Information-theoretic properties of special functions*

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Aim and motivation

Aim

- Study the spread of special functions of applied mathematics all over their domain of definition.
- Quantify the spread of orthogonal polynomials in a real variable along the orthogonality interval.

How?

By use of the following spreading measures:

- Information-theoretic lengths of Shannon, Rényi and Fisher types,
- Complexity measures of Fisher-Shannon and Cramér-Rao types,

of the Rakhmanov probability density $\rho(x)$ associated to the special function under consideration.

Aim and motivation

The information-theoretic-based spreading measures of the Rakhmanov density $\rho(x)$ allow us to

- grasp different facets of the special functions which are manifest in the great diversity and complexity of configuration shapes of the corresponding densities,
- measure how different are the special functions within a given class and among different classes,
- quantify the complexity of the special functions in various ways.

Rakhmanov density of special functions

- Hypergeometric functions $y_n(x)\sqrt{\omega(x)}$:

$$\rho_n(x) = [y_n(x)]^2\omega(x)$$

E.g.:

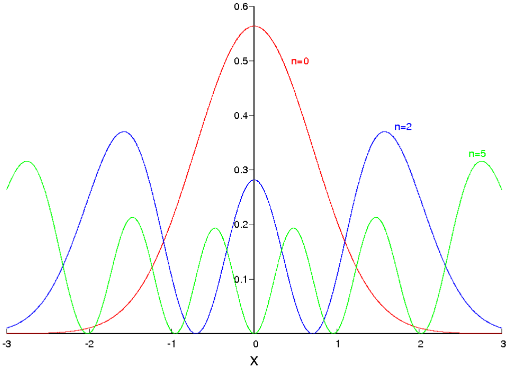
  - $V(x) \propto x^2$, then $y_n(x) \sim H_n(x)$ (Hermite polynomials),
  - $V(x) \propto x^{-1}$, then $y_n(x) \sim \mathcal{L}_n^{\alpha}(x)$ (Laguerre polynomials),
  - For a large class of confined potentials, $y_n(x) \sim P_n^{\alpha,\beta}(x)$ (Jacobi polynomials).

- For spherical harmonics $Y_{lm}(\theta,\phi)$:

$$\rho_{lm}(\theta) = |Y_{lm}(\theta,\phi)|^2$$

Rakhmanov density of Hermite polynomials

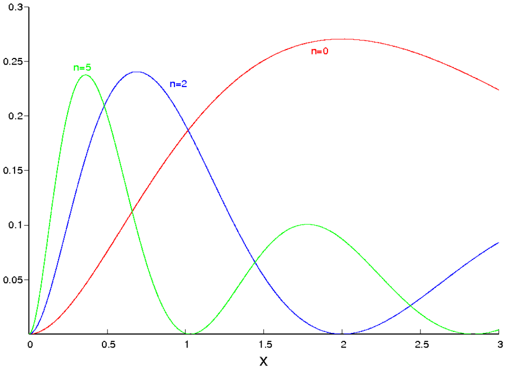
- Rakhmanov-Hermite density



$$\rho_n(x) = [H_n(x)]^2 \; e^{-x^2}$$

Rakhmanov density of Laguerre polynomials

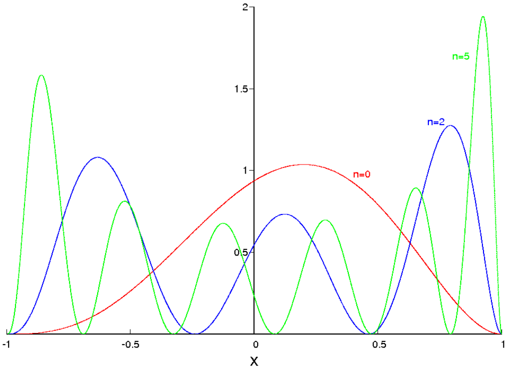
- Rakhmanov-Laguerre density for $\alpha = 2$



$$\rho_n(x) = \left[\mathcal{L}_n^{(2)}(x)\right]^2 x^2 e^{-x}$$

Rakhmanov density of Jacobi polynomials

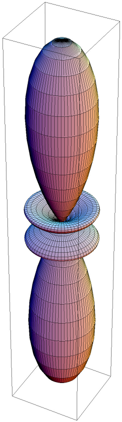
- Rakhmanov-Jacobi density for $\alpha = 2$ and $\beta = 3$



$$\rho_n(x) = \left[P_n^{(2,3)}(x)\right]^2 (1-x)^2(1+x)^3$$

Rakhmanov density of spherical harmonics

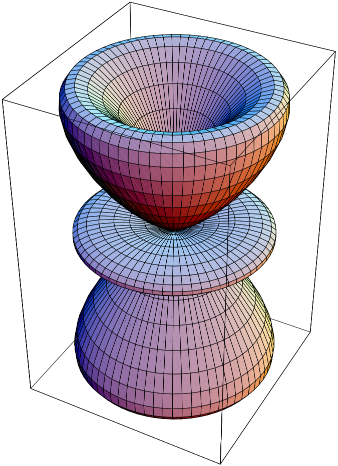
- Spherical harmonics with $l = 3$ and $m = 0$



$$|Y_{3,0}(\theta, \phi)|^2$$

Rakhmanov density of spherical harmonics

- Spherical harmonics with $l = 3$ and $m = 1$



$$|Y_{3,1}(\theta, \phi)|^2$$

Information-theoretic lengths of $\rho(x)$

- Standard deviation:

$$\Delta x = \left\{ \int_\Omega (x - \langle x \rangle)^2 \rho(x) dx \right\}^{\frac{1}{2}} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}; \quad \langle f(x) \rangle = \int_\Omega f(x) \rho(x) dx$$

- Rényi length of order q ($q > 0, q \neq 1$):

$$L_q^R[\rho] = exp\{R_q[\rho]\} = \langle [\rho(x)]^{q-1} \rangle^{-\frac{1}{q-1}} = \left\{ \int_\Omega [\rho(x)]^q \, dx \right\}^{-\frac{1}{q-1}}$$

- Shannon length:

$$L^S[\rho] = \lim_{q \to 1} L_q^R[\rho] = exp\{S[\rho]\} = exp\left\{ - \int_\Omega \rho(x) \, log \, \rho(x) \, dx \right\}$$

- Fisher length:

$$L^F[\rho] = \frac{1}{\sqrt{F[\rho]}} = \left\{ \int_\Omega \frac{\left[\rho'(x)\right]^2}{\rho(x)} dx \right\}^{-\frac{1}{2}}$$

Properties of the information-theoretic lengths

All these spreading lengths $\left(\Delta x, L_q^R[\rho], L^S[\rho], L^F[\rho]\right)$ share some common properties:

- Dimensions of length
- Linear scaling
- Vanishing in the limit of a Dirac delta density
- Reflection and translation invariance $(\Omega = (-\infty, +\infty))$

Mutual relationships:

$$L^F[\rho] \leq \Delta x$$

$$\sqrt{2\pi e} L^F[\rho] \leq L^S[\rho] \leq \sqrt{2\pi e}\, \Delta x$$

Complexity measures of $\rho(x)$

Cramér-Rao complexity

$$C_{CR}[\rho] = F[\rho] \times (\Delta x)^2$$

Fisher-Shannon complexity

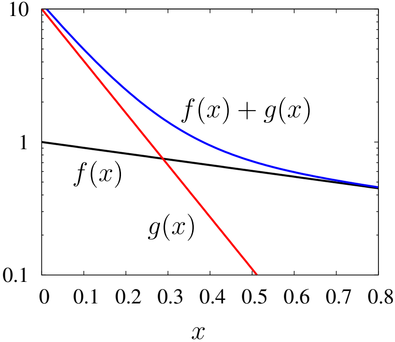
$$C_{FS}[\rho] = F[\rho] \times \frac{1}{2\pi e} \exp\left(2S[\rho]\right)$$

Properties of complexity measures

- Invariance under replication, translation and scaling transformations.
- Minimal values at the two extreme cases:
  - completely ordered systems (e.g. perfect crystal, Dirac delta distribution)
  - totally disordered systems (e.g. ideal gas, uniform distribution)

Remark

The complexity measures quantify how easily a system may be modelled!



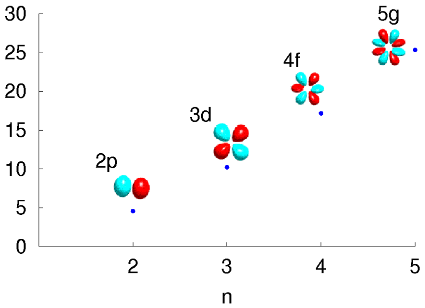
$$f(x) \sim e^{-ax}$$

$$g(x) \sim e^{-bx}$$

Why complexities?

- Fisher-Shannon complexity

$$C_{FS}[\rho_{nlm}] := F[\rho_{nlm}] \times \frac{1}{2\pi e} \exp\left(\frac{2}{3} S[\rho_{nlm}]\right)$$
$$= \frac{4(n-|m|)}{n^3} \frac{1}{2\pi e} e^{\frac{2}{3} B(n,l,m)}$$



Uncertainty-like relations

Heisenberg relation [1927]

$$\Delta x \Delta p \geq \frac{1}{2}$$

Shannon-length-based relation [1975]

$$L^S[\rho] \times L^S[\gamma] \geq e\pi$$

Rényi-length-based relation [2006]

$$L_q^R[\rho] \times L_r^R[\gamma] \geq \left(\frac{q}{\pi}\right)^{\frac{1}{2q-2}} \left(\frac{r}{\pi}\right)^{\frac{1}{2r-2}} ;\ q > 0, q \neq 1; r > 0, r \neq 1$$

Fisher-length-based relation [2011]

$$L^F[\rho] \times L^F[\gamma] \leq \frac{1}{2}$$

Fisher-Shannon complexity [2009]

$$C_{FS}[\rho] \times C_{FS}[\gamma] \geq 1$$

Standard deviation of C.O.P. : $(\Delta x)_n$

Theorem 1: The standard deviation of the c.o.p. $p_n(x)$ is given by

$$(\Delta x)_n = \begin{cases} \sqrt{n+\frac{1}{2}} & \text{Hermite} \quad H_n(x) \\ \sqrt{2n^2+2(\alpha+1)n+\alpha+1} & \text{Laguerre} \quad \mathcal{L}_n^{(\alpha)}(x) \\ \left[\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} + \right. \\ \left. + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+3)}\right]^{1/2} & \text{Jacobi} \quad P_n^{(\alpha,\beta)}(x) \end{cases}$$

Proof

Let the three-term recurrence relation of $p_n(x)$ be

$$x\, p_n(x) = \boldsymbol{\alpha_n}\, p_{n+1}(x) + \boldsymbol{\beta_n}\, p_n(x) + \boldsymbol{\gamma_n}\, p_{n-1}(x)$$

Then

$$\langle x \rangle_n = \int_a^b x\, \rho_n(x)\, dx = \frac{1}{d_n^2} \int_a^b x\, p_n^2(x)\, \omega(x)\, dx = \boldsymbol{\beta_n}$$

$$\langle x^2 \rangle_n = \int_a^b x^2\, \rho_n(x)\, dx = \frac{1}{d_n^2} \left(d_{n+1}^2\, \boldsymbol{\alpha_n}^2 + d_n^2\, \boldsymbol{\beta_n}^2 + d_{n-1}^2 \boldsymbol{\gamma_n}^2 \right)$$

$$\Rightarrow (\Delta x)_n = \sqrt{\langle x^2 \rangle_n - \langle x \rangle_n^2} = \frac{1}{d_n^2} \left(d_{n+1}^2\, \boldsymbol{\alpha_n}^2 + d_{n-1}^2\, \boldsymbol{\gamma_n}^2 \right)$$

Fisher length of C.O.P. $\{p_n(x)\}$

$$L^F[\rho] = \left\{ \int_\Omega \frac{\left[\rho'(x)\right]^2}{\rho(x)}\, dx \right\}^{-\frac{1}{2}}$$

The Fisher length of the C.O.P. $\{p_n(x)\}$ has the values

$$L^F[H_n(x)] = \frac{1}{\sqrt{4n+2}}$$

for Hermite polynomials $H_n(x)$, and

$$L^F\left[\mathcal{L}_n^{(\alpha)}(x)\right] = \begin{cases} \frac{1}{\sqrt{4n+1}}, & \alpha = 0 \\ \sqrt{\frac{\alpha^2-1}{(2n+1)\alpha+1}}, & \alpha > 1 \\ 0, & \alpha \in (-1,+1], \alpha \neq 0 \end{cases}$$

for Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x)$.

Fisher length of C.O.P. $\{p_n(x)\}$

The Fisher length of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is given by

$$L^F\left[P_n^{(\alpha,\beta)}(x)\right] = \left\{F\left[P_n^{(\alpha,\beta)}\right]\right\}^{-\frac{1}{2}}$$

with

$$F\left[P_n^{(\alpha,\beta)}\right] = \begin{cases} \frac{2n+\alpha+\beta+1}{4(n+\alpha+\beta-1)}\left[n(n+\alpha+\beta-1)\left(\frac{n+\alpha}{\beta+1}+2+\frac{n+\beta}{\alpha+1}\right)\right. \\ \left. +\,(n+1)(n+\alpha+\beta)\left(\frac{n+\alpha}{\beta-1}+2+\frac{n+\beta}{\alpha-1}\right)\right], & \alpha,\beta>1 \\ \frac{2n+\beta+1}{4}\left[\frac{n^2}{\beta+1}+n+(4n+1)(n+\beta+1)+\frac{(n+1)^2}{\beta-1}\right], & \alpha=0,\beta>1 \\ 2n(n+1)(2n+1), & \alpha,\beta=0 \\ \infty, & \text{otherwise} \end{cases}$$

Shannon length of C.O.P. $\{p_n(x)\}$

Definition:

$$L^S[\rho_n] = exp\{S[\rho_n]\}$$

where the Shannon entropy is given by

$$S[\rho_n] := -\int_\Omega \rho_n(x)\, log\, \rho_n(x)\, dx$$

$$= -\int_\Omega \omega(x)\, p_n^2(x)\, log\left[\omega(x)\, p_n^2(x)\right]\, dx$$

$$= J[p_n] + E[p_n]$$

where $J[p_n]$ and $E[p_n]$ are the entropic functionals

Shannon length of C.O.P. $\{p_n(x)\}$

Entropic functionals:

- $J\left[p_n\right] := -\int_\Omega \omega(x)\, p_n^2(x)\, log\left[\omega(x)\right]\, dx$

$$= \begin{cases} n + \frac{1}{2} \qquad \text{for Hermite} \quad H_n(x) \\ 2n + \alpha + 1 - \alpha\psi(\alpha + n + 1) \qquad \text{for Laguerre} \quad \mathcal{L}_n^{(\alpha)}(x) \\ -\alpha\psi(n + \alpha + 1) - \beta\psi(n + \beta + 1) + (\alpha + \beta) \\ \times \Big[-\ln 2 + \frac{1}{2n+\alpha+\beta+1} + 2\psi(2n + \alpha + \beta + 1) \\ -\psi(n + \alpha + \beta + 1) \Big] \qquad \text{for Jacobi} \quad P_n^{(\alpha,\beta)}(x) \end{cases}$$

- $E\left[p_n\right] := -\int_\Omega \omega(x)\, p_n^2(x)\, log\left[p_n^2(x)\right]\, dx$

Shannon length of C.O.P. $\{p_n(x)\}$

Entropic functionals: $E\left[\rho_n\right]$ can only be asymptotically $(n >> 1)$ computed by means of the l^2-method of Aptekarev, Buyarov and JSD.

Theorem 3:

The Shannon length of the C.O.P. $\{p_n(x)\}$ has the following asymptotical $(n >> 1)$ behavior:

$$L^S\left[\rho_n\right] = \begin{cases} \frac{\pi}{e}\sqrt{2n} + o(1) & \text{for Hermite} \quad H_n(x) \\ \frac{2\pi}{e}\, n + o(1) & \text{for Laguerre} \quad \mathcal{L}_n^{(\alpha)}(x) \\ \frac{\pi}{e} + o(1) & \text{for Jacobi} \quad P_n^{(\alpha,\beta)}(x) \end{cases}$$

Shannon length of C.O.P. $\{p_n(x)\}$

Corollary:

It is fulfilled that

$$L^S[\rho_n] \approx \frac{\pi\sqrt{2}}{e}(\Delta x)_n \qquad \text{for} \quad n >> 1$$

for all Hermite, Laguerre and Jacobi polynomials.

Remark: the linearity factor

- is the same for all real C.O.P.
- does not depend on the parameter of the polynomials.

This does not hold for general polynomials, i.e. for polynomials with orthogonalities other than Favard.

Cramér-Rao complexity measures

Definition:

$$C_{CR}[\rho] = F[\rho] \times (\Delta x)^2$$

Hermite polynomials:

$$C_{CR}\left[H_n\right] = 4n^2 + 4n + 1$$

Laguerre polynomials:

$$C_{CR}\left[\mathcal{L}_n^{(\alpha)}\right] = \begin{cases} 8n^3 + \left[8(\alpha+1)+2\right] n^2 + 6(\alpha+1)n + (\alpha+1), & \alpha = 0 \\ \frac{1}{\alpha^2-1}\left[4\alpha n^3 + (4\alpha^2+6\alpha+2)n^2 \right. \\ \left. +(4\alpha^2+6\alpha+2)n + (\alpha+1)^2\right], & \alpha > 1 \\ \infty, & \text{otherwise} \end{cases}$$

Cramér-Rao complexity measures

Jacobi polynomials:

$$C_{CR}\left[P_n^{(\alpha,\beta)}\right] =$$

$$\begin{cases} 2n(n+1)\left[\frac{(n+1)^2}{2n+3}+\frac{n^2}{2n-1}\right], & \alpha=\beta=0 \\ \left[\frac{(n+1)^2(n+\beta+1)^2}{(2n+\beta+2)^2(2n+\beta+3)}+\frac{n^2(n+\beta)^2}{(2n+\alpha-1)(2n+\beta)^2}\right] \\ \times\left[\frac{n^2}{\beta+1}+n+(4n+1)(n+\beta+1)+\frac{(n+1)^2}{\beta-1}\right], & \alpha=0,\beta>1 \\ \left[\frac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}+\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2}\right] \\ \times\frac{1}{n+\alpha+\beta-1}\left[n(n+\alpha+\beta-1)\left(\frac{n+\alpha}{\beta+1}+2+\frac{n+\beta}{\alpha+1}\right)\right. \\ \left.+(n+1)(n+\alpha+\beta)\left(\frac{n+\alpha}{\beta-1}+2+\frac{n+\beta}{\alpha-1}\right)\right], & \alpha>1,\beta>1 \\ \infty, & \text{otherwise,} \end{cases}$$

Fisher-Shannon complexity measures: Asymptotics

Definition:

$$C_{FS}[\rho] = F[\rho] \times \frac{1}{2\pi e} \exp\left(2S[\rho]\right)$$

Hermite polynomials:

$$C_{FS}\left[H_n\right] \approx 2^{7/6} \left(\frac{1}{\pi e^2}\right)^{2/3} n^{7/6}, \qquad n \gg 1$$

Laguerre polynomials:

$$C_{FS}\left[\mathcal{L}_n^{(\alpha)}\right] \approx \begin{cases} 2^{4/3} \left(\frac{1}{\pi e^2}\right)^{2/3} n^{4/3}, & \alpha = 0 \\ \frac{2^{1/3}\alpha}{\alpha^2-1} \left(\frac{1}{\pi e^2}\right)^{2/3} n^{4/3}, & \alpha > 1 \\ \infty, & \text{otherwise,} \end{cases}$$

Fisher-Shannon complexity measures: Asymptotics

Jacobi polynomials:

$$C_{FS}\left[P_n^{(\alpha,\beta)}\right] \approx \begin{cases} 2\left(\frac{1}{\pi e^2}\right)^{2/3} n^3, & \alpha=\beta=0 \\ \frac{1}{4}\left(\frac{1}{\pi e^2}\right)^{2/3}\left[\frac{1}{\beta+1}+4+\frac{1}{\beta-1}\right] n^3, & \alpha=0, \beta>1 \\ \frac{1}{4}\left(\frac{1}{\pi e^2}\right)^{2/3}\left[\frac{\beta}{\beta^2-1}+\frac{\alpha}{\alpha^2-1}\right] n^3, & \alpha>1, \beta>1 \\ \infty, & \text{otherwise,} \end{cases}$$

Comparison of C.O.P.complexities

$y_n(x)$	$C_{CR}[y_n(x)]$	$C_{FS}[y_n(x)]$
$H_n(x)$	$\sim n^2$	$\sim n^{7/6}$
$\mathcal{L}_n^{(\alpha)}(x)$	$\sim n^3$	$\sim n^{4/3}$
$P_n^{(\alpha,\beta)}(x)$	$\sim n^3$	$\sim n^3$

Conclusions

For the real classical orthogonal polynomials we have computed:

- the standard deviation, the Fisher length and the Cramér-Rao complexity (explicitly)
- the Shannon length and the Fisher-Shannon complexity (asymptotically)

Open problems

- To extend this study to the whole Askey scheme of o.p.
- To determine the information-theoretic lengths and complexities of special functions other than the c.o.p.
- To calculate the asymptotical behaviour of Rényi's lengths of c.o.p.

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