is it easy to show by induction that each coefficient $a_s(\zeta)$ is analytic at $\zeta = 0$, and from the subsequent error bounds that the asymptotic expansions are uniformly valid at infinity if $\psi^{(s)}(\zeta) = O\left(|\zeta|^{-s-(1/2)-\delta}\right)$ as $\zeta \to \pm\infty$.

Let

$$\hat{W}_{2n+1,2}(u,\zeta) = \left(d\hat{\zeta} / d\zeta\right)^{-1/2} \mathrm{Ai}\left(u^{2/3}\hat{\zeta}\right) + \hat{\varepsilon}_{2n+1,2}(u,\zeta) \tag{1.25}$$

be a solution of the canonical turning point equation $W'' = \left\{u^2\zeta + \psi(\zeta)\right\}W$. This is a useful form when studying the zeros.

Substitution of (1.25) into the ODE, then applying variation parameters, yields

$$h(\zeta)=\int_{\alpha}^{\zeta}\mathsf{K}(\zeta,t)\phi(t)\{J(t)+h(t)\}dt\ , \tag{1.26}$$

where

$$h(\zeta)=\left(d\hat{\zeta}/d\zeta\right)^{1/2}\hat{\varepsilon}_{2n+1,2}(u,\zeta), \tag{1.27}$$

$$J(\zeta)=\mathrm{Ai}\left(u^{2/3}\hat{\zeta}\right), \tag{1.28}$$

$$\phi(\zeta)=\left(d\hat{\zeta}/d\zeta\right)^{-3/2}\hat{\zeta}^{1/2}\left\{\hat{\psi}_n(\zeta)-\psi(\zeta)\right\}=O\left(u^{-2n}\right), \tag{1.29}$$

and

$$\mathsf{K}(\zeta,t)=\pi u^{-1}\left(u^{2/3}\hat{t}\right)^{1/2}\left[\mathrm{Bi}\left(u^{2/3}\hat{\zeta}\right)\mathrm{Ai}\left(u^{2/3}\hat{t}\right)-\mathrm{Ai}\left(u^{2/3}\hat{\zeta}\right)\mathrm{Bi}\left(u^{2/3}\hat{t}\right)\right], \tag{1.30}$$

in which $\hat{t}=t+\sum_{s=1}^{n}a_s(t)u^{-2s}$.

We need $\left|\mathsf{K}(\zeta,t)\right|\le P_0(\zeta)Q(t)$ for $t\ge\zeta$.

Let c be the largest negative root of $\mathrm{Ai}(x) = \mathrm{Bi}(x)$; then Olver defines a so-called weight function by

$$E(x)=1 \quad (-\infty < x \le c), \quad E(x)=\{\mathrm{Bi}(x)/\mathrm{Ai}(x)\}^{1/2} \quad (c \le x < \infty), \tag{1.31}$$

and the Airy functions are then expressed in the form

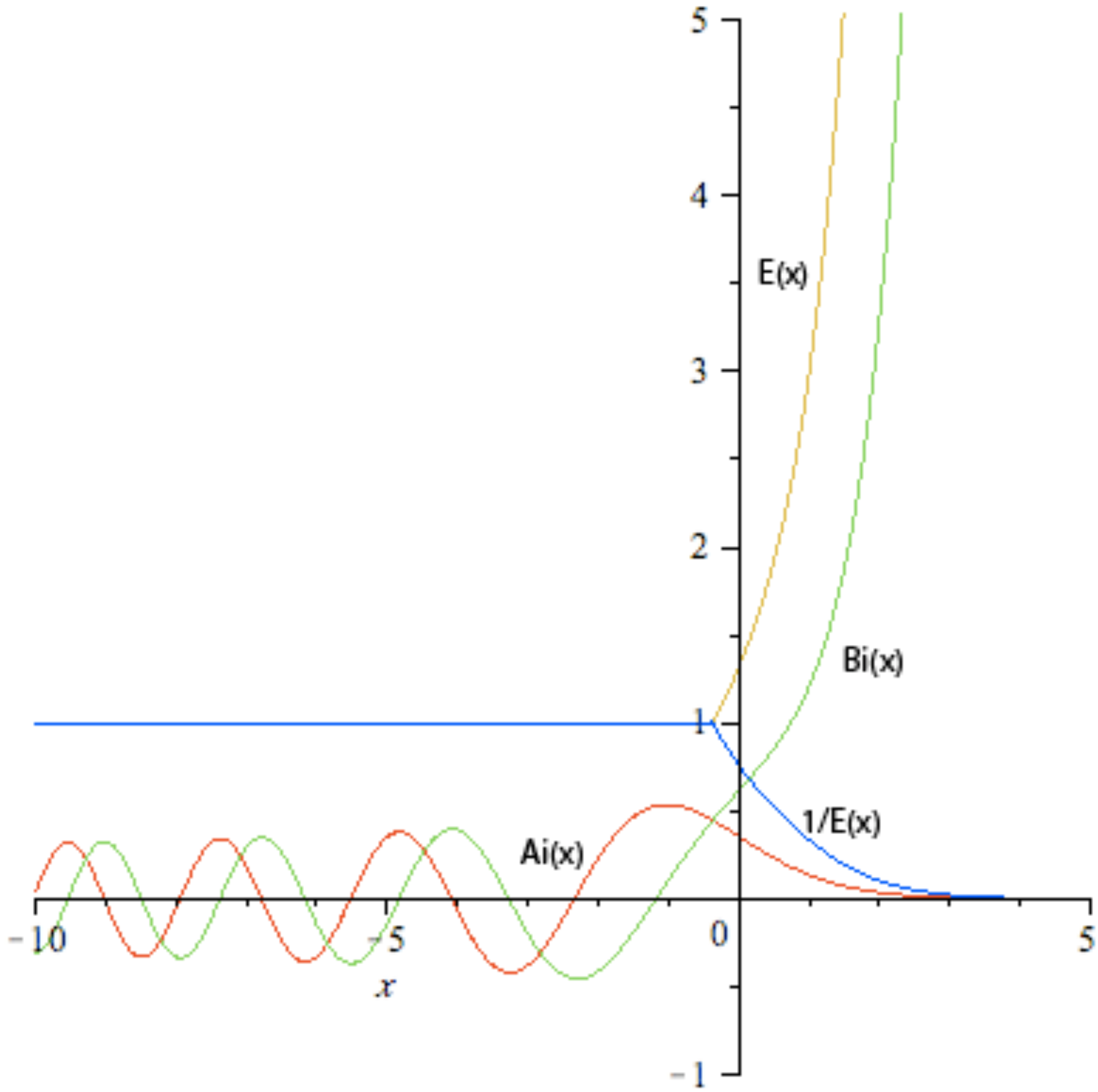
$$\mathrm{Ai}(x) = E^{-1}(x) M(x) \sin\{\theta(x)\}, \quad \mathrm{Bi}(x) = E(x) M(x) \cos\{\theta(x)\}. \tag{1.32}$$

As $x \to \infty$

$$E(x) \sim \sqrt{2} \exp\{\tfrac{2}{3} x^{3/2}\}, \quad M(x) \sim \pi^{-1/2} x^{-1/4}. \tag{1.33}$$

From these expressions it can be shown that

$$\begin{aligned} |\mathsf{K}(\zeta,t)| \le {} & E^{-1}\left(u^{2/3}\hat{\zeta}\right) M\left(u^{2/3}\hat{\zeta}\right) \\ & \times(\pi / u) \left|u^{2/3}\hat{t}\right|^{1/2} E\left(u^{2/3}\hat{t}\right) M\left(u^{2/3}\hat{t}\right) \quad (t \ge \zeta). \end{aligned} \tag{1.34}$$



5
4
3
2
1
−1
E(x)
Bi(x)
1/E(x)
Ai(x)
−10
−5
0
5
x

Using Theorem 1 we arrive at our desired bound

$$\left|\hat{\varepsilon}_{2n+1,2}(u,\zeta)\right| \le \left(\frac{d\hat{\zeta}}{d\zeta}\right)^{-1/2} \frac{E^{-1}\left(u^{2/3}\hat{\zeta}\right)M\left(u^{2/3}\hat{\zeta}\right)}{\lambda} \times\left[\exp\left\{\frac{\lambda\int_{\zeta}^{\beta}\left|\phi_n(u,t)\right|dt}{u}\right\}-1\right], \tag{1.35}$$

uniformly for $\alpha < \zeta < \beta$, where

$$\phi_n(u,t) = \left(d\hat{\zeta}/d\zeta\right)^{-3/2} \hat{\zeta}^{1/2}\left\{\hat{\psi}_n(\zeta)-\psi(\zeta)\right\} = O\left(u^{-2n}\right), \tag{1.36}$$

and

$$\lambda = \sup_{-\infty<x<\infty}\left\{\pi|x|^{1/2} M^2(x)\right\} = 1.04\cdots. \tag{1.37}$$

This shows

$$\hat{\varepsilon}_{2n+1,2}(u,\zeta) = E^{-1}\left(u^{2/3}\hat{\zeta}\right)M\left(u^{2/3}\hat{\zeta}\right)O\left(u^{-2n-1}\right), \tag{1.38}$$

uniformly for $\alpha < \zeta < \beta$ (which can be unbounded with appropriate conditions on $\psi(\zeta)$).

Now let's consider the *derivative* of a solution of an ODE having a turning point (c.f. Wong and Lang (1990)). We assume the parent equation is of the form

$$\frac{d^2y}{dz^2}+h(z)\frac{dy}{dz}-u^2 f(z)y=0\ , \tag{1.39}$$

where as before f has a simple zero at z_0. Dividing by $f(z)$ and differentiating yields a linear 2nd order ODE for $y'=dy/dz$. We obtain an ODE without the first derivative by defining

$$\tilde{w}(z)=f^{-1/2}(z)\exp\left\{\tfrac{1}{2}\int h(z)dz\right\}y'(z), \tag{1.40}$$

to arrive at

$$d^2\tilde{w}/dz^2=\left\{u^2 f(z)+\tilde{g}(z)\right\}\tilde{w}\ , \tag{1.41}$$

where

$$\tilde{g}(z)=\frac{3f'^2(z)}{4f^2(z)}+\frac{f'(z)h(z)-f''(z)}{2f(z)}+\frac{1}{4}h^2(z)-\frac{1}{2}h'(z)\,. \tag{1.42}$$

Note that as $z\to z_0$

$$\tilde{g}(z)=\frac{3}{4(z-z_0)^2}+O\left(\frac{1}{z-z_0}\right). \tag{1.43}$$

Again we use

$$\tfrac{2}{3}\zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t)\,dt \ , \tag{1.44}$$

and

$$\tilde{W}(\zeta) = \left(f(z)/\zeta\right)^{1/4} \tilde{w}(z) = \left(\zeta f(z)\right)^{-1/4} \exp\left\{\tfrac{1}{2}\int h(z)\,dz\right\} y'(z) \, , \tag{1.45}$$

to obtain

$$\frac{d^2\tilde{W}}{d\zeta^2} = \left\{u^2\zeta + \frac{3}{4\zeta^2} + \frac{\tilde{\psi}(\zeta)}{\zeta}\right\}\tilde{W} \ , \tag{1.46}$$

where $\tilde{\psi}(\zeta)$ is analytic at $\zeta = 0$.

The comparison equation

$$\frac{d^2\tilde{W}}{d\zeta^2} = \left\{u^2\zeta + \frac{3}{4\zeta^2}\right\}\tilde{W} \ , \tag{1.47}$$

has a solution $\tilde{W} = \zeta^{-1/2}\mathrm{Ai}'\left(u^{2/3}\zeta\right)$.

We need a companion solution to (1.47) which is recessive at $\zeta = 0$, and for brevity we consider $0 < \zeta < \beta$ here. Using

$$\mathrm{Ai}'(z) = \mathrm{Ai}'(0) + \tfrac{1}{2}\mathrm{Ai}(0)z^2 + O\left(z^3\right), \tag{1.48}$$

$$\mathrm{Bi}'(z) = \mathrm{Bi}'(0) + \tfrac{1}{2}\mathrm{Bi}(0)z^2 + O\left(z^3\right), \tag{1.49}$$

we define

$$\mathrm{Di}(z) = \frac{\mathrm{Bi}'(0)\mathrm{Ai}(z) - \mathrm{Ai}'(0)\mathrm{Bi}(z)}{\sqrt{\mathrm{Bi}'^2(0) + \mathrm{Ai}'^2(0)}} = \frac{\sqrt{3}}{2}\mathrm{Ai}(z) + \frac{1}{2}\mathrm{Bi}(z), \tag{1.50}$$

with the desired property

$$\mathrm{Di}'(z) = \frac{1}{2\left(3^{1/6}\right)\Gamma\left(\frac{2}{3}\right)} z^2 + O\left(z^5\right). \tag{1.51}$$

Thus $\zeta^{-1/2}\mathrm{Di}'\left(u^{2/3}\zeta\right)$ is the solution we seek. Let

$$\tilde{W}_2(u,\zeta) = \zeta^{-1/2}\mathrm{Ai}'\left(u^{2/3}\zeta\right) + \tilde{\varepsilon}_2(u,\zeta), \tag{1.52}$$

be a solution of

$$\frac{d^2\tilde{W}}{d\zeta^2} = \left\{u^2\zeta + \frac{3}{4\zeta^2} + \frac{\tilde{\psi}(\zeta)}{\zeta}\right\}\tilde{W}. \tag{1.53}$$

We find

$$\tilde{\eta}_2(u,\zeta) = \frac{2\pi}{u^{4/3}} \int_\zeta^\beta \tilde{\mathsf{K}}(\zeta,t) \frac{\tilde{\psi}(t)}{\left(u^{-1}+t^{3/2}\right)} \left\{ \mathrm{Ai}'\left(u^{2/3}t\right) + \tilde{\eta}_2(u,t) \right\} dt \ , \tag{1.54}$$

where

$$\tilde{\eta}_2(u,\zeta) = \zeta^{1/2} \tilde{\varepsilon}_2(u,\zeta) , \tag{1.55}$$

with

$$\left|\tilde{\mathsf{K}}(\zeta,t)\right| \le \left(u^{-1}+t^{3/2}\right) t^{-2} \left|\mathrm{Ai}'\left(u^{2/3}\zeta\right)\right| \mathrm{Di}'\left(u^{2/3}t\right) \quad (t \ge \zeta) . \tag{1.56}$$

The artificial factor $\left(u^{-1}+t^{3/2}\right)$ is a balancing function, introduced to sharpen the error bound. Using Theorem 1 we get

$$\left|\tilde{\eta}_{2n+1,2}(u,\zeta)\right| \le 2\pi \left|\mathrm{Ai}'\left(u^{2/3}\zeta\right)\right| \left[\exp\left\{ \frac{\tilde{\lambda}\Phi(u,\zeta)}{u} \right\} - 1 \right], \tag{1.57}$$

where

$$\tilde{\lambda} = \sup_{0<x<\infty} \left\{ x^{-2}\left(1+x^{3/2}\right) \left|\mathrm{Ai}'(x)\right| \mathrm{Di}'(x) \right\} = 0.1059\cdots, \tag{1.58}$$

and

$$\Phi(u,\zeta) = \int_{\zeta}^{\beta} \frac{\left|\tilde{\psi}(t)\right|}{\left(u^{-1} + t^{3/2}\right)} dt \; . \tag{1.59}$$

If $0 < \delta \le \zeta < \infty$ then

$$\Phi(u,\zeta) \le \int_{\delta}^{\beta} \frac{\left|\tilde{\psi}(t)\right|}{t^{3/2}} dt = O(1) \; . \tag{1.60}$$

Otherwise we have

$$\begin{aligned} \Phi(u,\zeta) &\le \int_{0}^{\delta} \frac{\left|\tilde{\psi}(t)\right|}{\left(u^{-1} + t^{3/2}\right)} dt + \int_{\delta}^{\beta} \frac{\left|\tilde{\psi}(t)\right|}{\left(u^{-1} + t^{3/2}\right)} dt \\ &= u^{1/3} \int_{0}^{u^{2/3}\delta} \frac{\left|\tilde{\psi}\left(u^{-2/3}t\right)\right|}{\left(1 + t^{3/2}\right)} dt + \int_{\delta}^{\beta} \frac{\left|\tilde{\psi}(t)\right|}{\left(u^{-1} + t^{3/2}\right)} dt \\ &\le u^{1/3} \int_{0}^{\infty} \frac{M}{\left(1 + t^{3/2}\right)} dt + \int_{\delta}^{\beta} \frac{\left|\tilde{\psi}(t)\right|}{t^{3/2}} dt = O\left(u^{1/3}\right), \end{aligned} \tag{1.61}$$

where $M = \sup_{0 \le t \le \delta} \left|\tilde{\psi}(t)\right|$.

In summary there is a solution y_2 (say) whose derivative satisfies

$$y_2'(z) = \left(\frac{f(z)}{\zeta}\right)^{1/4} \exp\left\{-\tfrac{1}{2}\int h(z)\,dz\right\}\left\{\mathrm{Ai}'\left(u^{2/3}\zeta\right) + \tilde{\eta}_2(u,\zeta)\right\}, \tag{1.62}$$

where

$$\tilde{\eta}_{2n+1,2}(u,\zeta) = \begin{cases} \mathrm{Ai}'\left(u^{2/3}\zeta\right) O\left(u^{-2/3}\right) & (0 \le \zeta \le \delta) \\ \mathrm{Ai}'\left(u^{2/3}\zeta\right) O\left(u^{-1}\right) & (0 < \delta \le \zeta < \beta) \end{cases}. \tag{1.63}$$

Application to Bessel functions. The equation

$$\frac{d^2\tilde{w}}{dx^2} = \left\{\nu^2 \frac{1-x^2}{x^2} + \frac{3x^4 + 10x^2 - 1}{4x^2\left(1-x^2\right)^2}\right\}\tilde{w}, \tag{1.64}$$

has a solution

$$\tilde{w}(x) = x^{3/2}\left(1-x^2\right)^{-1/2} J_\nu'(\nu x). \tag{1.65}$$

Let

$$\tfrac{2}{3}\zeta^{3/2} = \ln\left\{\frac{1+\left(1-x^2\right)^{1/2}}{x}\right\} - \left(1-x^2\right)^{1/2}, \tag{1.66}$$

with $0 < x \le 1$ mapped to $0 \le \zeta < \infty$. Then with
$\tilde{W} = x^{-1/2}\left(1-x^2\right)^{1/4}\zeta^{-1/4}\tilde{w}$ we get

$$\frac{d^2\tilde{W}}{d\zeta^2} = \left\{\nu^2\zeta + \frac{3}{4\zeta^2} + \frac{\tilde{\psi}(\zeta)}{\zeta}\right\}\tilde{W}, \tag{1.67}$$

where $\tilde{\psi}(\zeta)$ is analytic at $\zeta = 0$. We arrive at

$$J_\nu'(\nu x) = -\frac{2\sqrt{\pi}\nu^{\nu-(1/6)}e^{-\nu}}{x\Gamma(\nu+1)}\left(\frac{1-x^2}{\zeta}\right)^{1/4}\left\{\mathrm{Ai}'\left(\nu^{2/3}\zeta\right) + \tilde{\eta}_2(\nu,\zeta)\right\}, \tag{1.68}$$

where $\tilde{\eta}_2(\nu,\zeta)$ is bounded as above, uniformly for $0 \le \zeta < \infty$. The interval $-\infty < \zeta \le 0$ is similarly treated.

Less sharp and more complicated error bounds can also be computed by differentiating the original uniform approximations of Olver

$$J_\nu(\nu x)=\frac{2\sqrt{\pi}\nu^{\nu+(1/6)}e^{-\nu}}{\Gamma(\nu+1)}\left(\frac{\zeta}{1-x^2}\right)^{1/4}\left\{\mathrm{Ai}\left(\nu^{2/3}\zeta\right)+\varepsilon_{\mathrm{Olver}}(\nu,\zeta)\right\}, \qquad (1.69)$$

giving

$$J_\nu'(\nu x)=-\frac{2\sqrt{\pi}\nu^{\nu-(1/6)}e^{-\nu}}{x\Gamma(\nu+1)}\left(\frac{1-x^2}{\zeta}\right)^{1/4}\left\{\mathrm{Ai}'\left(\nu^{2/3}\zeta\right)+\eta_{\mathrm{Olver}}(\nu,\zeta)\right\} \qquad (1.70)$$

where

$$\eta_{\mathrm{Olver}}(\nu,\zeta)=\frac{1}{4\nu^{2/3}}\left[\frac{1}{\zeta}-\frac{2x^2\zeta^{1/2}}{\left(1-x^2\right)^{3/2}}\right]\left\{\mathrm{Ai}\left(\nu^{2/3}\zeta\right)+\varepsilon_{\mathrm{Olver}}(\nu,\zeta)\right\}+\frac{\varepsilon_{\mathrm{Olver}}'(\nu,\zeta)}{\nu^{2/3}}. \qquad (1.71)$$

Comparisons of relative error bounds vs. exact values ($\nu=100$):

x	$\left\|\tilde{\eta}_2(100,\zeta)/\mathrm{Ai}'\left(100^{2/3}\zeta\right)\right\|$	$\left\|\eta_{\mathrm{Olver}}(100,\zeta)/\mathrm{Ai}'\left(100^{2/3}\zeta\right)\right\|$	$\left\|\mathrm{exact}(100,\zeta)/\mathrm{Ai}'\left(100^{2/3}\zeta\right)\right\|$
0.1	0.0005169	0.0060655	0.0004601
0.5	0.0014428	0.0048714	0.0011554
0.9	0.0045035	0.0067972	0.0034557

If we try a Cherry-type expansion:

$$\tilde{W} = \left(d\hat{\zeta} / d\zeta\right)^{-1/2} \hat{\zeta}^{-1/2} \mathrm{Ai}'\left(u^{2/3}\hat{\zeta}\right), \tag{1.72}$$

we find it satisfies

$$\frac{d^2\hat{W}}{d\zeta^2} = \left\{u^2\hat{\zeta}'^2\hat{\zeta} + \frac{3\hat{\zeta}'^2}{4\hat{\zeta}^2} + \frac{3\hat{\zeta}''^2 - 2\hat{\zeta}'\hat{\zeta}'''}{4\hat{\zeta}'^2}\right\}\hat{W}\ . \tag{1.73}$$

We need $\hat{\zeta}$ to vanish at $\zeta = 0$, so we must choose

$$\hat{\zeta} = \zeta + \mathscr{A}_n(u,\zeta), \quad \mathscr{A}_n(u,\zeta) = \sum_{s=1}^{n} \frac{\zeta a_s(\zeta)}{u^{2s}}\ . \tag{1.74}$$

Thus

$$\frac{d^2\hat{W}}{d\zeta^2} = \left\{u^2\zeta + \frac{3}{4\zeta^2} + \frac{\psi_n(\zeta)}{\zeta}\right\}\hat{W}\ . \tag{1.75}$$

As before, we seek $a_s(\zeta)$ so that $\psi_n(\zeta) = \psi(\zeta) + O\left(u^{-2n}\right)$. This implies

$$a_1(\zeta) = \frac{1}{2\zeta^{3/2}} \int \frac{\psi(\zeta)}{\zeta^{3/2}}\, d\zeta\ , \tag{1.76}$$

with similar problems for subsequent coefficients.

Next consider case III

$$\frac{d^2W}{d\zeta^2} = \left\{ \frac{u^2}{4\zeta} + \frac{\nu^2 - 1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta} \right\} W . \tag{1.77}$$

Olver's expansions are of the form

$$W(u,\zeta) \sim \zeta^{1/2} I_\nu\left(u\zeta^{1/2}\right) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{u^{2s}} - \frac{\zeta I_{\nu+1}\left(u\zeta^{1/2}\right)}{u} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{u^{2s}} . \tag{1.78}$$

Instead we try a Cherry-type expansion

$$\hat{W}(u,\zeta) \sim \left(d\hat{\zeta} / d\zeta\right)^{-1/2} \hat{\zeta}^{1/2} I_\nu\left(u\hat{\zeta}^{1/2}\right), \tag{1.79}$$

where

$$\hat{\zeta} = \zeta + \mathscr{A}_n(u,\zeta), \quad \mathscr{A}_n(u,\zeta) = \sum_{s=1}^{\infty} \frac{\zeta a_s(\zeta)}{u^{2s}} . \tag{1.80}$$

We find that

$$a_1(\zeta) = \frac{2}{\zeta^{1/2}} \int_0^\zeta \frac{\psi(t)}{t^{1/2}} dt , \tag{1.81}$$

with the other coefficients also being analytic at $\zeta = 0$.

References

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Possible extensions: Convergent expansions.