parameter differentiation for Bessel and associated Legendre functions

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Outline

definitions: special functions

- **gamma function** (Re $z > 0$) $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$$

Euler reflection formula: $\quad \Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

factorial: (for $n \in \{0, 1, 2, \ldots\}$) $\quad \Gamma(n+1) = n!$

Pochhammer symbol: (for $n \in \{0, 1, 2, \ldots\}$) $\quad (a)_n := \dfrac{\Gamma(a+n)}{\Gamma(a)}$

- **digamma function – logarithmic derivative of gamma function**

$$\frac{d}{dz}\Gamma(z) =: \Gamma(z)\psi(z)$$

- **generalized factorial function**

$${}_pF_q\left((a_p);(b_q);z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^k}{k!}$$

definitions – Bessel functions (unrestricted order ν)

- **Bessel function of the first kind**

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\nu+1; -\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-z^2/4)^n}{n!\Gamma(\nu+n+1)}$$

- **Bessel function of the second kind (Weber's function)**

$$Y_\nu(z) = \cot(\pi\nu) J_\nu(z) - \csc(\pi\nu) J_{-\nu}(z)$$
$$Y_n(z) = \frac{1}{\pi} \left.\frac{\partial J_\nu(z)}{\partial \nu}\right|_{\nu=n} + \frac{(-1)^n}{\pi} \left.\frac{\partial J_\nu(z)}{\partial \nu}\right|_{\nu=-n}$$

- **modified Bessel function of the first kind**

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\nu+1; \frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z^2/4)^n}{n!\Gamma(\nu+n+1)}$$

- **modified Bessel function of the second kind (Macdonald's function)**

$$K_\nu(z) = \frac{\pi}{2} \csc(\pi\nu) I_{-\nu}(z) - \csc(\pi\nu) I_\nu(z)$$
$$K_n(z) = \frac{(-1)^n}{2} \left.\frac{\partial I_{-\nu}(z)}{\partial \nu}\right|_{\nu=n} - \frac{(-1)^n}{2} \left.\frac{\partial I_\nu(z)}{\partial \nu}\right|_{\nu=n}$$

associated Legendre functions $P_\nu^\mu, Q_\nu^\mu : \mathbf{C} \setminus \{-1,1\} \to \mathbf{C}$ (complex degree ν order μ)

- **two (Gauss hypergeometric) functions – associated Legendre d.e.**

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right] w = 0$$

- **associated Legendre function of the first kind** $(|1-z| < 2)$

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1}\right]^{\frac{\mu}{2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$

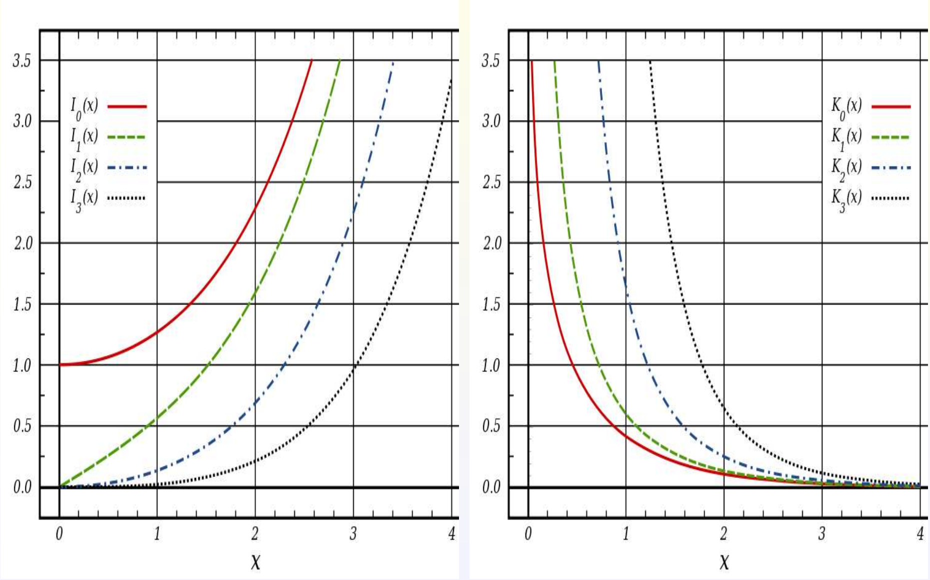
$$= -\frac{\sin(\pi\nu)}{\pi} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(-\nu+n)\Gamma(\nu+1+n)}{n!\Gamma(1-\mu+n)} \left(\frac{1-z}{2}\right)^n$$

- **associated Legendre function of the second kind** $(|z| > 1)$

$$Q_\nu^\mu(z) = \frac{\sqrt{\pi} e^{i\mu\pi} \Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right) z^{\nu+\mu+1}} {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

$$= \frac{1}{2} e^{i\pi\mu} (z^2-1)^{\frac{\mu}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\nu+\mu+2}{2}+n)\Gamma(\frac{\nu+\mu+1}{2}+n)}{n!\Gamma(\nu+\frac{3}{2}+n) z^{2n}}$$

order derivative illustration for modified Bessel functions



order derivatives for Bessel functions

- **Bessel function of the first kind**

$$\frac{\partial J_{\pm\nu}(z)}{\partial\nu} = \pm J_{\pm\nu}(z)\log\left(\frac{z}{2}\right) \mp \sum_{m=0}^{\infty}(-1)^m\left(\frac{z}{2}\right)^{\pm\nu+2m}\frac{\psi(\pm\nu+m+1)}{m!\Gamma(\pm\nu+m+1)}$$

- **Bessel function of the second kind (Weber's function)**

$$\frac{\partial Y_\nu(z)}{\partial\nu} = \cot(\pi\nu)\frac{\partial J_\nu(z)}{\partial\nu} - \csc(\pi\nu)\frac{\partial J_{-\nu}(z)}{\partial\nu} - \pi\csc(\pi\nu)Y_{-\nu}(z)$$

- **modified Bessel function of the first kind**

$$\frac{\partial I_\nu(z)}{\partial\nu} = I_\nu(z)\log\left(\frac{z}{2}\right) - \sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{\nu+2m}\frac{\psi(\nu+m+1)}{m!\Gamma(\nu+m+1)}$$

- **modified Bessel function of the second kind (Macdonald's function)**

$$\frac{\partial K_\nu(z)}{\partial\nu} = -\pi\cot(\pi\nu)K_\nu(z) + \frac{\pi}{2}\csc(\pi\nu)\left[\frac{\partial I_{-\nu}(z)}{\partial\nu} - \frac{\partial I_\nu(z)}{\partial\nu}\right]$$

Bessel function mirror symmetry about $\nu = 0$

- **Bessel function of the first kind**

$$J_{-\nu}(z) = J_\nu(z)\cos(\pi\nu) - Y_\nu(z)\sin(\pi\nu)$$

- **Bessel function of the second kind (Weber's function)**

$$Y_{-\nu}(z) = J_\nu(z)\sin(\pi\nu) + Y_\nu(z)\cos(\pi\nu)$$

- **modified Bessel function of the first kind**

$$I_{-\nu}(z) = I_\nu(z) + \frac{2}{\pi}K_\nu(z)\sin(\pi\nu)$$

- **modified Bessel function of the second kind (Macdonald's function)**

$$K_{-\nu}(z) = K_\nu(z)$$

order derivatives for Bessel functions (zero-order)

Derivatives of Bessel function with respect to the order evaluated at integer-orders is given in §3.2.3 in Magnus, Oberhettinger & Soni (1966)

- **Bessel function of the first kind**

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=0}=\frac{\pi}{2}Y_0(z)$$

- **Bessel function of the second kind (Weber's function)**

$$\left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=0}=-\frac{\pi}{2}J_0(z)$$

- **modified Bessel function of the first kind**

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=0}=-K_0(z)$$

- **modified Bessel function of the second kind (Macdonald's function)**

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=0}=0$$

order derivatives for Bessel functions (integer-order)

Derivatives of Bessel function with respect to the order evaluated at integer-orders is given by (§3.2.3 in MOS (1966))

- **Bessel function of the first kind**

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=\pm n} = \frac{\pi}{2}(\pm 1)^n Y_n(z) \pm (\pm 1)^n \frac{n!}{2}\sum_{k=0}^{n-1}\frac{(z/2)^{k-n}}{k!(n-k)}J_k(z)$$

- **Bessel function of the second kind (Weber's function)**

$$\left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=\pm n} = -\frac{\pi}{2}(\pm 1)^n J_n(z) \pm (\pm 1)^n \frac{n!}{2}\sum_{k=0}^{n-1}\frac{(z/2)^{k-n}}{k!(n-k)}Y_k(z)$$

- **modified Bessel function of the first kind**

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=\pm n} = (-1)^{n+1} K_n(z) \pm \frac{n!}{2}\sum_{k=0}^{n-1}\frac{(-z/2)^{k-n}}{k!(n-k)}I_k(z)$$

- **modified Bessel function of the second kind (Macdonald's function)**

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=\pm n} = \pm\frac{n!}{2}\sum_{k=0}^{n-1}\frac{(z/2)^{k-n}}{k!(n-k)}K_k(z)$$

order derivatives for Bessel functions (half-integer)

Derivatives of Bessel function with respect to the order evaluated at half-integer-orders is given by (§3.3.3 in MOS (1966))

- **Bessel function of the first kind**

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=\pm1/2}=\sqrt{\frac{2}{\pi z}}\left[\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}\mathsf{Ci}(2z)\mp\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix}\mathsf{Si}(2z)\right]$$

- **Bessel function of the second kind (Weber's function)**

$$\left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=\pm1/2}=\pm\sqrt{\frac{2}{\pi z}}\left[\begin{Bmatrix}\cos z\\ \sin z\end{Bmatrix}\mathsf{Ci}(2z)\pm\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}[\mathsf{Si}(2z)-\pi]\right]$$

- **modified Bessel function of the first kind**

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=\pm1/2}=\sqrt{\frac{1}{2\pi z}}\left[e^z\mathsf{Ei}(-2z)\mp e^{-z}\mathsf{Ei}(2z)\right]$$

- **modified Bessel function of the second kind (Macdonald's function)**

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=\pm1/2}=\mp\sqrt{\frac{\pi}{2z}}e^z\mathsf{Ei}(-2z)$$

order derivatives for Bessel functions

- **cosine integral:** $\mathrm{Ci}(z) := -\int_z^\infty \frac{\cos t}{t}\,dt$

  sine integral: $\mathrm{Si}(z) := \int_0^z \frac{\sin t}{t}\,dt$

  exponential integral: $\mathrm{Ei}(z) := -\int_{-z}^\infty \frac{e^{-t}}{t}\,dt$

- **order derivatives at half-odd integer orders**

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=1/2+n} = \mathrm{ci}(2z)J_{n+1/2}(z) - (-1)^n\mathrm{Si}(2z)J_{-n-1/2}(z)$$

$$+\frac{n!}{2}\sum_{k=0}^{n-1}\frac{(z/2)^{k-n}}{k!(n-k)}J_{k+1/2}(z) - \frac{n!\sqrt{\pi z}}{2}\sum_{k=1}^{n}\frac{(2/z)^k}{(n-k)!k}\sum_{p=0}^{k-1}\frac{z^p}{p!}$$

$$\times\left[J_{n-k+1/2}(z)J_{p-1/2}(2z) - (-1)^{n-k-p}J_{k-n-1/2}(z)J_{1/2-p}(2z)\right]$$

parameter differentiation for associated Legendre functions

- $w_1 = P_\nu^\mu : \{z \in \mathbf{C} : |z-1| < 2\} \to \mathbf{C}$

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)}{\pi}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(-\nu+n)\Gamma(\nu+1+n)}{n!\Gamma(1-\mu+n)}\left(\frac{1-z}{2}\right)^n$$

$$Q_\nu^\mu(z) = -\frac{\pi^{i\pi\mu}\Gamma(1+\nu+\mu)}{2\sin(\pi\mu)\Gamma(1+\nu-\mu)} P_\nu^{-\mu}(z) + \frac{\pi e^{i\pi\mu}}{2\sin(\pi\mu)} P_\nu^\mu(z)$$

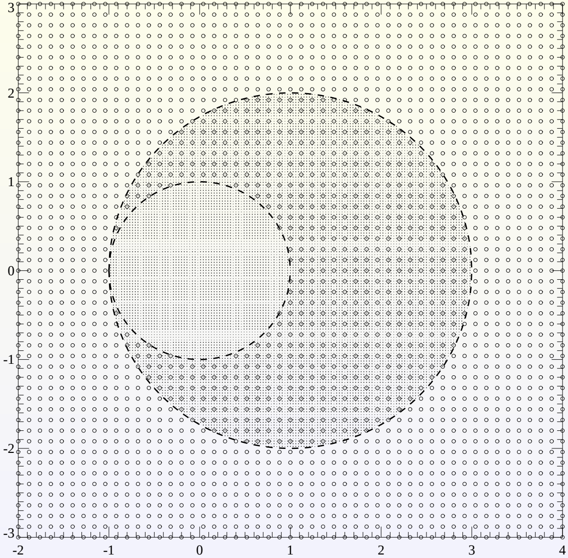
- $w_2 = Q_\nu^\mu : \{z \in \mathbf{C} : |z| > 1\} \to \mathbf{C}$

$$Q_\nu^\mu(z) = \tfrac{1}{2} e^{i\pi\mu}(z^2-1)^{\frac{\mu}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\nu+\mu+2}{2}+n)\Gamma(\frac{\nu+\mu+1}{2}+n)}{n!\Gamma(\nu+\frac{3}{2}+n)z^{2n}}$$

$$P_\nu^\mu(z) = \frac{e^{-i\pi\mu}\sin[\pi(\nu+\mu)]}{\pi\cos(\pi\nu)} Q_\nu^\mu(z) - \frac{e^{-i\pi\mu}\sin[\pi(\nu-\mu)]}{\pi\cos(\pi\nu)} Q_{-\nu-1}^\mu(z)$$

- **would like** $\dfrac{\partial P_\nu^\mu(z)}{\partial\nu}$, $\dfrac{\partial Q_\nu^\mu(z)}{\partial\nu}$, **and** $\dfrac{\partial P_\nu^\mu(z)}{\partial\mu}$, $\dfrac{\partial Q_\nu^\mu(z)}{\partial\mu}$, **for** $z \in \mathbf{C} \setminus \{-1,1\}$

domains in $\mathbf{C}$: $P_\nu^\mu(z) : |z-1| < 2 \qquad Q_\nu^\mu(z) : |z| > 1$



general degree derivatives for associated Legendre functions

- **general degree derivatives for** $\{z \in \mathbf{C} : |z-1| < 2\}$

$$\frac{\partial}{\partial\nu} P_\nu^\mu(z) = \pi \cot \pi\nu P_\nu^\mu(z) - \frac{1}{\pi}\mathbf{A}_\nu^\mu(z)$$

$$\begin{aligned} \frac{\partial}{\partial\nu} Q_\nu^\mu(z) = \pi \cot(\pi\nu) Q_\nu^\mu(z) \\ -\frac{\pi e^{i\pi\mu}\Gamma(1+\nu+\mu)}{2\sin(\pi\mu)\Gamma(1+\nu-\mu)} \left[\psi(1+\nu-\mu) - \psi(1+\nu-\mu)\right] P_\nu^{-\mu}(z) \\ +\frac{e^{i\pi\mu}\Gamma(1+\nu+\mu)}{2\sin(\pi\mu)\Gamma(1+\nu-\mu)} \mathbf{A}_\nu^{-\mu}(z) - \frac{e^{i\pi\mu}}{2\sin(\pi\mu)} \mathbf{A}_\nu^\mu(z) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\nu} Q_\nu^\mu(z) = -\frac{\pi^2}{2} P_\nu^\mu(z) + \frac{\pi \sin(\pi\mu)}{\sin(\pi\nu)\sin\pi(\nu+\mu)} Q_\nu^\mu(z) - \frac{1}{2}\cot\pi(\nu+\mu)\mathbf{A}_\nu^\mu(z) \\ +\frac{1}{2}\csc\pi(\nu+\mu)\mathbf{A}_\nu^\mu(-z) \end{aligned}$$

$$\mathbf{A}_\nu^\mu(z) = \sin(\pi\nu)\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(-\nu+n)\Gamma(\nu+1+n)}{n!\Gamma(1-\mu+n)} \left[\psi(\nu+n+1) - \psi(-\nu+n)\right] \left(\frac{1-z}{2}\right)^n$$

$$P_\nu^\mu(-z) = e^{\mp i\pi\nu} P_\nu^\mu(z) - \frac{2}{\pi} e^{-i\pi\mu} \sin[\pi(\nu+\mu)] Q_\nu^\mu(z)$$

$$Q_\nu^\mu(-z) = -e^{\pm i\pi\nu} Q_\nu^\mu(z)$$

general order derivatives for associated Legendre functions

- **general order derivatives for** $\{z \in \mathbf{C} : |z-1| < 2\}$

$$\frac{\partial}{\partial\mu} P_\nu^\mu(z) = \frac{1}{2} P_\nu^\mu(z) \log\frac{z+1}{z-1} - \frac{1}{\pi}\mathbf{B}_\nu^\mu(z)$$

$$\frac{\partial}{\partial\mu} Q_\nu^\mu(z) = -\tfrac{\pi\Gamma(1+\nu+\mu)e^{i\pi\mu}}{2\Gamma(1+\nu-\mu)\sin(\pi\mu)}\left[\psi(1+\nu+\mu)+\psi(1+\nu-\mu)\right] P_\nu^{-\mu}(z)$$
$$+\pi[i-\cot(\pi\mu)]Q_\nu^\mu(z) + \tfrac{1}{2}\log\tfrac{z+1}{z-1} Q_\nu^\mu(z)$$
$$+\tfrac{\Gamma(1+\nu+\mu)e^{i\pi\mu}}{\Gamma(1+\nu-\mu)\sin(\pi\mu)}\mathbf{B}_\nu^{-\mu}(z) - \tfrac{e^{i\pi\mu}}{2\sin(\pi\mu)}\mathbf{B}_\nu^\mu(z)$$

$$\mathbf{B}_\nu^\mu(z) = \sin(\pi\nu)\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \sum_{n=0}^\infty \frac{\Gamma(\nu+n+1)\psi(n-\mu+1)}{n!\Gamma(\nu-n+1)\Gamma(n-\mu+1)}\left(\frac{1-z}{2}\right)^n$$

general order derivatives for associated Legendre functions

- **general order derivatives for** $\{z \in \mathbf{C} : |z| > 1\}$

$$\frac{\partial}{\partial\mu} Q_\nu^\mu(z) = [i\pi + \frac{1}{2}\log(z^2-1)]Q_\nu^\mu(z) + \frac{1}{4}\mathbf{C}_\nu^\mu(z)$$

$$\begin{aligned}\frac{\partial}{\partial\mu} P_\nu^\mu(z) &= [i\pi + \tfrac{1}{2}\log(z^2-1)]P_\nu^\mu(z) \\ &\quad + \tfrac{e^{-2i\pi\mu}}{\cos(\pi\nu)}\left[e^{-i\pi\nu}Q_\nu^\mu(z) + e^{i\pi\nu}Q_{-\nu-1}^\mu(z)\right] \\ &\quad + \tfrac{e^{-i\pi\nu}}{4\pi\cos(\pi\nu)}\left\{\sin[\pi(\nu+\mu)]\mathbf{C}_\nu^\mu(z) - \sin[\pi(\nu-\mu)]\mathbf{C}_{-\nu-1}^\mu(z)\right\}\end{aligned}$$

$$\begin{aligned}\mathbf{C}_\nu^\mu(z) &= e^{i\pi\mu}(z^2-1)^{\frac{\mu}{2}} \\ &\quad \times \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\nu+\mu+2}{2}+n\right)\Gamma\left(\frac{\nu+\mu+1}{2}+n\right)}{n!\Gamma(\nu+\frac{3}{2}+n)z^{2n}}\left[\psi\left(\frac{\nu+\mu+2}{2}+n\right) + \psi\left(\frac{\nu+\mu+1}{2}+n\right)\right]\end{aligned}$$

general degree derivatives for associated Legendre functions

- **general degree derivatives for** $\{z \in \mathbf{C} : |z| > 1\}$

$$\frac{\partial}{\partial \nu} Q_\nu^\mu(z) = \frac{1}{4} \mathbf{C}_\nu^\mu(z) - \frac{1}{2} \mathbf{D}_\nu^\mu(z)$$

$$\begin{aligned}\frac{\partial}{\partial \nu} P_\nu^\mu(z) = & \frac{e^{-i\pi\mu}\cos(\pi\mu)}{\cos^2(\pi\nu)} \left[Q_\nu^\mu(z) - Q^\mu_{-\nu-1}(z) \right] \\ & + \frac{e^{-i\pi\mu}}{4\pi\cos(\pi\nu)} \left\{ \sin[\pi(\nu+\mu)] \mathbf{C}_\nu^\mu(z) - \sin[\pi(\nu-\mu)] \mathbf{C}^\mu_{-\nu-1}(z) \right\} \\ & - \frac{e^{-i\pi\mu}}{2\pi\cos(\pi\nu)} \left\{ \sin[\pi(\nu+\mu)] \mathbf{D}_\nu^\mu(z) - \sin[\pi(\nu-\mu)] \mathbf{D}^\mu_{-\nu-1}(z) \right\}\end{aligned}$$

$$\mathbf{D}_\nu^\mu(z) = e^{i\pi\mu}(z^2-1)^{\frac{\mu}{2}} \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\nu+\mu+2}{2}+n\right)\Gamma\left(\frac{\nu+\mu+1}{2}+n\right)}{n!\Gamma(\nu+\frac{3}{2}+n) z^{2n}} \psi(\nu + \tfrac{3}{2} + n)$$

What are these $\mathbf{A}_\nu^\mu(z)$, $\mathbf{B}_\nu^\mu(z)$, $\mathbf{C}_\nu^\mu(z)$, $\mathbf{D}_\nu^\mu(z)$?

- **most likely independent set of 2 functions**
- **derivatives with respect to parameters of** ${}_2F_1(a,b;c;z)$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{n!(c)_n}\psi(a+n)=\psi(a){}_2F_1(a,b;c;z)+\frac{\partial}{\partial a}{}_2F_1(a,b;c;z)$$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{n!(c)_n}\psi(c+n)=\psi(c){}_2F_1(a,b;c;z)-\frac{\partial}{\partial c}{}_2F_1(a,b;c;z)$$

- **Brychkov & Geddess (2004) "Differentiation of generalized hypergeometric functions with respect to parameters does not, in general, lead to generalized hypergeometric functions"**
- **Ancarani & Gasaneo (2008,2009,2010) compute derivatives with respect to parameters of** ${}_pF_q$ **and show relation to Kampé de Fériet double hypergeometric series**

differentiation with respect to parameters of Gauss hypergeometric functions in terms of Kampé de Fériet multiple hypergeometric series

$$\frac{\partial}{\partial a}{}_2F_1(a,b;c;z)=\frac{zb}{c}F^{2:2;1}_{2:1;0}\left[\begin{matrix}a+1,b+1:1,a;1;\\ 2,c+1:a+1;-;\end{matrix}z,z\right]$$

$$\frac{\partial}{\partial c}{}_2F_1(a,b;c;z)=-\frac{zab}{c^2}F^{2:2;1}_{2:1;0}\left[\begin{matrix}a+1,b+1:1,c;1;\\ 2,c+1:c+1;-;\end{matrix}z,z\right]$$

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_nz^n}{n!(c)_n}\psi(a+n)=\psi(a){}_2F_1(a,b;c;z)+\frac{zb}{c}F^{2:2;1}_{2:1;0}\left[\begin{matrix}a+1,b+1:1,a;1;\\ 2,c+1:a+1;-;\end{matrix}z,z\right]$$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_nz^n}{n!(c)_n}\psi(c+n)=\psi(c){}_2F_1(a,b;c;z)+\frac{zab}{c^2}F^{2:2;1}_{2:1;0}\left[\begin{matrix}a+1,b+1:1,c;1;\\ 2,c+1:c+1;-;\end{matrix}z,z\right]$$

Szmytkowski's formulas (see Szmytkowski (2009))

$$\begin{aligned}
\left[\frac{\partial}{\partial\nu}P_\nu^m(z)\right]_{\nu=p} &= P_p^m(z)\log\frac{z+1}{2} \\
&+ \left[2\psi(2p+1)-\psi(p+1)-\psi(p-m+1)\right]P_p^m(z) \\
&+ (-1)^{p+m}\sum_{k=0}^{p-m-1}(-1)^k\frac{2k+2m+1}{(p-m-k)(p+m+k+1)} \\
&\qquad\times\left[1+\frac{k!(p+m)!}{(k+2m)!(p-m)!}\right]P_{k+m}^m(z) \\
&+ (-1)^p\frac{(p+m)!}{(p-m)!}\sum_{k=0}^{m-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k^{-m}(z)
\end{aligned}$$

where $p, m \in \{0, 1, 2, \ldots\}$ **and** $0 \le m \le p$

Szmytkowski's formulas

Some special cases include for $m = 0$

$$\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=p} = P_p(z)\log\frac{z+1}{2} + 2\left[\psi(2p+1) - \psi(p+1)\right]P_p(z) + 2(-1)^p\sum_{k=0}^{p-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k(z)$$

and for $m = p$

$$\left[\frac{\partial}{\partial\nu}P_\nu^p(z)\right]_{\nu=p} = P_p^p(z)\log\frac{z+1}{2} + \left[2\psi(2p+1) - \psi(p+1) + \gamma\right]P_p^p(z) + (-1)^p(2p)!\sum_{k=0}^{p-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k^{-p}(z)$$

where $\gamma \approx 0.57721566$ **is Euler's constant**

Application of Szmytkowski's formula

Recently I proved:

$$\|\mathbf{x}-\mathbf{x}'\|^{\nu} = \frac{e^{i\pi(\nu+d-1)/2}\Gamma\left(\frac{d-2}{2}\right)}{\sqrt{\pi}\Gamma\left(-\frac{\nu}{2}\right)} \frac{\left(r_>^2 - r_<^2\right)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}}$$

$$\times \sum_{\lambda=0}^{\infty} \left(\lambda + \frac{d}{2} - 1\right) Q_{\lambda+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right) C_{\lambda}^{d/2-1}(\cos\gamma)$$

for $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$ **with** $r = \|\mathbf{x}\|$, $r' = \|\mathbf{x}'\|$, **and** $r_{\lessgtr} = \begin{smallmatrix}\min\\ \max\end{smallmatrix}\{r, r'\}$, **then using**

$$\lim_{\nu\to 0} \frac{\partial}{\partial\nu} \|\mathbf{x}-\mathbf{x}'\|^{\nu} = \|\mathbf{x}-\mathbf{x}'\|^{2p} \log\|\mathbf{x}-\mathbf{x}'\|$$

and Whipple's transformation of Legendre functions to obtain a Gegenbauer expansion for singular logarithmic kernels associated with fundamental solutions of the even-dimensional polyharmonic equation in Euclidean space

Cohl's formulas (see Cohl (2010,2011))

In the sequel $m, n \in \{0, 1, 2, \dots\}$

$$\frac{\Gamma(\nu \mp m + \frac{1}{2})}{\Gamma(\nu - m + \frac{1}{2})} \left[\frac{\partial}{\partial\mu} P^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m}$$
$$= Q^{m}_{\nu-1/2}(z) + \psi\left(\nu \mp m + \frac{1}{2}\right) P^{m}_{\nu-1/2}(z)$$
$$\pm m! \sum_{k=0}^{m-1} \frac{(-1)^{k-m}\left(z^2-1\right)^{(k-m)/2}}{2^{k-m+1} k!(m-k)} P^{k}_{\nu+k-m-1/2}(z)$$

$$\frac{\Gamma(\nu \mp m + \frac{1}{2})}{\Gamma(\nu - m + \frac{1}{2})} \left[\frac{\partial}{\partial\mu} Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m}$$
$$= \left[i\pi + \psi\left(\nu \mp m + \tfrac{1}{2}\right)\right] Q^{m}_{\nu-1/2}(z)$$
$$\pm m! \sum_{k=0}^{m-1} \frac{(-1)^{k-m}\left(z^2-1\right)^{(k-m)/2}}{k!(m-k)2^{k-m+1}} Q^{k}_{\nu+k-m-1/2}(z)$$

Cohl's formulas (see Cohl (2010,2011))

In the sequel $m, n \in \{0, 1, 2, \ldots\}$

$$\pm \left[\frac{\partial}{\partial\nu} P^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = \left[\psi\left(\mu + n + \frac{1}{2}\right) - \psi\left(\mu - n + \frac{1}{2}\right)\right] P^{\mu}_{n-1/2}(z)$$

$$+ \sum_{k=0}^{n-1} \frac{n!\,\Gamma\left(\mu - n + 1/2\right)\left(z^2-1\right)^{(n-k)/2}}{\Gamma\left(\mu + n - 2k + \frac{1}{2}\right) k!(n-k)2^{k-n+1}} P^{\mu+n-k}_{k-1/2}(z)$$

$$\left[\frac{\partial}{\partial\nu} Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n}$$

$$= -\sqrt{\frac{\pi}{2}}\, e^{i\pi\mu}\, \Gamma\left(\mu - n + \frac{1}{2}\right)\left(z^2-1\right)^{-1/4} Q^{n}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right)$$

$$\pm n! \sum_{k=0}^{n-1} \frac{\left(z^2-1\right)^{(n-k)/2}}{2^{k-n+1} k!(n-k)} Q^{\mu+k-n}_{k-1/2}(z)$$

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Future work in this classical area of Legendre functions

- **organize the known results, which are scattered, for parameter derivatives in particular cases**
- **find connections and pursue properties of $\mathrm{A}_\nu^\mu, \mathrm{B}_\nu^\mu, \mathrm{C}_\nu^\mu, \mathrm{D}_\nu^\mu$ functions**
- **study multiple derivatives with respect to parameters**
- **investigate anti-derivatives with respect to parameters**