On a generalization of the complementary error function

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The function V

We consider the following function:

$$V_{\nu,\mu}(\alpha,\beta,z) = \int_0^\infty \mathrm{e}^{-zt}(t+\alpha)^\nu (t+\beta)^\mu \mathrm{d}t,$$

where $\beta > \alpha > 0$ and $\Re z > 0$.

The function V is introduced in López, Pérez Sinusía and Temme (2006) and López and Pérez Sinusía (2007) as a first order approximation to the following singular perturbation problem:

$$-\epsilon \Delta U + U_z = 0$$

where $(x,y,z) \in \mathbb{R} \times \mathbb{R} \times (0,\infty)$, ϵ is a small parameter and we have the condition

$$U(x,y,0) = \begin{cases} 1 & (x,y) \in (-1,1) \times (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

A singular perturbation problem

A first order approximation to the solution can be given in terms of functions of the form

$$F(u_i, v, \lambda) = \int_0^\infty \frac{r\mathrm{e}^{-\lambda r^2}}{\sqrt{r^2 + u_i^2(r^2 + v^2)}} \mathrm{d}r, \qquad i = 1, 2,$$

where

$$u_1 = \frac{\eta z}{\sqrt{\xi^2 + z^2}}, \quad u_2 = \frac{\xi z}{\sqrt{\xi^2 + z^2}}, \quad v = \sqrt{\xi^2 + \eta^2},$$

and $\xi = \pm 1 + x$, $\eta = \pm 1 + y$. Additionally,

$$\lambda = \frac{\sqrt{\xi^2 + \eta^2 + z^2}}{4z^2} \frac{1}{\epsilon}.$$

The functions $F(u_i, v, \lambda)$ can be written in terms of the V function, namely

$$F(u_i, v, \lambda) = \frac{1}{2} V_{-\frac{1}{2}, -1}(u_i^2, v^2, 1).$$

A singular perturbation problem

It is of interest to analyse the behaviour of the solution $U(x, y, z)$ near the boundary and corners of the unit square, where boundary layers appear for small values of ϵ (i.e. large values of λ).

- If x or y (but not both) are close to ± 1, then ξ or η (and then u_i) are small, and the integrand in

$$F(u, v, \lambda) = \int_0^\infty \frac{r\mathrm{e}^{-\lambda r^2}}{\sqrt{r^2 + u^2}(r^2 + v^2)} \mathrm{d}r$$

  has a saddle point $r = 0$ coalescing with two poles $r = \pm \mathrm{i} v$ or with two algebraic singularities $r = \pm \mathrm{i} u$. The complementary error function can be used to describe these cases.

- Near the corners of $[-1, 1] \times [-1, 1]$ the saddle point coalesces with two poles and two algebraic singularities, and further analysis is needed.

Some particular cases

- Kummer U function, when $\alpha = 0$ or $\beta = 0$:

$$V_{\nu,\mu}(0,\beta,z) = \beta^{\mu+\nu+1}\Gamma(\nu+1)U(\nu+1,\mu+\nu+2,\beta z), \quad \mathrm{Re}\,\nu > -1,$$

- Incomplete Gamma function, when $\mu = 0$ or $\nu = 0$:

$$V_{\nu,0}(\alpha,\beta,z) = \alpha^{\nu+1}U(1,\nu+2,\alpha z) = z^{-\nu-1}e^{\alpha z}\Gamma(\nu+1,\alpha z),$$

or when $\alpha = \beta$:

$$V_{\nu,\mu}(\alpha,\alpha,z) = \alpha^{\nu+\mu+1}U(1,\nu+\mu+1,\alpha z) = z^{-\mu-\nu-1}e^{\alpha z}\Gamma(\nu+\mu+1,\alpha z).$$

- Complementary error function:

$$V_{-\frac{1}{2},-1}(0,1,z) = \pi e^{z}\mathrm{erfc}\sqrt{z}, \quad V_{0,-\frac{1}{2}}(0,1,z) = \sqrt{\frac{\pi}{z}}e^{z}\mathrm{erfc}\sqrt{z}.$$

Some related integrals

- Generalized Goodwin–Staton integral:

$$I(\mu, z) = \int_0^\infty \frac{t^\mu \mathrm{e}^{-t^2}}{t+z} \mathrm{d}t, \qquad 0 < \arg\, z < \pi, \qquad \mathrm{Re}\, \mu > -1.$$

- The integral

$$H(u, v) = \int_0^\infty \frac{(u+t)^r G(t) \mathrm{e}^{-t^2}}{v+t} \mathrm{d}t, \quad |\arg\, u|, |\arg\, v| < \pi, \quad -1 < r < 1,$$

considered in Ciarkowski (1989). Here $G(t)$ is regular in a neighborhood of the positive real axis.

- A more general case

$$\int_0^\infty \frac{t^n \mathrm{e}^{-xt^m}}{(t-\lambda)^k} \mathrm{d}t, \quad \mathrm{Re}\, x > 0, \quad m, k \in \mathbb{N}, \quad \lambda \in \mathbb{C} \backslash (\mathbb{R}^+ \cup \{0\}), \quad n = 0, 1, \ldots$$

studied by López and Pagola (2011).

Some properties of the function V

Recall that

$$V_{\nu,\mu}(\alpha,\beta,z) = \int_0^\infty \mathrm{e}^{-zt}(t+\alpha)^\nu(t+\beta)^\mu \mathrm{d}t,$$

where $\beta > \alpha > 0$ and $\Re z > 0$. Observe first that

$$V_{\nu,\mu}(\alpha,\beta,z) = V_{\mu,\nu}(\beta,\alpha,z),$$

so one can restrict the analysis to the case $\beta > \alpha$. Similarly,

$$V_{\nu,\mu}(\alpha,\beta,z) = z^{-1-\nu-\mu}V_{\nu,\mu}(\alpha z,\beta z,1).$$

For the evaluation of the V function, we can use the following ideas:

- Power series expansions for small values of z.
- Asymptotic and modified asymptotic expansions for large values of z.

Power series expansions

We observe that we can write

$$V_{\nu,\mu}(\alpha,\beta,z) = e^{\alpha z}\int_{\alpha}^{\infty} e^{-zt} t^{\nu}(t+\beta-\alpha)^{\mu}\,dt,$$

so if we define

$$G_{\nu,\mu}(\alpha,\beta,z) = \int_{0}^{\alpha} e^{(\alpha-t)z} t^{\nu}(t+\beta-\alpha)^{\mu}\,dt,$$

then we have

$$G_{\nu,\mu}(\alpha,\beta,z) + V_{\nu,\mu}(\alpha,\beta,z)$$
$$= (\beta-\alpha)^{\nu+\mu+1} e^{\alpha z}\Gamma(\nu+1)\,U(\nu+1,\nu+\mu+2,(\beta-\alpha)z).$$

Power series expansions

Expanding the exponential function we obtain

$$G_{\nu,\mu}(\alpha,\beta,z) = (\beta-\alpha)^{\nu+\mu+1}\sum_{k=0}^{\infty}\frac{[(\beta-\alpha)z]^k}{k!}\phi_k(\nu,\mu,\gamma)$$

where $\gamma = (\beta-\alpha)/\alpha$.

Here

$$\phi_k(\nu,\mu,\gamma) = \int_0^{\gamma}(\gamma-t)^k t^{\nu}(t+1)^{\mu}\,dt, \qquad k = 0,1,2,\ldots$$

which can be identified as a Gauss hypergeometric function:

$$\phi_k(\nu,\mu,\gamma) = \gamma^{\nu+k+1}\frac{\Gamma(k+1)\Gamma(\nu+1)}{\Gamma(k+\nu+2)}\;{}_2F_1\left(\begin{matrix}-\mu,\nu+1\\ \nu+k+2\end{matrix};-\gamma\right),$$

for $k = 0,1,2,\ldots$.

Power series expansions

Therefore

$$G_{\nu,\mu}(\alpha,\beta,z) = (\beta-\alpha)^{\nu+\mu+1}\gamma^{\nu+1}\Gamma(\nu+1)\sum_{k=0}^{\infty} d_k H_k(\mu,\nu,\gamma),$$

where $d_k = (\alpha z)^k$ and

$$H_k(\mu,\nu,\gamma) = \frac{1}{\Gamma(\nu+k+2)}\,{}_2F_1\left(\begin{matrix}-\mu,\nu+1\\ \nu+k+2\end{matrix};-\gamma\right), \quad k=0,1,2,\ldots.$$

If we write

$$\sum_{k=0}^{\infty} d_k H_k(\nu,\mu,\gamma) = \sum_{k=0}^{K-1} d_k H_k(\nu,\mu,\gamma) + R_K, \tag{1}$$

then the remainder can be estimated using the integral representation of the H_k functions:

$$R_K \approx \frac{(\alpha z)^K}{\Gamma(\nu+K+2)}.$$

Power series expansions

The functions H_k satisfy a three-term recurrence relation for increasing k:

$$H_{k+1} + b_k H_k + a_k H_{k-1} = 0,$$

see DLMF 15.5.18, and H_k is the minimal solution, so the continued fraction

$$\frac{H_k}{H_{k-1}} = \frac{-a_k}{b_k+}\ \frac{-a_{k+1}}{b_{k+1}+}\ \frac{-a_{k+2}}{b_{k+2}+}\ \cdots$$

converges (Pincherle).

Therefore, we write the sum in Horner form

$$\sum_{k=0}^{K} d_k H_k = d_0 H_0 \left(1 + \frac{d_1}{d_0}\frac{H_1}{H_0}\left(1 + \frac{d_2}{d_1}\frac{H_2}{H_1}\left(\ldots + \left(1 + \frac{d_K}{d_{K-1}}\frac{H_K}{H_{K-1}}\right)\right)\right)\right),$$

and we use the continued fraction expansion for the ratio H_K/H_{K-1}, plus backward evaluation. Note that $d_k/d_{k-1} = \alpha z$.

Power series expansions

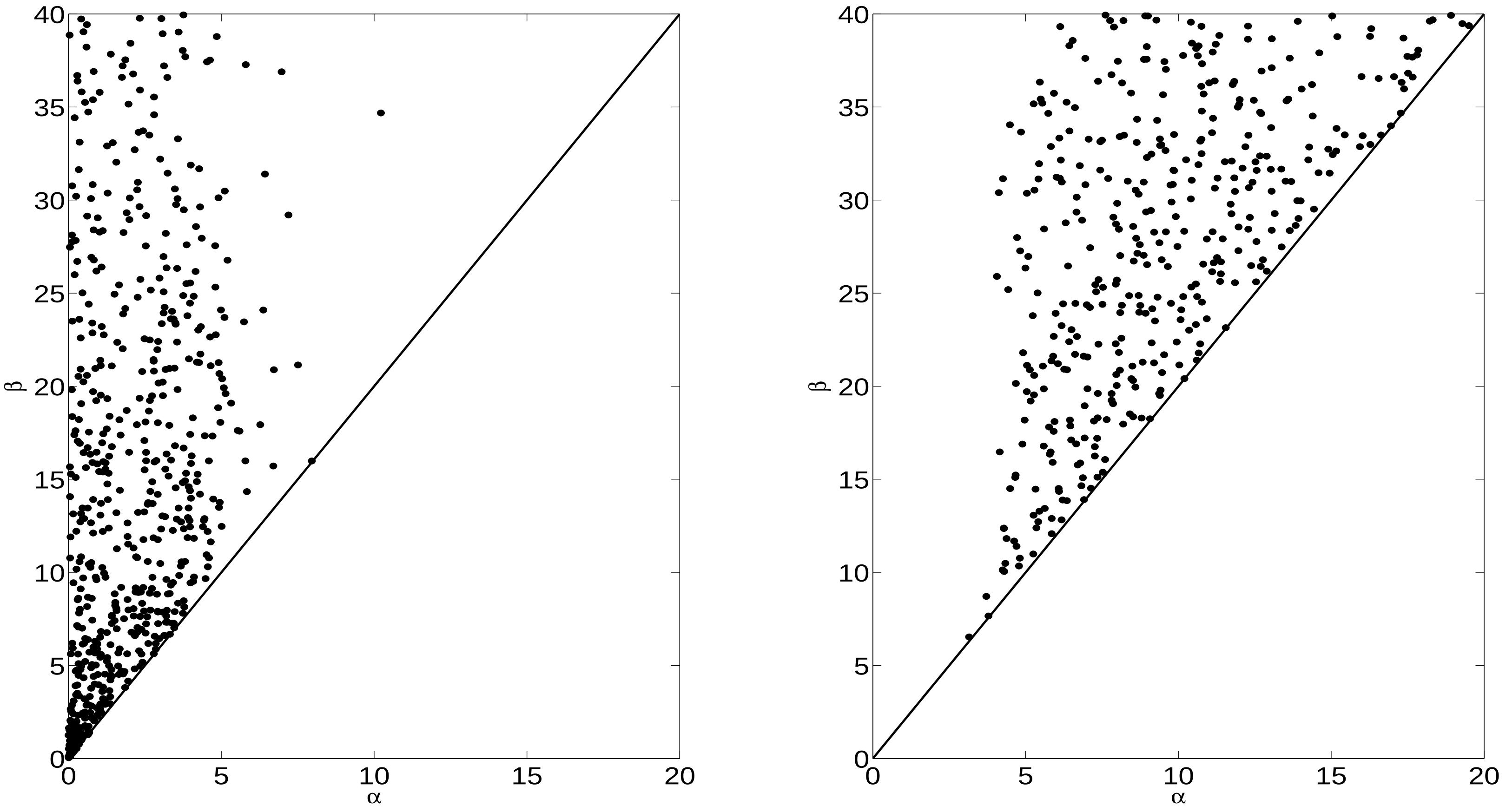

Figure 1: Absolute error in $V_{-1/2,-1}(\alpha, \beta, z)$ function using power series. Black dots indicate values for which $|e| \leq 10^{-14}$ (left), and points for which $|e| \geq 10^{-14}$ (right). Here $z = 0.87$, and the maximum error is 2.7×10^{-9}.

Power series expansions

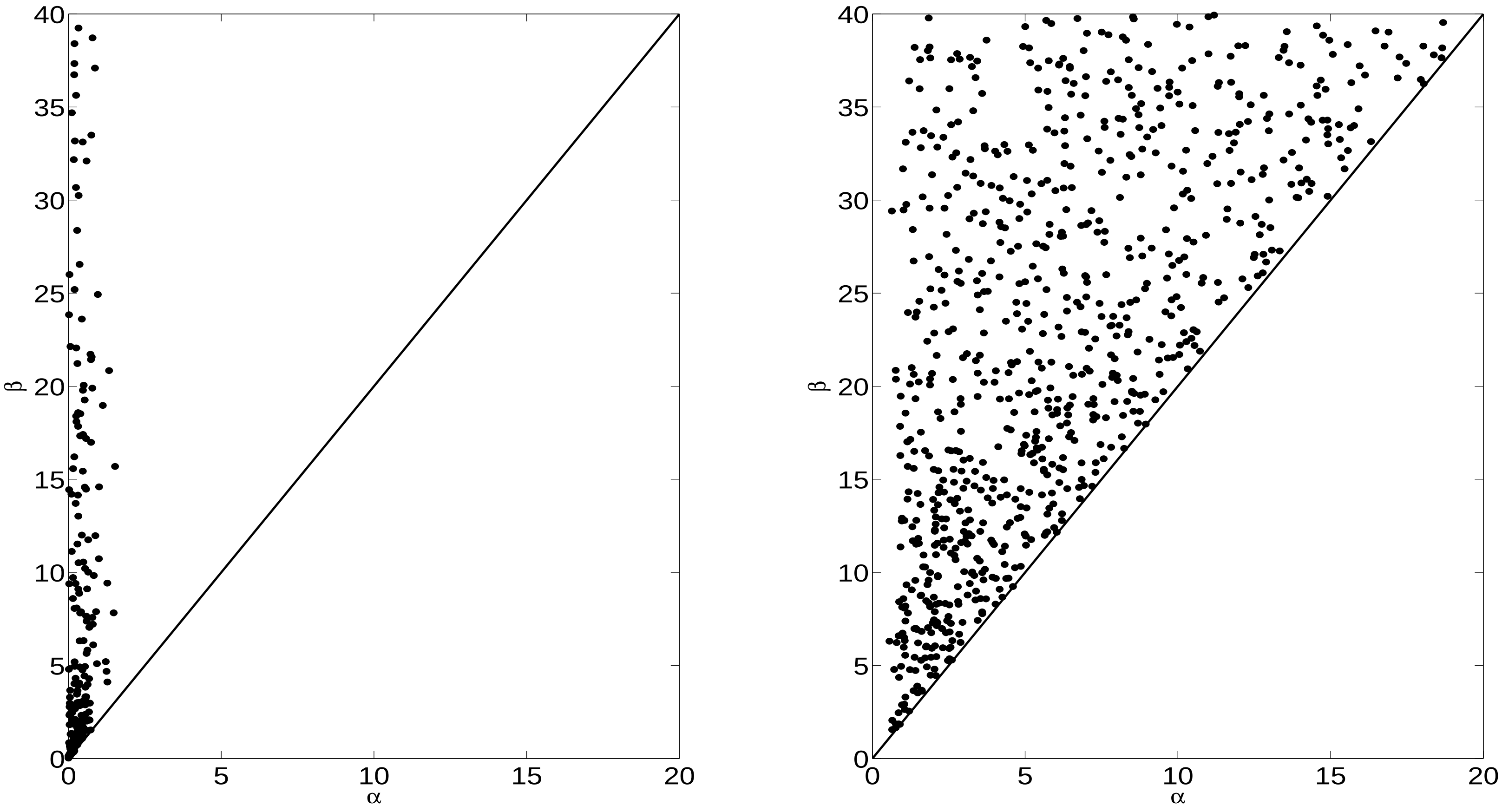

Figure 2: Same as in the previous figure but with $z = 4.31$. The maximum error in this example is 2.82×10^{19}.

Power series expansions

Results get considerably worse when α and z grow. Large α will produce cancellations when subtracting

$$V_{\nu,\mu}(\alpha,\beta,z) = (\beta-\alpha)^{\nu+\mu+1}e^{\alpha z}\Gamma(\nu+1)U(\nu+1,\nu+\mu+2,(\beta-\alpha)z) - G_{\nu,\mu}(\alpha,\beta,z). \tag{2}$$

Similarly, large values of z will be harmful, since

$$V_{\nu,\mu}(\alpha,\beta,z) \sim \frac{\alpha^{\nu}\beta^{\mu}}{z}, \qquad z\to\infty,$$

using Watson's lemma, whereas both terms on the right hand side of (2) are exponentially large when z is large.

On the other hand, for small values of the parameters the results are very satisfactory.

Asymptotic expansions for large z

One can derive asymptotic expansions of the function $V_{\nu,\mu}(\alpha,\beta,z)$ for large z using Watson's lemma. For α and β bounded away from 0, expand

$$(t+\alpha)^\nu (t+\beta)^\mu = \alpha^\nu \beta^\mu \sum_{k=0}^{\infty} c_k t^k,$$

and integration term by term gives

$$V_{\nu,\mu}(\alpha,\beta,z) \sim \alpha^\nu \beta^\mu \sum_{k=0}^{\infty} c_k \frac{k!}{z^{k+1}}, \quad z \to \infty,$$

which is valid when $|\arg z| < 3\pi/2$. The first coefficients are

$$c_0 = 1, \qquad c_1 = \frac{\mu\alpha + \nu\beta}{\alpha\beta}, \qquad c_2 = \frac{2\alpha\beta\nu\mu + \alpha^2\mu(\mu-1) + \beta^2\nu(\nu-1)}{2\alpha^2\beta^2},$$

but their computation soon becomes a bit cumbersome.

Modified asymptotic expansions

An alternative expansion follows from the ideas in D. and Temme (2008), see also Gil, Segura and Temme (2007):

$$(t+\alpha)^\nu = \left(\frac{\alpha}{\beta}\right)^\nu (t+\beta)^\nu \sum_{k=0}^{\infty} d_k \left(\frac{t}{t+\beta}\right)^k,$$

where

$$d_k = \left(\frac{\beta-\alpha}{\alpha}\right)^k \binom{\nu}{k}.$$

If we substitute this into the integral representation of $V_{\nu,\mu}(\alpha,\beta,z)$, we get

$$V_{\nu,\mu}(\alpha,\beta,z) = \alpha^\nu \beta^{\mu+1} \sum_{k=0}^{\infty} d_k \Phi_k,$$

where

$$\Phi_k = k!U(k+1,\nu+\mu+2,\beta z)$$

are confluent hypergeometric functions.

Modified asymptotic expansions

Some properties of the previous expansion:

- It is convergent if $0 < \beta < 2\alpha$.
- The $U-$functions in these series can be computed by using a backward recursion scheme:

$$\sum_{k=0}^{K} d_k\Phi_k = d_0\Phi_0\left(1+\frac{d_1}{d_0}\frac{\Phi_1}{\Phi_0}\left(1+\frac{d_2}{d_1}\frac{\Phi_2}{\Phi_1}\left(\ldots+\left(1+\frac{d_K}{d_{K-1}}\frac{\Phi_K}{\Phi_{K-1}}\right)\right)\right)\right),$$

plus continued fractions for the ratios. More precisely, we evaluate $r_K = \Phi_K/\Phi_{K-1}$ and then we update it:

$$r_k = \frac{-\alpha_k}{\beta_k + r_{k+1}}, \qquad j = K-1, K-2, \ldots, 1.$$

The coefficients α_k and β_k are those of the $(1,0)$ recursion for U-functions. Then we only need to compute $\Phi_0 = U(1, \nu+\mu+2, \beta z)$.

Modified asymptotic expansions

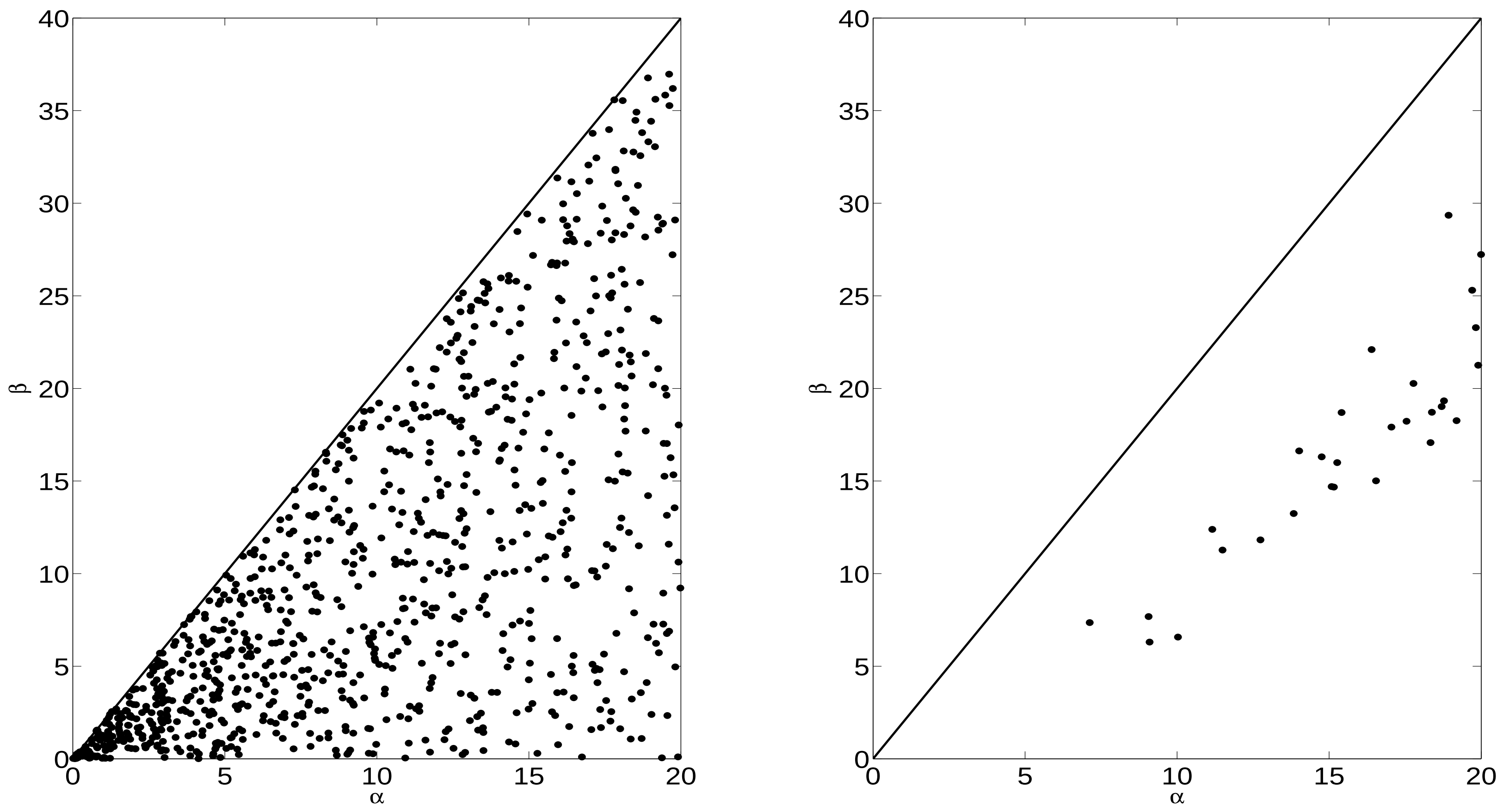

Figure 3: Absolute error in $V_{-1/2,-1}(\alpha, \beta, z)$ using modified asymptotic series. Black dots indicate values for which $|e| \leq 10^{-14}$ (left), and $|e| \geq 10^{-14}$ (right). Here $z = 10.45$. The maximum error in this example is 1.78×10^{-14}.

Modified asymptotic expansions

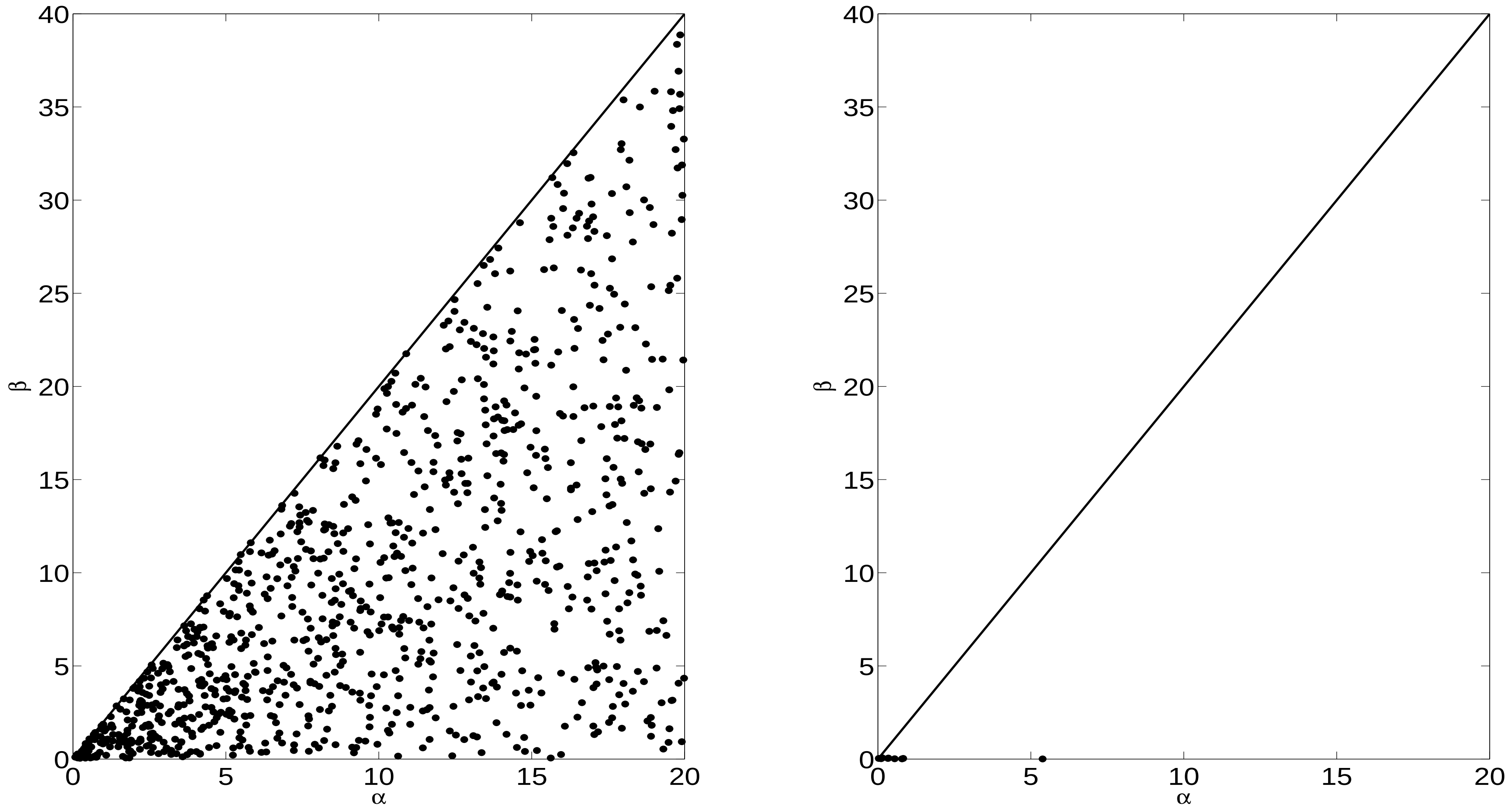

Figure 4: Same as in the previous figure but with $z = 0.45$. The maximum error in this example is 1.05×10^{-13}, for values of β very close to 0.

Recurrence relations for general ν and μ

Integration by parts gives

$$\mu V_{\nu+1,\mu-1} + [(\beta-\alpha)z + \nu - \mu]V_{\nu,\mu} - \nu V_{\nu-1,\mu+1} = \alpha^\nu \beta^\mu (\beta-\alpha).$$

In general, if we set

$$\nu = a + \epsilon_1 n, \qquad \mu = c + \epsilon_2 n,$$

with $\epsilon_j = 0, \pm 1$ (not both equal to 0), then the function

$$v_n(z) = V_{a+\epsilon_1 n, c+\epsilon_2 n}(\alpha, \beta, z)$$

satisfies an inhomogeneous three term recurrence relation:

$$v_{n+1}(z) + b_n v_n(z) + a_n v_{n-1}(z) = d_n.$$

The solutions of the homogeneous recurrence can be written in terms of confluent hypergeometric functions. Can we use such a recursion for computations?

Recurrence relations for general ν and μ

We consider $\epsilon_{1,2} = 0, \pm 1$ (not both equal to 0), and large values of n in

$$V_{a+\epsilon_1 n, c+\epsilon_2 n}(\alpha, \beta, nz) = \int_0^\infty (t+\alpha)^a (t+\beta)^c \mathrm{e}^{-n\phi(t)} \mathrm{d}t,$$

where

$$\phi(t) = zt - \epsilon_1 \log(t+\alpha) - \epsilon_2 \log(t+\beta).$$

The zeros of $\phi'(t)$ are given by

$$t_\pm = -\frac{(\alpha+\beta)z - \epsilon_1 - \epsilon_2}{2z} \pm \frac{\sqrt{[(\alpha+\beta)z - \epsilon_1 - \epsilon_2]^2 - 4z\Delta}}{2z},$$

where $\Delta = \alpha\beta z - \epsilon_1 \beta - \epsilon_2 \alpha$. When

$$z = z_0 = \frac{\epsilon_1}{\alpha} + \frac{\epsilon_2}{\beta},$$

then t_+ coalesces with the endpoint $t = 0$.

Example. The $(\epsilon_1, \epsilon_2) = (1, -1)$ recursion

In this case,

$$t_\pm = -\frac{\alpha + \beta}{2} \pm \frac{\sqrt{z(\beta - \alpha)(z(\beta - \alpha) + 4)}}{2z}.$$

If $z > 0$, then both roots are real and $t_- < 0$. The factor Δ vanishes at

$$z_0 = \frac{1}{\alpha} - \frac{1}{\beta} > 0.$$

- If $z > z_0$ then both t_+ and t_- are negative, and asymptotics follow from Watson's lemma at $t = 0$.
- When $z < z_0$ then t_+ is positive and becomes relevant in the asymptotic analysis for large n.
- When $z = z_0$ we have t_+ coalescing with $t = 0$, and we need the complementary error function in the analysis, see Wong (2001).

Example. The $(\epsilon_1, \epsilon_2) = (1, -1)$ recursion

It can be shown that for small z, the function

$$V_{a+n,c-n}(\alpha, \beta, nz)$$

is minimal for increasing n, namely

$$V_{a+n,c-n}(\alpha, \beta, nz) = \mathcal{O}\left(e^{-2\sqrt{n(\beta-\alpha)z}}\right), \qquad n \to \infty,$$

and a Miller-type algorithm should be used.

As normalization, we can use a formula like

$$\sum_{n=0}^{\infty} \frac{(-c)_n}{n!} V_{a+n,c-n}(\alpha, \beta, z) = (\beta - \alpha)^c z^{-a-1} e^{\alpha z} \Gamma(a+1, \alpha z),$$

obtained by summation in n.

The end

Thank you for your attention!