Inequalities for eigenfunctions of the p-Laplacian

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Abstract

Motivated by the work of P. Lindqvist, we study eigenfunctions $\sin_p$ **of the one-dimensional** p**-Laplace operator, and prove several inequalities for these and** p**-analogues of other trigonometric functions and their inverse functions. Similar inequalities are given also for the** p**-analogues of the hyperbolic functions and their inverses.**

This talk is based on:
[BV] B. A. Bhayo and M. Vuorinen: *Inequalities for eigenfunctions of the p-Laplacian*.**- January 2011, 23 pp. arXiv math.CA 1101.3911.**

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In a highly cited paper P. Lindqvist [L] studied generalized trigonometric functions depending on a parameter $p > 1$ which for the case $p = 2$ reduce to the familiar functions. Numerous later authors, see e.g. [LP], [BEM1, BEM2], [DM] and the bibliographies of these papers, have extended this work in various directions including the study of generalized hyperbolic functions and their inverses. Our goal here to study these p-trigonometric and p-hyperbolic functions and to prove several inequalities for them. In our proofs we use the classical gamma function $\Gamma(x)$, the psi function $\psi(x)$ and the beta function $B(x,y)$. For $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$, these functions are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt,\ \ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},\ \ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively. The hypergeometric function is denoted by $F(a,b;c;x)$.

We start by discussing eigenfunctions of the so-called one-dimensional p-Laplacian Δ_p on $(0,1)$, $p \in (1,\infty)$. The eigenvalue problem ([DM])

$$-\Delta_p u = -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u,\; u(0) = u(1) = 0,$$

has eigenvalues

$$\lambda_n = (p-1)(n\pi_p)^p,$$

and eigenfunctions

$$\sin_p(n\pi_p t),\; n \in \mathbb{N},$$

where

$$\pi_p = \int_0^1 (1-s)^{-1/p} s^{1/p-1} ds = \frac{2}{p} B\left(1-\frac{1}{p}, \frac{1}{p}\right) = \frac{2\pi}{p\sin(\pi/p)}.$$

P. Lindqvist studied these generalized trigonometric functions in [L]. Motivated by Lindqvist's work, P. J. Bushell and D. E. Edmunds [BE] found very recently many new results for these functions. Several authors considered also various other p-analogues of trigonometric and hyperbolic functions and their inverses (see [LP], [BEM2], [DM]). In particular, we considered the following homeomorphisms,

$$\sin_p : (0, a_p) \to I,\ \cos_p : (0, a_p) \to I,\ \tan_p : (0, b_p) \to I,$$

$$\sinh_p : (0, c_p) \to I,\ \tanh_p : (0, \infty) \to I\ ,$$

where $I = (0,1)$ **and** $a_p = \pi_p/2$**,**

$$b_p = \frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right) = 2^{-1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}, \frac{1}{2}\right) .$$

$$c_p = \left(\frac{1}{2}\right)^{1/p} F\left(1\,, \frac{1}{p}\,; 1 + \frac{1}{p}\,; \frac{1}{2}\right) .$$

For $x \in I$, their inverse functions are defined as

$$\begin{aligned}
\operatorname{arcsin}_p x &= \int_0^x (1-t^p)^{-1/p} dt = x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right) \\
&= x\,(1-x^p)^{(p-1)/p} F\left(1,1;1+\frac{1}{p};x^p\right), \\
\operatorname{arctan}_p x &= \int_0^x (1+t^p)^{-1} dt = x\,F\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right), \\
\operatorname{arsinh}_p x &= \int_0^x (1+t^p)^{-1/p} dt = x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};-x^p\right), \\
\operatorname{artanh}_p x &= \int_0^x (1-t^p)^{-1} dt = x\,F\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right),
\end{aligned}$$

and by [BE, Prop 2.2] $\operatorname{arccos}_p(x) = \operatorname{arcsin}_p((1-x^p)^{1/p})$. For the particular case $p = 2$ one obtains the familiar elementary functions.

In mathematica we define the functions *sin_p, cos_p, tan_p, $sinh_p$, $tanh_p$*, **in the following way:**

$$\mathtt{sinp}[p\_, y\_] := x/.\mathtt{FindRoot}[\mathtt{arcsinp}[p, x] == y, \{x, 0.5\}]$$

$$\mathtt{cosp}[p\_, y\_] := x/.\mathtt{FindRoot}[\mathtt{cosp}[p, x] == y, \{x, 0.5\}]$$

$$\mathtt{tanp}[p\_, y\_] := x/.\mathtt{FindRoot}[\mathtt{arctanp}[p, x] == y, \{x, 0.5\}]$$

$$\mathtt{sinhp}[p\_, y\_] := x/.\mathtt{FindRoot}[\mathtt{arsinhp}[p, x] == y, \{x, 0.5\}]$$

$$\mathtt{tanhp}[p\_, y\_] := x/.\mathtt{FindRoot}[\mathtt{artanhp}[p, x] == y, \{x, 0.5\}]\,.$$

sin_p and $\arcsin_p$

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cos_p and $\arccos_p$

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tan_p and $\arctan_p$

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$sinh_p$ and arsinh_p

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$tanh_p$ and artanh_p

Main results

Theorem 1.1

For $p > 1$ and $x \in (0,1)$, we have

1. $\left(1 + \frac{x^p}{p(1+p)}\right) x < \mathrm{arcsin}_p\, x < \frac{\pi_p}{2}\, x,$
2. $\left(1 + \frac{1-x^p}{p(1+p)}\right) (1 - x^p)^{1/p} < \mathrm{arccos}_p\, x < \frac{\pi_p}{2}\, (1 - x^p)^{1/p},$
3. $\frac{(p(1+p)(1+x^p) + x^p)x}{p(1+p)(1+x^p)^{1+1/p}} < \mathrm{arctan}_p\, x < 2^{1/p}\, b_p\, \frac{x}{(1+x^p)^{1/p}}.$

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$$\left(1+\frac{x^p}{p(1+p)}\right)x < \arcsin_p x < \frac{\pi_p}{2}x.$$

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$$\left(1+\frac{1-x^p}{p(1+p)}\right)(1-x^p)^{1/p} < \operatorname{arccos}_p x < \frac{\pi_p}{2}(1-x^p)^{1/p}.$$

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$$\frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}} < \operatorname{arctan}_p x < 2^{1/p}\, b_p \,\frac{x}{(1+x^p)^{1/p}}.$$

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Theorem 1.2

For $p > 1$ and $x \in (0,1)$, we have

$$y\left(1+\frac{\log(1+x^p)}{1+p}\right) < \operatorname{arsinh}_p x < y\left(1+\frac{1}{p}\log(1+x^p)\right), \quad (1)$$

where $y = \left(\frac{x^p}{1+x^p}\right)^{1/p}$, and

$$x\left(1-\frac{1}{1+p}\log(1-x^p)\right) < \operatorname{artanh}_p x < x\left(1-\frac{1}{p}\log(1-x^p)\right). \quad (2)$$

Theorem 1.3

For $x > 0$ and $z = \pi x/2$, the function $g(k) = f(z^k)^{1/k}$ is decreasing in $k \in (0,\infty)$, where
$f(z) \in \{\operatorname{arsinh}(z), \operatorname{arcosh}(z), \operatorname{artanh}(2z/\pi)\}$.

$$z\left(1+\frac{\log(1+x^p)}{1+p}\right) < \operatorname{arsinh}_p x < z\left(1+\frac{1}{p}\log(1+x^p)\right).$$

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$$x\left(1-\frac{1}{1+p}\log(1-x^p)\right) < \operatorname{artanh}_p x < x\left(1-\frac{1}{p}\log(1-x^p)\right).$$

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Remark

For the particular case $p = 2$. Zhu [Z] has proved for $x > 0$

$$\frac{6\sqrt{2}(\sqrt{1+x^2}-1)^{1/2}}{4+\sqrt{2}(\sqrt{1+x^2}+1)^{1/2}} < \operatorname{arsinh}(x).$$

When $p = 2$, our bound in Theorem 1.2(1) differs from this bound roughly 0.01 when $x \in (0, 1)$.

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Difference of Zhu's bound and $y\left(1+\frac{\log(1+x^p)}{1+p}\right) < \operatorname{arsinh}_p x$.

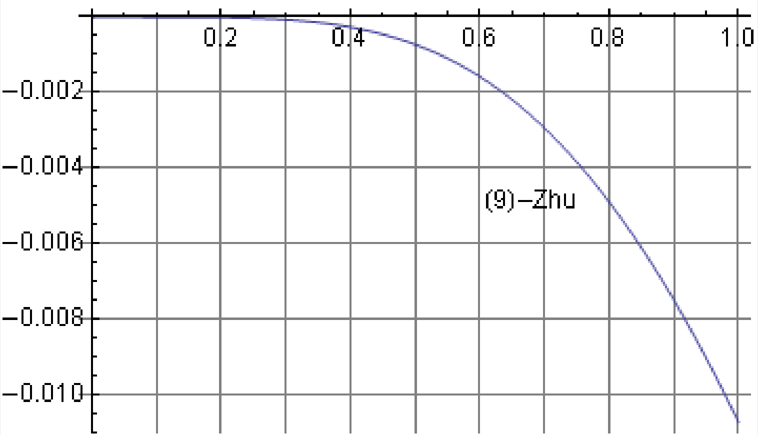

Figure: 11

For convenience, we use the notation $\mathbb{R}_+ = (0,\infty)$.

Lemma 2.1 [N2, Thm 2.1]

Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable, log-convex function and let $a\geq 1$. Then $g(x)=(f(x))^a/f(a\,x)$ decreases on its domain. In particular, if $0\leq x\leq y$, then the following inequalities

$$\frac{(f(y))^a}{f(a\,y)}\leq\frac{(f(x))^a}{f(a\,x)}\leq(f(0))^{a-1}$$

hold true. If $0<a\leq 1$, then the function g is an increasing function on $\mathbb{R}_+$ and inequalities are reversed.

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Lemma 2.2

1. **For** $a, b, c > 0$, $c < a + b$, **and** $|x| < 1$,

$$F(a, b; c; x) = (1 - x)^{c-a-b} F(c - a, c - b; c; x) \,.$$

2. **For** $a, x \in (0, 1)$, **and** $b, c \in (0, \infty)$

$$F(-a, b; c; x) < 1 - \frac{a\,b}{c}\, x \,.$$

3. **For** $a, x \in (0, 1)$, **and** $b, c \in (0, \infty)$

$$F(a, b; c; x) + F(-a, b; c; x) > 2 \,.$$

4. **Let** $a, b, c \in (0, \infty)$ **and** $c > a + b$. **Then for** $x \in [0, 1]$,

$$F(a, b; c; x) \leq \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \,.$$

5. **For** $a, b > 0$, **the following function**

$$f(x)=\frac{F(a,b;a+b;x)-1}{\log(1/(1-x))}$$

is strictly increasing from $(0,1)$ **onto** $(a\,b/(a+b), 1/B(a,b))$**. (see [AVV1, Thms 1.19(10), 1.52(1), Lems, 1.33, 1.35]).**

Lemma 2.3

For $p>1$ **and** $x\in(0,1)$**, the following functions**

$$f(k)=(\mathrm{arcsin}_p(x^k))^{1/k} \quad \text{and} \quad g(k)=(\mathrm{artanh}_p(x^k))^{1/k}$$

are decreasing in $k\in(0,\infty)$**. In particular, for** $k\geq 1$

$$\sqrt[k]{\mathrm{arcsin}_p(x^k)}\leq \mathrm{arcsin}_p(x)\leq(\mathrm{arcsin}_p\sqrt[k]{x})^k\,,$$

$$\sqrt[k]{\mathrm{artanh}_p(x^k)}\leq \mathrm{artanh}_p(x)\leq(\mathrm{artanh}_p\sqrt[k]{x})^k\,.$$

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Theorem 2.4

For $p > 1$, the following inequalities hold

1. $\operatorname{arcsin}_p(r\,s) \le \sqrt{\operatorname{arcsin}_p(r^2)\,\operatorname{arcsin}_p(s^2)} \le \operatorname{arcsin}_p(r)\,\operatorname{arcsin}_p(s),\;\; r,s \in (0,1),$
2. $\operatorname{artanh}_p(r\,s) \le \sqrt{\operatorname{artanh}_p(r^2)\,\operatorname{artanh}_p(s^2)} \le \operatorname{artanh}_p(r)\,\operatorname{artanh}_p(s),\;\; r,s \in (0,1),$
3. $\sqrt{\operatorname{arsinh}_p(r^2)\,\operatorname{arsinh}_p(s^2)} \le \operatorname{arsinh}_p(r\,s),\;\; r,s \in (0,\infty)\,.$

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Remark

The derivative of the function $h(x) = \log(1/\mathrm{arsinh}_p(e^x)),\ x > 0$ by Mathematica is

$$h'(x) = -1/F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; u\right) = D_1,\ \ u = e^{px}/(1 + e^{px}),$$

Let $v = e^{p\,x} + 1$, the manuallay found derivative of $h(x)$ is

$$h'(x) = -\frac{1}{v^2}\left(\frac{(v-1)\,F\,(2, 1 + 1/p; 2 + 1/p; (v-1)/v)}{(1/p+1)\ F\,(1, 1/p; 1 + 1/p; (v-1)/v)} + v\right) = D_2\,.$$

By Lemma 2.2(1)

$$D_2 = -\frac{(1-u)(1+p)F(1, 1/p; 1 + 1/p; u) + p\,u\,F(1, 1/p; 2 + 1/p; u)}{(1+p)F(1, 1/p; 1 + 1/p; u)}\,.$$

We observe that $D_1 = D_2$.

E. Neuman [N1] proved several inequalities involving trigonometric, hyperbolic functions and their inverses. In the following Lemmas 2.5 and 2.7 we have similar inequalities in the generalized form.

Lemma 2.5

For $k, p > 1$ **and** $r \geq s$**, we have**

$$\left(\frac{\mathrm{arcsin}_p(s)}{\mathrm{arcsin}_p(r)}\right)^k \leq \frac{\mathrm{arcsin}_p(s^k)}{\mathrm{arcsin}_p(r^k)}, \quad r, s \in (0,1),$$

$$\left(\frac{\mathrm{artanh}_p(s)}{\mathrm{artanh}_p(r)}\right)^k \leq \frac{\mathrm{artanh}_p(s^k)}{\mathrm{artanh}_p(r^k)}, \quad r, s \in (0,1),$$

$$\frac{\mathrm{arsinh}_p(s^k)}{\mathrm{arsinh}_p(r^k)} \leq \left(\frac{\mathrm{arsinh}_p(s)}{\mathrm{arsinh}_p(r)}\right)^k, \quad r, s \in (0,\infty).$$

Lemma 2.6 [K, Thm 2, p.151]

Let $J \subset \mathbb{R}$ be an open interval, and let $f : J \to \mathbb{R}$ be strictly monotonic function. Let $f^{-1} : f(J) \to J$ be the inverse to f then

1. **if f is convex and increasing, then f^{-1} is concave;**
2. **if f is convex and decreasing, then f^{-1} is convex;**
3. **if f is concave and increasing, then f^{-1} is convex;**
4. **if f is concave and decreasing, then f^{-1} is concave.**

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Lemma 2.7

For $k, p > 1$ and $r \geq s$, we have

$$\left(\frac{\sin_p(r)}{\sin_p(s)}\right)^k \leq \frac{\sin_p(r^k)}{\sin_p(s^k)}, \quad r, s \in (0,1),$$

$$\left(\frac{\tanh_p(r)}{\tanh_p(s)}\right)^k \leq \frac{\tanh_p(r^k)}{\tanh_p(s^k)}, \quad r, s \in (0,\infty),$$

$$\left(\frac{\sinh_p(r)}{\sinh_p(s)}\right)^k \geq \frac{\sinh_p(r^k)}{\sinh_p(s^k)}, \quad r, s \in (0,1).$$

Inequalities reverse for $k \in (0,1)$.

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Lemma 2.8

For $p > 1$, the following inequalities hold

1. $\sqrt{\sin_p(r^2)\sin_p(s^2)} \le \sin_p(r\,s)\,,\ r, s \in (0, \pi_p/2)$
2. $\sqrt{\tanh_p(r^2)\tanh_p(s^2)} \le \tanh_p(r\,s)\,,\ r, s \in (0, \infty)$
3. $\sinh_p(r\,s) \le \sqrt{\sinh_p(r^2)\sinh_p(s^2)}\,,\ r, s \in (0, \infty)\,.$

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Lemma 2.9

For $p > 1$, the following relations hold

1. $\sqrt{\sin_p(r)\sin_p(s)} \leq \sin_p((r+s)/2),\ \ r,s \in (0,\pi_p/2),$
2. $\sqrt{\sinh_p(r)\sinh_p(s)} \leq \sinh_p((r+s)/2),\ \ r,s \in (0,\infty)\,.$

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In 2006, Á. Baricz [B, Corollary 1.26] proved the following inequality:

$$\cosh(\sqrt{x\,y}) \le \sqrt{\cosh(x)\,\cosh(y)}$$

for all $x, y \in (0, \infty)$, with equality if and only if $x = y$.

In the following lemma we prove the similar inequalities:

Lemma 2.10

For $r, s \in (0, \infty)$, the following relations hold

1. $\sqrt{\mathrm{arsinh}_p(r)\,\mathrm{arsinh}_p(s)} \le \mathrm{arsinh}_p(\sqrt{r\,s})$
2. $\sqrt{\mathrm{artanh}_p(r)\,\mathrm{artanh}_p(s)} \ge \mathrm{artanh}_p(\sqrt{r\,s})$.

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Lemma 2.11

For $p > 1$, the following inequalities hold

1. $\sin_p(r+s) \le \sin_p(r) + \sin_p(s)\,, \;\; r, s \in (0, \pi_p/4)\,,$
2. $\tan_p(r+s) \ge \tan_p(r) + \tan_p(s)\,, \;\; r, s \in (0, b_p/2)\,,$
3. $\sinh_p(r+s) \ge \sinh_p(r) + \sinh_p(s)\,, \;\; r, s \in (0, c_p/2)\,,$
4. $\tanh_p(r+s) \le \tanh_p(r) + \tanh_p(s)\,, \;\; r, s \in (0, \infty)\,.$

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For easy reference we recall the following identity

$$F(a,b;c;z) = (1-z)^{-b} F(b, c-a; c; -z/(1-z)), \qquad (3)$$

(see [AS, 15.3.5]).

Theorem 1.1

For $p > 1$ and $x \in (0,1)$, we have

1. $\left(1 + \frac{x^p}{p(1+p)}\right) x < \mathrm{arcsin}_p\, x < \frac{\pi_p}{2}\, x$,
2. $\left(1 + \frac{1-x^p}{p(1+p)}\right) (1-x^p)^{1/p} < \mathrm{arccos}_p\, x < \frac{\pi_p}{2}\,(1-x^p)^{1/p}$,
3. $\frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}} < \mathrm{arctan}_p\, x < 2^{1/p}\, b_p\, \frac{x}{(1+x^p)^{1/p}}$.

Proof

By Lemma 2.2(3),(2) we get

$$\left(1+\frac{x^p}{p\,(1+p)}\right) < F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right),$$

and the first inequality of (1) holds. For the second one we get

$$\operatorname{arcsin}_p x = x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)$$

$$\leq \frac{x\Gamma(1+1/p)\Gamma(1+1/p-1/p-1/p)}{\Gamma(1+1/p-1/p)\Gamma(1+1/p-1/p)} = x\,\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)$$

$$= x\,\frac{1}{p}\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right) = x\,\frac{1}{p}B\left(1-\frac{1}{p},\frac{1}{p}\right) = x\,\frac{\pi_p}{2}$$

by Lemma 2.2(4) and [AS, 6.1.17].

From [BE, Prop (2.11)], we know that $\operatorname{arccos}_p x = \operatorname{arcsin}_p\left((1-x^p)^{1/p}\right)$, and (2) follows from (1). For (3), we get

$$\begin{aligned}
\operatorname{arctan}_p x &= \left(\frac{x}{1+x^p}\right) F\left(1,1;1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&= \left(\frac{x}{1+x^p}\right)\left(\frac{1}{1+x^p}\right)^{1/p-1} F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&= \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&\le 2^{1/p}\, b_p \left(\frac{x^p}{1+x^p}\right)^{1/p}
\end{aligned}$$

by identity (3) and Lemma 2.2(1),(4).

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For the lower bound we get

$$\begin{aligned}\operatorname{arctan}_p x &> \left(\frac{x^p}{1+x^p}\right)^{1/p}\left(2 - F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\right)\\ &> \frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}}\end{aligned}$$

from Lemma 2.2(3),(2).

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Theorem 1.2

For $p > 1$ and $x \in (0, 1)$, we have

$$z\left(1 + \frac{\log(1 + x^p)}{1 + p}\right) < \operatorname{arsinh}_p x < z\left(1 + \frac{1}{p}\log(1 + x^p)\right), \quad (4)$$

where $z = \left(\frac{x^p}{1 + x^p}\right)^{1/p}$, and

$$x\left(1 - \frac{1}{1 + p}\log(1 - x^p)\right) < \operatorname{artanh}_p x < x\left(1 - \frac{1}{p}\log(1 - x^p)\right). \quad (5)$$

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Proof

For (4), we replace $b = 1/p$, $c - a = 1/p$, $c = 1 + 1/p$ **and** $x^p = z/(1-z)$ **in (3) we get**

$$\operatorname{arsinh}_p x = x\,F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; -x^p\right)$$

$$= \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; \left(\frac{x^p}{1+x^p}\right)\right).$$

We get

$$\frac{\log\left(1+x^p\right)}{1+p}\left(\frac{x^p}{1+x^p}\right)^{1/p}$$

$$< x\,F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; -x^p\right)$$

$$< \left(1 - \frac{1}{p}\log\left(1 - \left(\frac{x^p}{1+x^p}\right)\right)\right)\left(\frac{x^p}{1+x^p}\right)^{1/p}$$

from Lemma 2.2(2) and making observation that

$$B(1,1/p)=\frac{\Gamma(1)\Gamma(1/p)}{\Gamma(1+1/p)}=\frac{\Gamma(1/p)}{(1/p)\Gamma(1/p)}=p,$$

this implies (4). For (5) we get from Lemma 2.2(5)

$$\frac{1}{1+p}\log\left(\frac{1}{1-x^p}\right)+1<F\left(1\,,\frac{1}{p}\,,1+\frac{1}{p}\,;x^p\right)<\frac{1}{p}\log\left(\frac{1}{1-x^p}\right)+1\,,$$

which is equivalent to

$$x\left(1-\frac{1}{1+p}\log(1-x^p)\right)<x\,F\left(1\,,\frac{1}{p}\,,1+\frac{1}{p}\,;x^p\right)$$

$$<x\left(1-\frac{1}{p}\log(1-x^p)\right)\,,$$

and the result follows.

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Lemma 2.12

For $p > 1$ and $x \in (0, 1)$, the following inequalities hold

1. $\mathrm{arctan}_p(x) < \mathrm{arsinh}_p(x) < \mathrm{arcsin}_p(x) < \mathrm{artanh}_p(x)\,,$
2. $\tanh_p(z) < \sin_p(z) < \sinh_p(z) < \tan_p(z)\,,$

the first and the second inequality hold for $z \in (0, \pi_p/2)$, and the third holds for $z \in (0, b_p)$.

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Lemma 2.13

For $p > 1$, we have

$$\frac{6p^2}{3p^2 - 2} \leq \pi_p \leq \frac{12p^2}{6p^2 - \pi^2}, \qquad \pi_p = \frac{2\pi}{p \sin(\pi/p)}.$$

Lemma 2.14

For $a \in (0,1)$ and $k, r, s \in (1, \infty)$, the following inequalities hold

1. $\pi_{r\,s} \leq \sqrt{\pi_{r^2}\,\pi_{s^2}} \leq \sqrt{\pi_r\,\pi_s}$,
2. $\pi_{r^a\,s^{1-a}} \leq a\,\pi_r + (1-a)\pi_s$,
3. $\left(\dfrac{\pi_s}{\pi_r}\right)^k \leq \dfrac{\pi_{s^k}}{\pi_{r^k}}$, $r \leq s$.

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Lemma 2.15

For $p > 1$ and $x \in (0,1)$, we have

$$\operatorname{arcsin}_p\left(\frac{x}{\sqrt[p]{1+x^p}}\right) = \operatorname{arctan}_p(x)\,,$$

$$\operatorname{arcsin}_p(x) = \operatorname{arctan}_p\left(\frac{x}{\sqrt[p]{1-x^p}}\right)\,,$$

$$\operatorname{arccos}_p(x) = \operatorname{arctan}_p\left(\frac{\sqrt[p]{1-x^p}}{x}\right)\,,$$

$$\operatorname{arccos}_p\left(\frac{1}{\sqrt[p]{1+x^p}}\right) = \operatorname{arctan}_p(x)\,.$$

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In the following tables we give some specific values of the p-analogues functions with $p = 3$.

x	$\arcsin_p x$	$\arccos_p x$	$\arctan_p x$	$\text{arsinh}_p x$	$\text{artanh}_p x$
0.00000	0.00000	1.20920	0.00000	0.00000	0.00000
0.25000	0.25033	1.17782	0.24903	0.24968	0.25099
0.50000	0.50547	1.07974	0.48540	0.49502	0.51685
0.75000	0.78196	0.88660	0.68570	0.72710	0.85661
1.00000	1.20920	0.00000	0.83565	0.93771	∞

x	$\sin_p x$	$\cos_p x$	$\tan_p x$	$\sinh_p x$	$\tanh_p x$
0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
0.25000	0.24967	0.99478	0.25098	0.25033	0.24903
0.50000	0.49476	0.95788	0.51652	0.50518	0.48517
0.75000	0.72304	0.85362	0.84704	0.77588	0.68283
1.00000	0.91139	0.62399	1.46058	1.08009	0.82304

Conjecture

For a fixed $x \in (0,1)$, the functions

$$\sin_p(\pi_p x/2),\ \tan_p(\pi_p x/2),\ \sinh_p(c_p x)$$

are monotone in $p \in (1,\infty)$. For fixed $x > 0$, $\tanh_p(x)$ is increasing in $p \in (1,\infty)$.

Open broblem

Analogues of addition formulas for p-functions e.g in the form of an inequality.

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Lemma 3.1

For $x \in (0,1)$ **and** $z \in (0,\infty)$**, the following functions**

$$f_1(k) = \sin(x^k)^{1/k}\,, f_2(k) = \cos(x^k)^{1/k}\,,$$

$$f_3(k) = \arctan(x^k)^{1/k}\,, f_4(k) = \tanh(z^k)^{1/k}$$

are increasing in $(0,\infty)$**, and**

$$g_1(k) = \tan(x^k)^{1/k}\,, g_2(k) = \sinh(z^k)^{1/k}$$

are decreasing in $(0,\infty)$**.**

Lemma 3.2

The following inequalities hold

1. $\sqrt{\operatorname{arccos}(r^2)\operatorname{arccos}(s^2)} < \operatorname{arccos}(r\,s)$
2. $\arctan(r)\arctan(s) < \sqrt{\arctan(r^2)\arctan(s^2)} < \arctan(r\,s)$
3. $\sqrt{\operatorname{arcosh}(r^2)\,\operatorname{arcosh}(s^2)} < \operatorname{arcosh}(r\,s)$

(1) and (2) hold for $r, s \in (0, 1)$ and (3) holds for $r, s \in (1, \infty)$.

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Lemma 3.3

For $r, s \in (0, \infty)$**, we have**

1. $\sinh(r\,s) < \sqrt{\sinh(r^2)\sinh(s^2)}$
2. $\cosh(r\,s) < \sqrt{\cosh(r^2)\cosh(s^2)}$
3. $\tanh(r)\tanh(s) < \sqrt{\tanh(r^2)\tanh(s^2)} < \tanh(r\,s)\,.$

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Lemma 3.4

For $x \in (0,1)$ **, the following functions**

$$f(k) = \sin\left(\frac{\pi}{2}x^k\right)^{1/k},$$

$$g(k) = \tan\left(\frac{\pi}{2}x^k\right)^{1/k},$$

$$h(k) = \sinh\left(x^k\right)^{1/k}.$$

are decreasing in $(0,\infty)$**. In particular, for** $k \geq 1$

$$\sqrt[k]{\sin\left(\frac{\pi}{2}x^k\right)} \leq \sin\left(\frac{\pi}{2}x\right) \leq \sin\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k,$$

$$\sqrt[k]{\tan\left(\frac{\pi}{2}x^k\right)} \leq \tan\left(\frac{\pi}{2}x\right) \leq \tan\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k,$$

$$\sqrt[k]{\sinh\left(x^k\right)} \leq \sinh(x) \leq \sinh\left(\sqrt[k]{x}\right)^k.$$

Lemma 3.5

The following functions

$$f(k) = \cos\left(\frac{\pi}{2}x^{1/k}\right)^k \; x \in (0,1),$$

$$g(k) = \cosh\left(x^k\right)^{1/k}, \; x \in (0,1),$$

$$h(k) = \operatorname{arcosh}\left(\frac{\pi}{2}x^k\right)^{1/k}, \; x \in (1,\infty)$$

are decreasing in $(0,\infty)$. **In particular, for** $k \geq 1$

$$\cos\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k \leq \cos\left(\frac{\pi}{2}x\right) \leq \sqrt[k]{\cos\left(\frac{\pi}{2}x^k\right)},$$

$$\sqrt[k]{\cosh\left(x^k\right)} \leq \cosh(x) \leq \cosh\left(\sqrt[k]{x}\right)^k,$$

$$\sqrt[k]{\operatorname{arcosh}\left(\frac{\pi}{2}x^k\right)} \leq \operatorname{arcosh}\left(\frac{\pi}{2}x\right) \leq \operatorname{arcosh}\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k.$$

Lemma 3.6

The following relations hold

1. $\sin(r)\sin(s) < \sqrt{\sin(r^2)\sin(s^2)}$ **for** $r, s \in (0,1)$ **,**
2. $\cos(r)\cos(s) < \sqrt{\cos(r^2)\cos(s^2)} < \cos(r\,s)$,
3. $\tan(r)\tan(s) > \sqrt{\tan(r^2)\tan(s^2)} > \tan(r\,s)$**,** **the first inequality in (2) and (3) holds for** $r, s \in (0, \sqrt{\pi/2})$**, and second for** $r, s \in (0,1)$**.**

For $x \in (0,1)$**, the function** $g(k) = (\cos(kx) + \sin(kx))^{1/k}$ **is decreasing in** $(0,1)$**.**

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