On a generalization of the complementary error function

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The function $V$

We consider the following function:

$$V_{\nu,\mu}(\alpha, \beta, z) = \int_0^\infty e^{-zt}(t + \alpha)^\nu(t + \beta)^\mu dt,$$

where $\beta > \alpha > 0$ and $\Re z > 0$.

The function $V$ is introduced in Lópex, Pérez Sinusía and Temme (2006) and Lópex and Pérez Sinusía (2007) as a first order approximation to the following singular perturbation problem:

$$-\epsilon \Delta U + U_z = 0$$

where $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$, $\epsilon$ is a small parameter and we have the condition

$$U(x, y, 0) = \begin{cases} 1 & (x, y) \in (-1, 1) \times (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$
A singular perturbation problem

A first order approximation to the solution can be given in terms of functions of the form

\[
F(u_i, v, \lambda) = \int_0^\infty \frac{re^{-\lambda r^2}}{\sqrt{r^2 + u_i^2(r^2 + v^2)}}dr, \quad i = 1, 2,
\]

where

\[
u_1 = \frac{\eta z}{\sqrt{\xi^2 + z^2}}, \quad u_2 = \frac{\xi z}{\sqrt{\xi^2 + z^2}}, \quad v = \sqrt{\xi^2 + \eta^2},
\]

and \(\xi = \pm 1 + x, \eta = \pm 1 + y\). Additionally,

\[
\lambda = \frac{\sqrt{\xi^2 + \eta^2 + z^2}}{4z^2} \frac{1}{\epsilon}.
\]

The functions \(F(u_i, v, \lambda)\) can be written in terms of the \(V\) function, namely

\[
F(u_i, v, \lambda) = \frac{1}{2} V_{-\frac{1}{2}, -1}(u_i^2, v^2, 1).
\]
A singular perturbation problem

It is of interest to analyse the behaviour of the solution $U(x, y, z)$ near the boundary and corners of the unit square, where boundary layers appear for small values of $\varepsilon$ (i.e. large values of $\lambda$).

- If $x$ or $y$ (but not both) are close to $\pm 1$, then $\xi$ or $\eta$ (and then $u_i$) are small, and the integrand in

$$F(u, v, \lambda) = \int_0^\infty \frac{r e^{-\lambda r^2}}{\sqrt{r^2 + u^2 (r^2 + v^2)}} dr$$

has a saddle point $r = 0$ coalescing with two poles $r = \pm iv$ or with two algebraic singularities $r = \pm iu$. The complementary error function can be used to describe these cases.

- Near the corners of $[-1, 1] \times [-1, 1]$ the saddle point coalesces with two poles and two algebraic singularities, and further analysis is needed.
Some particular cases

- Kummer $U$ function, when $\alpha = 0$ or $\beta = 0$:

$$V_{\nu,\mu}(0, \beta, z) = \beta^{\mu+\nu+1} \Gamma(\nu+1) U(\nu+1, \mu+\nu+2, \beta z), \quad \text{Re} \nu > -1,$$

- Incomplete Gamma function, when $\mu = 0$ or $\nu = 0$:

$$V_{\nu,0}(\alpha, \beta, z) = \alpha^{\nu+1} U(1, \nu+2, \alpha z) = z^{-\nu-1} e^{\alpha z} \Gamma(\nu+1, \alpha z),$$

or when $\alpha = \beta$:

$$V_{\nu,\mu}(\alpha, \alpha, z) = \alpha^{\nu+\mu+1} U(1, \nu+\mu+1, \alpha z) = z^{-\mu-\nu-1} e^{\alpha z} \Gamma(\nu+\mu+1, \alpha z).$$

- Complementary error function:

$$V_{-\frac{1}{2}, -1}(0, 1, z) = \pi e^z \text{erfc} \sqrt{z}, \quad V_{0, -\frac{1}{2}}(0, 1, z) = \sqrt{\frac{\pi}{z}} e^z \text{erfc} \sqrt{z}.$$
Some related integrals

- Generalized Goodwin–Staton integral:

\[ I(\mu, z) = \int_{0}^{\infty} \frac{t^{\mu} e^{-t^2}}{t + z} \, dt, \quad 0 < \arg z < \pi, \quad \Re \mu > -1. \]

- The integral

\[ H(u, v) = \int_{0}^{\infty} \frac{(u + t)^r G(t) e^{-t^2}}{v + t} \, dt, \quad |\arg u|, |\arg v| < \pi, \quad -1 < r < 1, \]

considered in Ciarkowski (1989). Here \( G(t) \) is regular in a neighborhood of the positive real axis.

- A more general case

\[ \int_{0}^{\infty} \frac{t^n e^{-xt^m}}{(t - \lambda)^k} \, dt, \quad \Re x > 0, \quad m, k \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus (\mathbb{R}^+ \cup \{0\}), \quad n = 0, 1, \ldots \]

studied by López and Pagola (2011).
Some properties of the function $V$

Recall that

$$V_{\nu,\mu}(\alpha, \beta, z) = \int_0^\infty e^{-zt} (t + \alpha)^\nu (t + \beta)^\mu \, dt,$$

where $\beta > \alpha > 0$ and $\Re z > 0$. Observe first that

$$V_{\nu,\mu}(\alpha, \beta, z) = V_{\mu,\nu}(\beta, \alpha, z),$$

so one can restrict the analysis to the case $\beta > \alpha$. Similarly,

$$V_{\nu,\mu}(\alpha, \beta, z) = z^{-1-\nu-\mu} V_{\nu,\mu}(\alpha z, \beta z, 1).$$

For the evaluation of the $V$ function, we can use the following ideas:

- Power series expansions for small values of $z$.
- Asymptotic and modified asymptotic expansions for large values of $z$. 
Power series expansions

We observe that we can write

\[ V_{\nu,\mu}(\alpha, \beta, z) = e^{\alpha z} \int_{\alpha}^{\infty} e^{-zt} t^{\nu}(t + \beta - \alpha)^{\mu} \, dt, \]

so if we define

\[ G_{\nu,\mu}(\alpha, \beta, z) = \int_{0}^{\alpha} e^{(\alpha-t)z} t^{\nu}(t + \beta - \alpha)^{\mu} \, dt, \]

then we have

\[ G_{\nu,\mu}(\alpha, \beta, z) + V_{\nu,\mu}(\alpha, \beta, z) \]
\[ = (\beta - \alpha)^{\nu+\mu+1} e^{\alpha z} \Gamma(\nu + 1) U(\nu + 1, \nu + \mu + 2, (\beta - \alpha)z). \]
Expanding the exponential function we obtain

\[ G_{\nu,\mu}(\alpha, \beta, z) = (\beta - \alpha)^{\nu+\mu+1} \sum_{k=0}^{\infty} \frac{[(\beta - \alpha)z]^k}{k!} \phi_k(\nu, \mu, \gamma) \]

where \( \gamma = (\beta - \alpha)/\alpha \).

Here

\[ \phi_k(\nu, \mu, \gamma) = \int_{0}^{\gamma} (\gamma - t)^k t^\nu (t + 1)^\mu dt, \quad k = 0, 1, 2, \ldots \]

which can be identified as a Gauss hypergeometric function:

\[ \phi_k(\nu, \mu, \gamma) = \gamma^{\nu+k+1} \frac{\Gamma(k+1)\Gamma(\nu+1)}{\Gamma(k+\nu+2)} \binom{-\mu, \nu+1}{\nu+k+2} \binom{-\gamma}{} \]

for \( k = 0, 1, 2, \ldots \).
Power series expansions

Therefore

\[ G_{\nu,\mu}(\alpha, \beta, z) = (\beta - \alpha)^{\nu+\mu+1} \gamma^{\nu+1} \Gamma(\nu + 1) \sum_{k=0}^{\infty} d_k H_k(\mu, \nu, \gamma), \]

where \( d_k = (\alpha z)^k \) and

\[ H_k(\mu, \nu, \gamma) = \frac{1}{\Gamma(\nu + k + 2)} \binom{-\mu, \nu + 1}{\nu + k + 2} \frac{\Gamma(\nu+k+2)}{\Gamma(\nu+K+2)} \]

If we write

\[ \sum_{k=0}^{\infty} d_k H_k(\nu, \mu, \gamma) = \sum_{k=0}^{K-1} d_k H_k(\nu, \mu, \gamma) + R_K, \] (1)

then the remainder can be estimated using the integral representation of the \( H_k \) functions:

\[ R_K \approx \frac{(\alpha z)^K}{\Gamma(\nu + K + 2)}. \]
The functions $H_k$ satisfy a three-term recurrence relation for increasing $k$:

$$H_{k+1} + b_k H_k + a_k H_{k-1} = 0,$$

see DLMF 15.5.18, and $H_k$ is the minimal solution, so the continued fraction

$$\frac{H_k}{H_{k-1}} = \frac{-a_k}{b_k} \frac{-a_{k+1}}{b_{k+1}} \frac{-a_{k+2}}{b_{k+2}} \ldots$$

converges (Pincherle).

Therefore, we write the sum in Horner form

$$\sum_{k=0}^{K} d_k H_k = d_0 H_0 \left(1 + \frac{d_1}{d_0} \frac{H_1}{H_0} \left(1 + \frac{d_2}{d_1} \frac{H_2}{H_1} \left(\ldots + \left(1 + \frac{d_K}{d_{K-1}} \frac{H_K}{H_{K-1}}\right)\right)\right)\right),$$

and we use the continued fraction expansion for the ratio $H_K/H_{K-1}$, plus backward evaluation. Note that $d_k/d_{k-1} = \alpha z$. 
Figure 1: Absolute error in $V_{-1/2,-1}(\alpha, \beta, z)$ function using power series. Black dots indicate values for which $|e| \leq 10^{-14}$ (left), and points for which $|e| \geq 10^{-14}$ (right). Here $z = 0.87$, and the maximum error is $2.7 \times 10^{-9}$. 
Figure 2: Same as in the previous figure but with \( z = 4.31 \). The maximum error in this example is \( 2.82 \times 10^{19} \).
Power series expansions

Results get considerably worse when $\alpha$ and $z$ grow. Large $\alpha$ will produce cancellations when subtracting

$$V_{\nu,\mu}(\alpha, \beta, z) = (\beta - \alpha)^{\nu+\mu+1} e^{\alpha z} \Gamma(\nu + 1) U(\nu + 1, \nu + \mu + 2, (\beta - \alpha)z)$$

$$- G_{\nu,\mu}(\alpha, \beta, z).$$

(2)

Similarly, large values of $z$ will be harmful, since

$$V_{\nu,\mu}(\alpha, \beta, z) \sim \frac{\alpha^\nu \beta^\mu}{z}, \quad z \to \infty,$$

using Watson’s lemma, whereas both terms on the right hand side of (2) are exponentially large when $z$ is large.

On the other hand, for small values of the parameters the results are very satisfactory.
Asymptotic expansions for large \( z \)

One can derive asymptotic expansions of the function \( V_{\nu,\mu}(\alpha, \beta, z) \) for large \( z \) using Watson’s lemma. For \( \alpha \) and \( \beta \) bounded away from 0, expand

\[
(t + \alpha)^\nu(t + \beta)^\mu = \alpha^\nu \beta^\mu \sum_{k=0}^{\infty} c_k t^k,
\]

and integration term by term gives

\[
V_{\nu,\mu}(\alpha, \beta, z) \sim \alpha^\nu \beta^\mu \sum_{k=0}^{\infty} c_k \frac{k!}{z^{k+1}}, \quad z \to \infty,
\]

which is valid when \( |\arg z| < 3\pi/2 \). The first coefficients are

\[
c_0 = 1, \quad c_1 = \frac{\mu \alpha + \nu \beta}{\alpha \beta}, \quad c_2 = \frac{2\alpha \beta \nu \mu + \alpha^2 \mu (\mu - 1) + \beta^2 \nu (\nu - 1)}{2\alpha^2 \beta^2},
\]

but their computation soon becomes a bit cumbersome.
An alternative expansion follows from the ideas in D. and Temme (2008), see also Gil, Segura and Temme (2007):

\[(t + \alpha)^\nu = \left(\frac{\alpha}{\beta}\right)^\nu (t + \beta)^\nu \sum_{k=0}^{\infty} d_k \left(\frac{t}{t + \beta}\right)^k,\]

where

\[d_k = \left(\frac{\beta - \alpha}{\alpha}\right)^k \binom{\nu}{k}.\]

If we substitute this into the integral representation of \(V_{\nu,\mu}(\alpha, \beta, z)\), we get

\[V_{\nu,\mu}(\alpha, \beta, z) = \alpha^\nu \beta^{\mu+1} \sum_{k=0}^{\infty} d_k \Phi_k,\]

where

\[\Phi_k = k!U(k + 1, \nu + \mu + 2, \beta z)\]

are confluent hypergeometric functions.
Modified asymptotic expansions

Some properties of the previous expansion:

- It is convergent if \( 0 < \beta < 2\alpha \).

- The \( U \)–functions in these series can be computed by using a backward recursion scheme:

\[
\sum_{k=0}^{K} d_k \Phi_k = d_0 \Phi_0 \left( 1 + \frac{d_1 \Phi_1}{d_0 \Phi_0} \left( 1 + \frac{d_2 \Phi_2}{d_1 \Phi_1} \left( \ldots + \left( 1 + \frac{d_K \Phi_K}{d_{K-1} \Phi_{K-1}} \right) \right) \right) \right),
\]

plus continued fractions for the ratios. More precisely, we evaluate

\( r_K = \frac{\Phi_K}{\Phi_{K-1}} \) and then we update it:

\[
r_k = \frac{-\alpha_k}{\beta_k + r_{k+1}}, \quad j = K - 1, K - 2, \ldots, 1.
\]

The coefficients \( \alpha_k \) and \( \beta_k \) are those of the \((1, 0)\) recursion for \( U \)-functions. Then we only need to compute \( \Phi_0 = U(1, \nu + \mu + 2, \beta z) \).
Figure 3: Absolute error in $V_{-1/2,-1}(\alpha, \beta, z)$ using modified asymptotic series. Black dots indicate values for which $|\epsilon| \leq 10^{-14}$ (left), and $|\epsilon| \geq 10^{-14}$ (right). Here $z = 10.45$. The maximum error in this example is $1.78 \times 10^{-14}$. 
Figure 4: Same as in the previous figure but with $z = 0.45$. The maximum error in this example is $1.05 \times 10^{-13}$, for values of $\beta$ very close to 0.
Recurrence relations for general $\nu$ and $\mu$

Integration by parts gives

$$
\mu V_{\nu+1, \mu-1} + [(\beta - \alpha)z + \nu - \mu]V_{\nu, \mu} - \nu V_{\nu-1, \mu+1} = \alpha^\nu \beta^\mu (\beta - \alpha).
$$

In general, if we set

$$
\nu = a + \epsilon_1 n, \quad \mu = c + \epsilon_2 n,
$$

with $\epsilon_j = 0, \pm 1$ (not both equal to 0), then the function

$$
v_n(z) = V_{a+\epsilon_1 n, c+\epsilon_2 n}(\alpha, \beta, z)
$$

satisfies an inhomogeneous three term recurrence relation:

$$
v_{n+1}(z) + b_n v_n(z) + a_n v_{n-1}(z) = d_n.
$$

The solutions of the homogeneous recurrence can be written in terms of confluent hypergeometric functions. Can we use such a recursion for computations?
Recurrence relations for general $\nu$ and $\mu$

We consider $\epsilon_{1,2} = 0, \pm 1$ (not both equal to 0), and large values of $n$ in

$$V_{a+\epsilon_1 n, c+\epsilon_2 n}(\alpha, \beta, n z) = \int_0^\infty (t + \alpha)^a (t + \beta)^c e^{-n\phi(t)} dt,$$

where

$$\phi(t) = zt - \epsilon_1 \log(t + \alpha) - \epsilon_2 \log(t + \beta).$$

The zeros of $\phi'(t)$ are given by

$$t_\pm = -\frac{(\alpha + \beta)z - \epsilon_1 - \epsilon_2}{2z} \pm \frac{\sqrt{[(\alpha + \beta)z - \epsilon_1 - \epsilon_2]^2 - 4z\Delta}}{2z},$$

where $\Delta = \alpha \beta z - \epsilon_1 \beta - \epsilon_2 \alpha$. When

$$z = z_0 = \frac{\epsilon_1}{\alpha} + \frac{\epsilon_2}{\beta},$$

then $t_+\ coalesces with the endpoint $t = 0$. 
Example. The \((\epsilon_1, \epsilon_2) = (1, -1)\) recursion

In this case,

\[
t_{\pm} = -\frac{\alpha + \beta}{2} \pm \frac{\sqrt{z(\beta - \alpha)(z(\beta - \alpha) + 4)}}{2z}.
\]

If \(z > 0\), then both roots are real and \(t_- < 0\). The factor \(\Delta\) vanishes at

\[
z_0 = \frac{1}{\alpha} - \frac{1}{\beta} > 0.
\]

- If \(z > z_0\) then both \(t_+\) and \(t_-\) are negative, and asymptotics follow from Watson’s lemma at \(t = 0\).

- When \(z < z_0\) then \(t_+\) is positive and becomes relevant in the asymptotic analysis for large \(n\).

- When \(z = z_0\) we have \(t_+\) coalescing with \(t = 0\), and we need the complementary error function in the analysis, see Wong (2001).
Example. The $(\epsilon_1, \epsilon_2) = (1, -1)$ recursion

It can be shown that for small $z$, the function

$$V_{a+n, c-n}(\alpha, \beta, nz)$$

is minimal for increasing $n$, namely

$$V_{a+n, c-n}(\alpha, \beta, nz) = O \left( e^{-2\sqrt{n(\beta-\alpha)z}} \right), \quad n \to \infty,$$

and a Miller-type algorithm should be used. As normalization, we can use a formula like

$$\sum_{n=0}^{\infty} \frac{(-c)^n}{n!} V_{a+n, c-n}(\alpha, \beta, z) = (\beta - \alpha)^c z^{-a-1} e^{\alpha z} \Gamma(a + 1, \alpha z),$$

obtained by summation in $n$. 
Thank you for your attention!