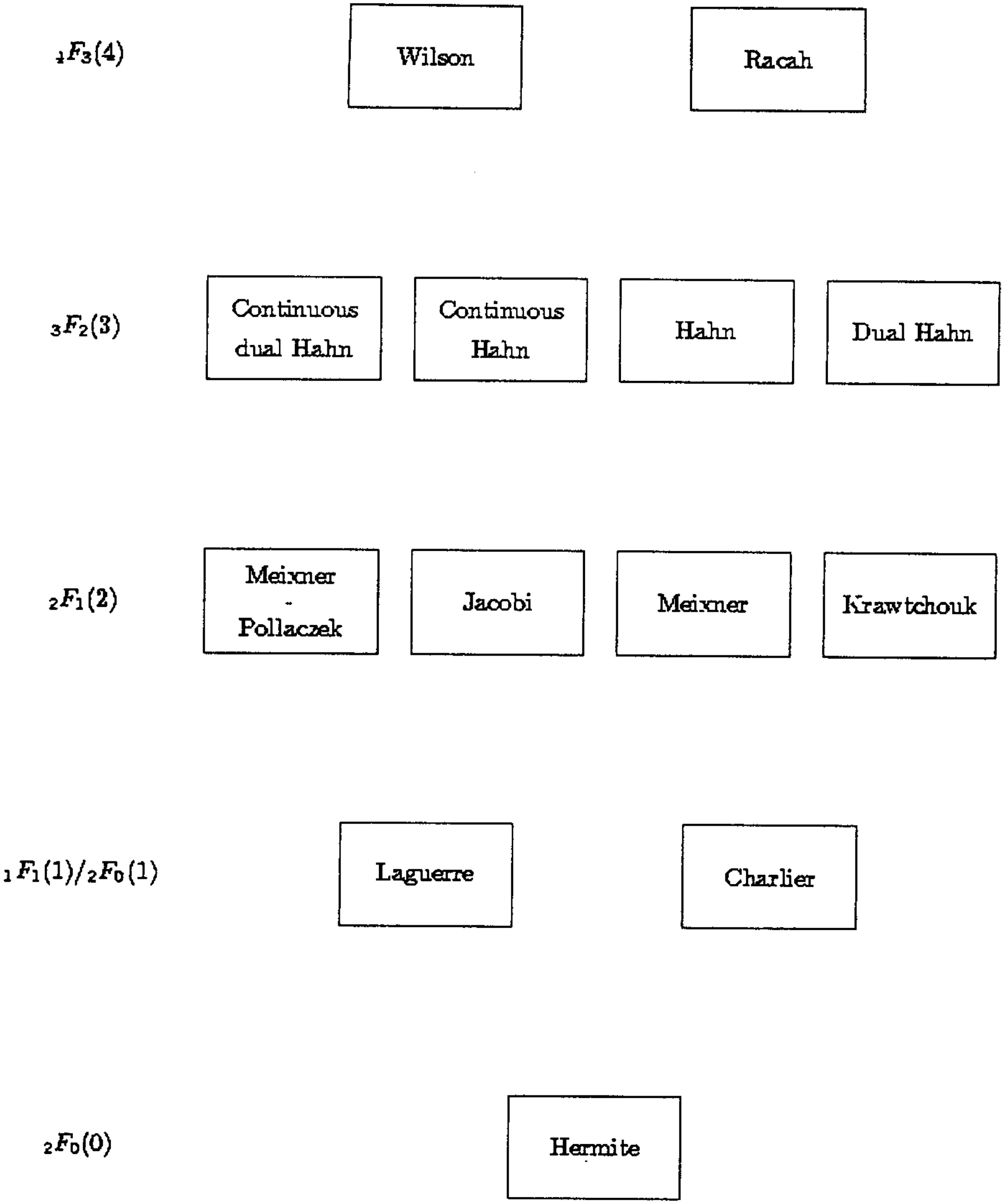


$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner - Pollaczek
Jacobi
Meixner
Krawtchouk
$_1F_1(1)/_2F_0(1)$
Laguerre
Charlier
$_2F_0(0)$
Hermite

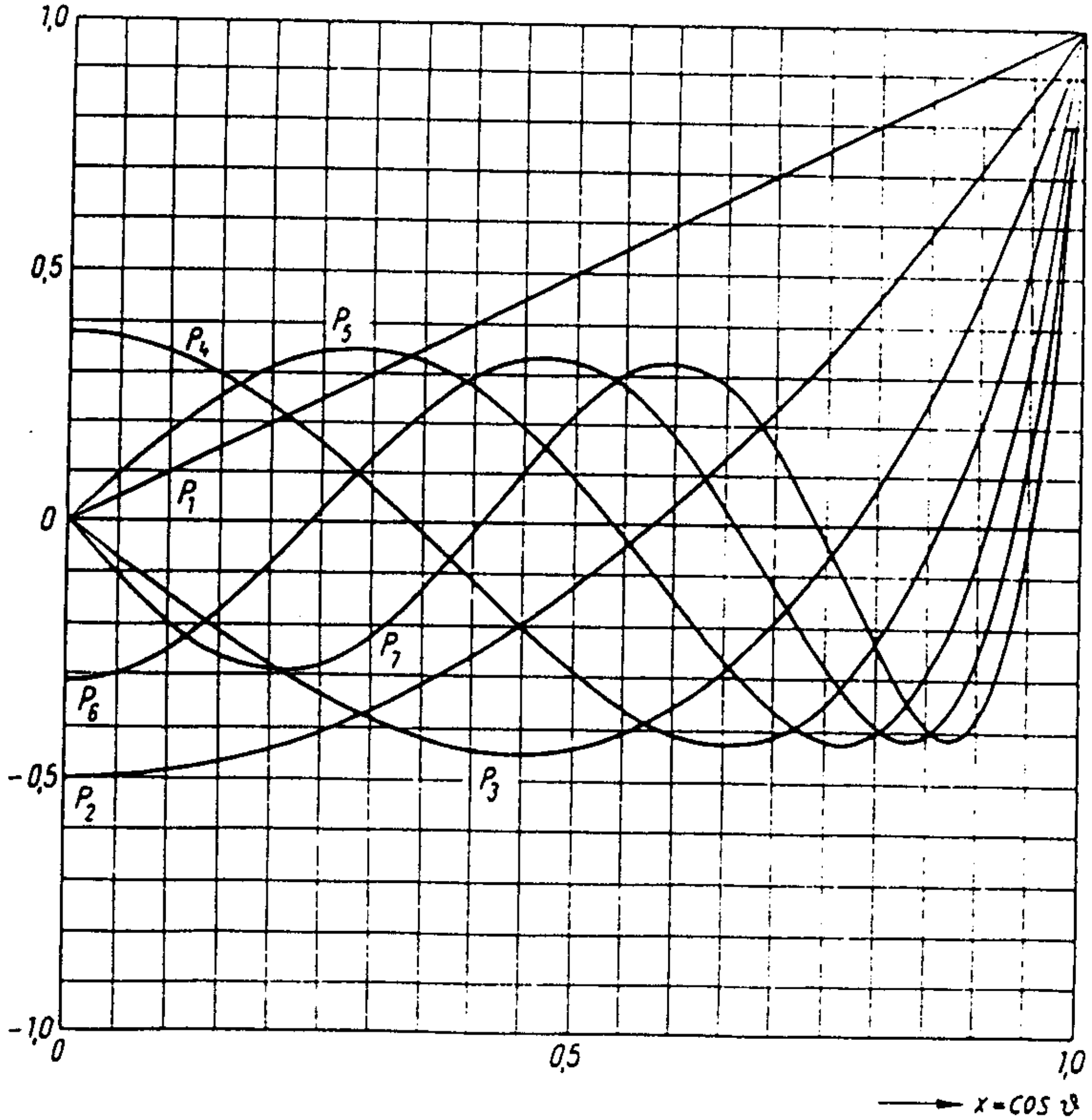


Fig. 69
Die Kugelfunktionen 1. Art $P_n(x)$

Fig. 69
Legendre functions of the 1st kind $P_n(x)$

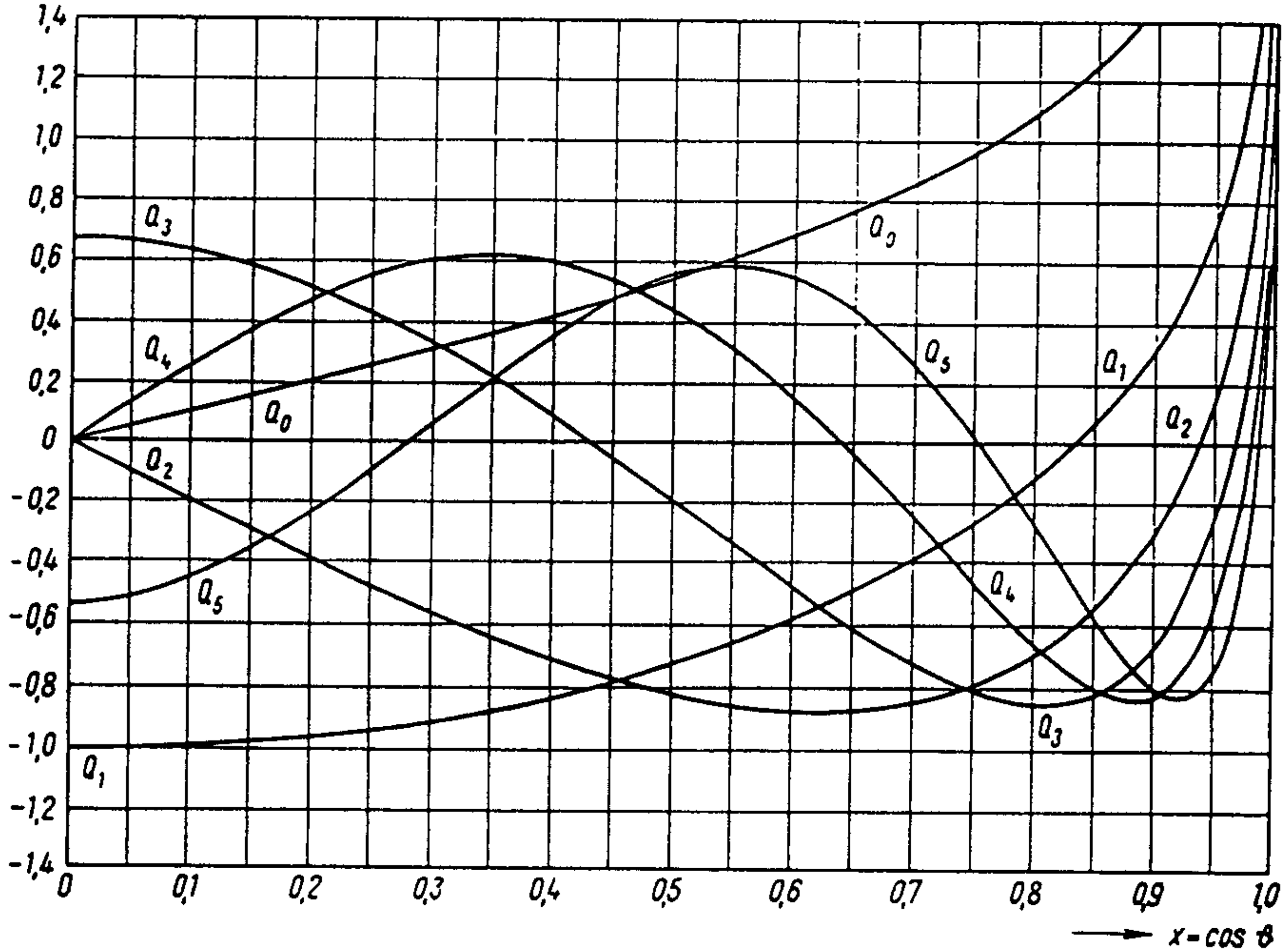


Fig. 71 Die Kugelfunktionen 2. Art $Q_n(x)$

Fig. 71 Legendre functions of the 2^{nd} kind $Q_n(x)$

22. Orthogonal Polynomials

Urs W. Hochstrasser [1]

Contents

22.17. Orthogonal Polynomials of a Discrete Variable

In this section some polynomials $f_n(x)$ are listed which are orthogonal with respect to the scalar product

22.17.1 $$(f_n, f_m)=\sum_i w^*(x_i) f_n(x_i) f_m(x_i).$$

The x_i are the integers in the interval $a \leq x_i \leq b$ and $w^*(x_i)$ is a positive function such that $\sum_i w^*(x_i)$ is finite. The constant factor which is still free in each polynomial when only the orthogonality condition is given is defined here by the explicit representation (which corresponds to the Rodrigues' formula)

22.17.2 $$f_n(x)=\frac{1}{r_n w^*(x)} \Delta^n[w^*(x) g(x, n)]$$

where $g(x, n)=g(x)g(x-1) \ldots g(x-n+1)$ and $g(x)$ is a polynomial in x independent of n.

Name	a	b	$w^*(x)$	r_n	$g(x, n)$	Remarks
Chebyshev	0	$N-1$	1	$1/n!$	$\binom{x}{n}\binom{x-N}{n}$	
Krawtchouk	0	N	$p^x q^{N-x}\binom{N}{x}$	$(-1)^n n!$	$\frac{q^n x!}{(x-n)!}$	$p, q>0;$ $p+q=1$
Charlier	0	∞	$\frac{e^{-a} a^x}{x!}$	$(-1)^n \sqrt{a^n n!}$	$\frac{x!}{(x-n)!}$	$a>0$
Meixner	0	∞	$\frac{c^x \Gamma(b+x)}{\Gamma(b) x!}$	c^n	$\frac{x!}{(x-n)!}$	$b>0,\ 0<c<1$
Hahn	0	~~∞~~	$\frac{\Gamma(b)\Gamma(c+x)\Gamma(d+x)}{x!\Gamma(b+x)\Gamma(c)\Gamma(d)}$	$n!$	$\frac{x!\Gamma(b+x)}{(x-n)!\Gamma(b+x-n)}$	

For a more complete list of the properties of these polynomials see [22.5] and [22.17].

Ch.5 Gamma Function

Properties

§5.11 Asymptotic Expansions §5.13 Integrals

§ 5.12. Beta Function

In this section all fractional powers have their principal values, except where noted otherwise. In (5.12.1)–(5.12.4) it is assumed $\Re a > 0$ and $\Re b > 0$.

¶ Euler's Beta Integral

5.12.1 $$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

5.12.2 $$\int_0^{\pi/2} \sin^{2a-1}\theta\cos^{2b-1}\theta\,d\theta = \tfrac{1}{2}B(a, b).$$

5.12.3 $$\int_0^{\infty} \frac{t^{a-1}\,dt}{(1+t)^{a+b}} = B(a, b).$$

5.12.4 $$\int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{(t+z)^{a+b}}\,dt = B(a, b)(1+z)^{-a}z^{-b},$$

$|\mathrm{ph}\, z| < \pi.$

5.12.5 $$\int_0^{\pi/2} (\cos t)^{a-1}\cos(bt)\,dt = \frac{\pi}{2^a}\frac{1}{aB\left(\frac{1}{2}(a+b+1), \frac{1}{2}(a-b+1)\right)},$$

$\Re a > 0.$

5.12.6 $$\int_0^{\pi} (\sin t)^{a-1}\,e^{ibt}\,dt = \frac{\pi}{2^{a-1}}\frac{e^{i\pi b/2}}{aB\left(\frac{1}{2}(a+b+1), \frac{1}{2}(a-b+1)\right)},$$

$\Re a > 0.$

5.12.7 $$\int_0^{\infty} \frac{\cosh(2bt)}{(\cosh t)^{2a}}\,dt = 4^{a-1}B(a+b, a-b),$$

$\Re a > |\Re b|.$

5.12.8 $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{(w+it)^a(z-it)^b} = \frac{(w+z)^{1-a-b}}{(a+b-1)B(a, b)},$$

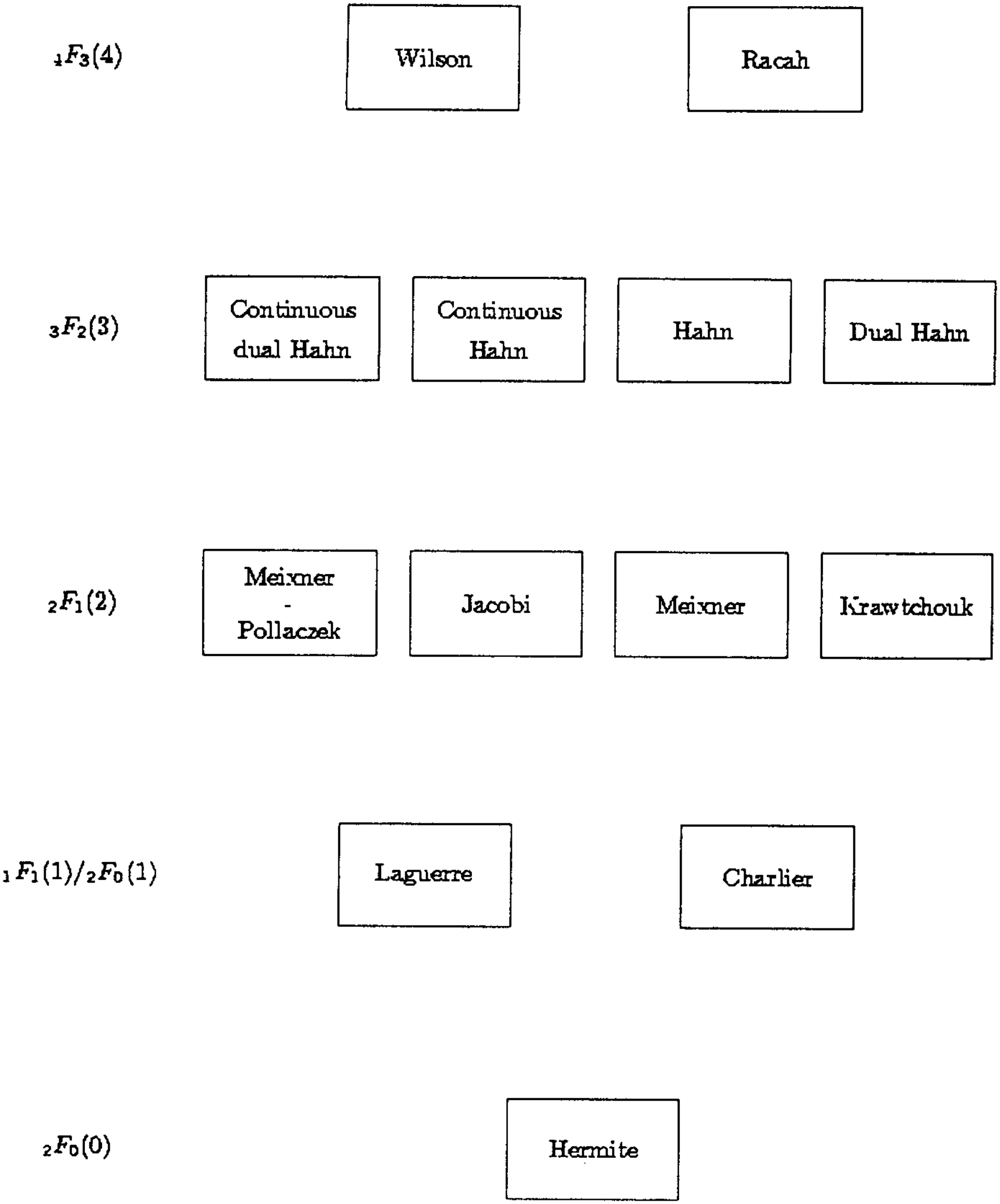
$\Re(a+b) > 1, \Re w > 0, \Re z > 0.$

In (5.12.8) the fractional powers have their principal values when $w > 0$ and $z > 0$, and are continued via continuity.

5.12.9 $$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} t^{-a}(1-t)^{-1-b}\,dt = \frac{1}{bB(a, b)},$$

$0 < c < 1, \Re(a+b) > 0.$

5.12.10 $$\frac{1}{2\pi i}\int_0^{(1+)} t^{a-1}(t-1)^{b-1}\,dt = \frac{\sin(\pi b)}{\pi}B(a, b),$$


$_4F_3(4)$
Wilson
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Krawtchouk
$_1F_1(1)/_2F_0(1)$
Laguerre
Charlier
$_2F_0(0)$
Hermite

Ch.5 Gamma Function
Properties

§5.12 Beta Function §5.14 Multidimensional Integrals

§ 5.13. Integrals

In (5.13.1) the integration path is a straight line parallel to the imaginary axis.

5.13.1
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s+a)\Gamma(b-s)z^{-s}\,ds=\frac{\Gamma(a+b)z^{a}}{(1+z)^{a+b}},$$
$$\Re(a+b)>0,\ -\Re a<c<\Re b,\ |\operatorname{ph}z|<\pi.$$

5.13.2
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Gamma(a+it)|^{2}e^{(2b-\pi)t}\,dt=\frac{\Gamma(2a)}{(2\sin b)^{2a}},$$
$$a>0,\ 0<b<\pi.$$

¶ Barnes' Beta Integral

5.13.3
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)\,dt=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$
$$\Re a,\Re b,\Re c,\Re d>0.$$

¶ Ramanujan's Beta Integral

5.13.4
$$\int_{-\infty}^{\infty}\frac{dt}{\Gamma(a+t)\Gamma(b+t)\Gamma(c-t)\Gamma(d-t)}=\frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)},$$
$$\Re(a+b+c+d)>3.$$

¶ de Branges–Wilson Beta Integral

5.13.5
$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\prod_{k=1}^{4}\Gamma(a_k+it)\Gamma(a_k-it)}{\Gamma(2it)\Gamma(-2it)}\,dt=\frac{\prod_{1\le j<k\le 4}\Gamma(a_j+a_k)}{\Gamma(a_1+a_2+a_3+a_4)},$$
$$\Re(a_k)>0,\ k=1,2,3,4.$$

For compendia of integrals of gamma functions see Apelblat (1983, pp. 124–127 and 129–130), Erdélyi *et al.* (1954a, b), Gradshteyn and Ryzhik (2000, pp. 644–652), Oberhettinger (1974, pp. 191–204), Oberhettinger and Badii (1973, pp. 307–316), Prudnikov *et al.* (1986b, pp. 57–64), Prudnikov *et al.* (1992a, pp. 127–130), and Prudnikov *et al.* (1992b, pp. 113–123).

§5.12 Beta Function

§5.14 Multidimensional Integrals

§ 5.2(iii). Pochhammer's Symbol

5.2.4
$$(a)_0 = 1,$$
$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$

5.2.5
$$(a)_n = \Gamma(a+n)/\Gamma(a),$$
$$a \neq 0, -1, -2, \ldots.$$

16.2.1
$${}_pF_q\left({a_1, \ldots, a_p \atop b_1, \ldots, b_q}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}.$$

For the definition of hypergeometric and generalized hypergeometric functions see §16.2.

18.20.5
$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left({-n, n+\alpha+\beta+1, -x \atop \alpha+1, -N}; 1\right),$$
$$n = 0, 1, \ldots, N.$$

18.20.6
$$K_n(x; p, N) = {}_2F_1\left({-n, -x \atop -N}; p^{-1}\right),$$
$$n = 0, 1, \ldots, N.$$

18.20.7
$$M_n(x; \beta, c) = {}_2F_1\left({-n, -x \atop \beta}; 1 - c^{-1}\right).$$

18.20.8
$$C_n(x, a) = {}_2F_0\left({-n, -x \atop -}; -a^{-1}\right).$$

18.20.9
$$p_n(x; a, b, \overline{a}, \overline{b}) = \frac{i^n (a+\overline{a})_n (a+\overline{b})_n}{n!} {}_3F_2\left({-n, n + 2\Re(a+b) - 1, a + ix \atop a+\overline{a}, a+\overline{b}}; 1\right).$$

(For symmetry properties of $p_n(x; a, b, \overline{a}, \overline{b})$ with respect to $a, b, \overline{a}, \overline{b}$ see Andrews *et al.* (1999, Corollary 3.3.4).)

18.20.10
$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left({-n, \lambda + ix \atop 2\lambda}; 1 - e^{-2i\phi}\right).$$

Ch.18 Orthogonal Polynomials
Askey Scheme

§18.18 Sums §18.20 Hahn Class: Explicit Representations

§ 18.19. Hahn Class: Definitions

¶ Hahn, Krawtchouk, Meixner, and Charlier

Tables 18.19.1 and 18.19.2 provide definitions via orthogonality and normalization (§§18.2(i), 18.2(iii)) for the Hahn polynomials $Q_n(x;\alpha,\beta,N)$, Krawtchouk polynomials $K_n(x;p,N)$, Meixner polynomials $M_n(x;\beta,c)$, and Charlier polynomials $C_n(x,a)$.

Table 18.19.1. Orthogonality properties for Hahn, Krawtchouk, Meixner, and Charlier OP's: discrete sets, weight functions, normalizations, and parameter constraints.

$p_n(x)$	X	w_x	h_n
$Q_n(x;\alpha,\beta,N)$, $n=0,1,\dots,N$	$\{0,1,\dots,N\}$	$\dfrac{(\alpha+1)_x(\beta+1)_{N-x}}{x!(N-x)!}$, $\alpha,\beta>-1$ or $\alpha,\beta<-N$	$\dfrac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}$ If $\alpha,\beta<-N$, then $(-1)^N w_x>0$ and $(-1)^N h_n>0$.
$K_n(x;p,N)$, $n=0,1,\dots,N$	$\{0,1,\dots,N\}$	$\binom{N}{x}p^x(1-p)^{N-x}$, $0<p<1$	$\left(\dfrac{1-p}{p}\right)^n / \binom{N}{n}$
$M_n(x;\beta,c)$	$\{0,1,2,\dots\}$	$(\beta)_x c^x/x!$, $\beta>0, 0<c<1$	$\dfrac{c^{-n}n!}{(\beta)_n(1-c)^\beta}$
$C_n(x,a)$	$\{0,1,2,\dots\}$	$a^x/x!, a>0$	$a^{-n}e^a n!$

Table 18.19.2. Hahn, Krawtchouk, Meixner, and Charlier OP's: leading coefficients.

$p_n(x)$	k_n
$Q_n(x;\alpha,\beta,N)$	$\dfrac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N)_n}$
$K_n(x;p,N)$	$p^{-n}/(-N)_n$
$M_n(x;\beta,c)$	$(1-c^{-1})^n/(\beta)_n$
$C_n(x,a)$	$(-a)^{-n}$

¶ Continuous Hahn

These polynomials are orthogonal on $(-\infty,\infty)$, and with $\Re a>0$, $\Re b>0$ are defined as follows.

18.19.1 $$p_n(x)=p_n(x;a,b,\bar{a},\bar{b}),$$

18.19.2 $$w(z;a,b,\bar{a},\bar{b})=\Gamma(a+iz)\Gamma(b+iz)\Gamma(\bar{a}-iz)\Gamma(\bar{b}-iz),$$

18.19.3 $$w(x) = w(x; a, b, \bar{a}, \bar{b}) = |\Gamma(a + i x)\Gamma(b + i x)|^2,$$

18.19.4 $$h_n = \frac{2\pi\Gamma(n + a + \bar{a})\Gamma(n + b + \bar{b})|\Gamma(n + a + \bar{b})|^2}{(2n + 2\Re(a + b) - 1)\Gamma(n + 2\Re(a + b) - 1)n!},$$

18.19.5 $$k_n = \frac{(n + 2\Re(a + b) - 1)_n}{n!}.$$

¶ Meixner–Pollaczek

These polynomials are orthogonal on $(-\infty, \infty)$, and are defined as follows.

18.19.6 $$p_n(x) = P_n^{(\lambda)}(x; \phi),$$

18.19.7 $$w^{(\lambda)}(z; \phi) = \Gamma(\lambda + i z)\Gamma(\lambda - i z)\, e^{(2\phi - \pi)z},$$

18.19.8 $$w(x) = w^{(\lambda)}(x; \phi) = |\Gamma(\lambda + i x)|^2\, e^{(2\phi - \pi)x},$$

$$\lambda > 0,\ 0 < \phi < \pi,$$

18.19.9 $$h_n = \frac{2\pi\Gamma(n + 2\lambda)}{(2\sin\phi)^{2\lambda} n!}, \quad k_n = \frac{(2\sin\phi)^n}{n!}.$$

j D

§ 34.2. Definition: 3j Symbol

The quantities j_1, j_2, j_3 in the 3j symbol are called *angular momenta*. Either all of them are nonnegative integers, or one is a nonnegative integer and the other two are half-odd positive integers. They must form the sides of a triangle (possibly degenerate). They therefore satisfy the *triangle conditions*

34.2.1 $$|j_r - j_s| \le j_t \le j_r + j_s,$$

where r, s, t is any permutation of 1, 2, 3. The corresponding *projective quantum numbers* m_1, m_2, m_3 are given by

34.2.2 $$m_r = -j_r, -j_r + 1, \dots, j_r - 1, j_r, \qquad r = 1, 2, 3,$$

and satisfy

34.2.3 $$m_1 + m_2 + m_3 = 0.$$

See Figure 34.2.1 for a schematic representation.

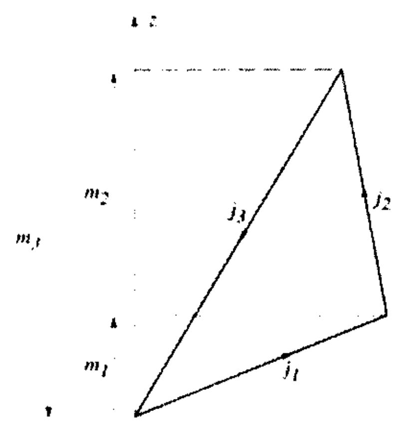


Figure 34.2.1. Angular momenta j_r and projective quantum numbers m_r, $r = 1, 2, 3$.

If either of the conditions (34.2.1) or (34.2.3) is not satisfied, then the 3j symbol is zero. When both conditions are satisfied the 3j symbol can be expressed as the finite sum

34.2.4 $$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3} \Delta(j_1 j_2 j_3) \left((j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j_3 + m_3)!(j_3 - m_3)! \right)^{\frac{1}{2}} \times \sum_s \frac{(-1)^s}{s!(j_1 + j_2 - j_3 - s)!(j_1 - m_1 - s)!(j_2 + m_2 - s)!(j_3 - j_2 + m_1 + s)!(j_3 - j_1 - m_2 + s)!},$$

where

34.2.5 $$\Delta(j_1 j_2 j_3) = \left(\frac{(j_1 + j_2 - j_3)!(j_1 - j_2 + j_3)!(-j_1 + j_2 + j_3)!}{(j_1 + j_2 + j_3 + 1)!} \right)^{\frac{1}{2}},$$

and the summation is over all nonnegative integers s such that the arguments in the factorials are nonnegative.

Equivalently,

34.2.6
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-m_1+m_3} \frac{(j_1+j_2+m_3)!(j_2+j_3-m_1)!}{\Delta(j_1j_2j_3)(j_1+j_2+j_3+1)!} \left(\frac{(j_1+m_1)!(j_3-m_3)!}{(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!}\right)^{\frac{1}{2}} \times {}_3F_2(-j_1-j_2-j_3-1, -j_1+m_1, -j_3-m_3; -j_1-j_2-m_3, -j_2-j_3+m_1; 1),$$

where ${}_3F_2$ is defined as in §16.2.

For alternative expressions for the 3j symbol, written either as a finite sum or as other terminating generalized hypergeometric series ${}_3F_2$ of unit argument, see Varshalovich *et al.* (1988, §§8.21, 8.24–8.26).

§ 5.13. Integrals

In (5.13.1) the integration path is a straight line parallel to the imaginary axis.

5.13.1
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s+a)\Gamma(b-s)z^{-s}\,ds=\frac{\Gamma(a+b)z^{a}}{(1+z)^{a+b}},$$
$\Re(a+b)>0$, $-\Re a<c<\Re b$, $|\operatorname{ph} z|<\pi$.

5.13.2
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Gamma(a+it)|^{2}\,e^{(2b-\pi)t}\,dt=\frac{\Gamma(2a)}{(2\sin b)^{2a}},$$
$a>0$, $0<b<\pi$.

¶ Barnes' Beta Integral

5.13.3
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)\,dt=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$
$\Re a, \Re b, \Re c, \Re d>0$.

¶ Ramanujan's Beta Integral

5.13.4
$$\int_{-\infty}^{\infty}\frac{dt}{\Gamma(a+t)\Gamma(b+t)\Gamma(c-t)\Gamma(d-t)}=\frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)},$$
$\Re(a+b+c+d)>3$.

¶ de Branges–Wilson Beta Integral

5.13.5
$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\prod_{k=1}^{4}\Gamma(a_k+it)\Gamma(a_k-it)}{\Gamma(2it)\Gamma(-2it)}\,dt=\frac{\prod_{1\le j<k\le 4}\Gamma(a_j+a_k)}{\Gamma(a_1+a_2+a_3+a_4)},$$
$\Re(a_k)>0$, $k=1,2,3,4$.

For compendia of integrals of gamma functions see Apelblat (1983, pp. 124–127 and 129–130), Erdélyi *et al.* (1954a, b), Gradshteyn and Ryzhik (2000, pp. 644–652), Oberhettinger (1974, pp. 191–204), Oberhettinger and Badii (1973, pp. 307–316), Prudnikov *et al.* (1986b, pp. 57–64), Prudnikov *et al.* (1992a, pp. 127–130), and Prudnikov *et al.* (1992b, pp. 113–123).

§ 34.4. Definition: 6j Symbol

The 6j symbol is defined by the following double sum of products of 3j symbols:

34.4.1
$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \sum_{m_r m'_s} (-1)^{l_1+m'_1+l_2+m'_2+l_3+m'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & m'_2 & -m'_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -m'_1 & m_2 & m'_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ m'_1 & -m \end{pmatrix}$$

where the summation is taken over all admissible values of the m's and m''s for each of the four 3j symbols; compare (34.2.2) and (34.2.3).

Except in degenerate cases the combination of the triangle inequalities for the four 3j symbols in (34.4.1) is equivalent to the existence of a tetrahedron (possibly degenerate) with edges of lengths $j_1, j_2, j_3, l_1, l_2, l_3$; see Figure 34.4.1.

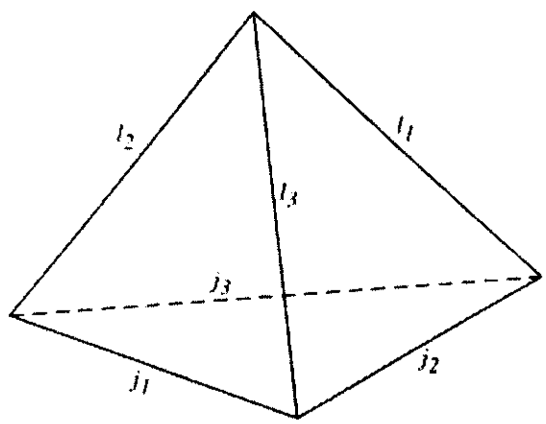


Figure 34.4.1. Tetrahedron corresponding to 6j symbol.

The 6j symbol can be expressed as the finite sum

34.4.2
$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \sum_s \frac{(-1)^s (s+1)!}{(s-j_1-j_2-j_3)!(s-j_1-l_2-l_3)!(s-l_1-j_2-l_3)!(s-l_1-l_2-j_3)!} \times \frac{1}{(j_1+j_2+l_1+l_2-s)!(j_2+j_3+l_2+l_3-s)!(j_3+j_1+l_3+l_1-s)!},$$

where the summation is over all nonnegative integers s such that the arguments in the factorials are nonnegative.

Equivalently,

34.4.3
$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+j_3+l_1+l_3} \times \frac{\Delta(j_1 j_2 j_3)\Delta(j_2 l_1 l_3)(j_1 - j_2 + l_1 + l_2)!(-j_2 + j_3 + l_2 + l_3)!(j_1 + j_3 + l_1 + l_3 + 1)!}{\Delta(j_1 l_2 l_3)\Delta(j_3 l_1 l_2)(j_1 - j_2 + j_3)!(-j_2 + l_1 + l_3)!(j_1 + l_2 + l_3 + 1)!(j_3 + l_1 + l_2 + 1)!} \times {}_4F_3\left({-j_1 + j_2 - j_3,\ j_2 - l_1 - l_3,\ -j_1 - l_2 - l_3 - 1,\ -j_3 - l_1 - l_2 - 1 \atop -j_1 + j_2 - l_1 - l_2,\ j_2 - j_3 - l_2 - l_3,\ -j_1 - j_3 - l_1 - l_3 - 1};1\right),$$

where ${}_4F_3$ is defined as in §16.2.

For alternative expressions for the 6j symbol, written either as a finite sum or as other terminating generalized hypergeometric series ${}_4F_3$ of unit argument, see Varshalovich *et al.* (1988, §§9.2.1, 9.2.3).

34.5.10
$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(j_2+l_2+j_3-l_3) & \frac{1}{2}(j_1-l_1+j_3+l_3) & \frac{1}{2}(j_1+l_1+j_2-l_2) \\ \frac{1}{2}(j_2+l_2-j_3+l_3) & \frac{1}{2}(-j_1+l_1+j_3+l_3) & \frac{1}{2}(j_1+l_1-j_2+l_2) \end{Bmatrix}.$$

Equations (34.5.9) and (34.5.10) are called *Regge symmetries*. Additional symmetries are obtained by applying (34.5.8) to (34.5.9) and (34.5.10). See Srinivasa Rao and Rajeswari (1993, pp. 102–103) and references given there.

§ 34.5(iii). Recursion Relations

In the following equation it is assumed that the triangle conditions are satisfied.

34.5.11
$$(2j_1+1)((J_3+J_2-J_1)(L_3+L_2-J_1)-2(J_3L_3+J_2L_2-J_1L_1))\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$
$$= j_1E(j_1+1)\begin{Bmatrix} j_1+1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} + (j_1+1)E(j_1)\begin{Bmatrix} j_1-1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix},$$

where

34.5.12
$$J_r = j_r(j_r+1), \qquad L_r = l_r(l_r+1),$$

34.5.13
$$E(j) = ((j^2-(j_2-j_3)^2)((j_2+j_3+1)^2-j^2)(j^2-(l_2-l_3)^2)((l_2+l_3+1)^2-j^2))^{\frac{1}{2}}.$$

For further recursion relations see Varshalovich *et al.* (1988, §9.6) and Edmonds (1974, pp. 98–99).

§ 34.5(iv). Orthogonality

34.5.14
$$\sum_{j_3}(2j_3+1)(2l_3+1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3' \end{Bmatrix} = \delta_{l_3,l_3'}.$$

§ 34.5(v). Generating Functions

For generating functions for the 6j symbol see Biedenharn and van Dam (1965, p. 255, eq. (4.18)).

§ 34.5(vi). Sums

34.5.15
$$\sum_j (-1)^{j+j'+j''}(2j+1)\begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j \\ j_4 & j_3 & j'' \end{Bmatrix} = \begin{Bmatrix} j_1 & j_4 & j' \\ j_2 & j_3 & j'' \end{Bmatrix},$$

34.5.16
$$(-1)^{j_1+j_2+j_3+j_1'+j_2'+l_1+l_2}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} j_1' & j_2' & j_3 \\ l_1 & l_2 & l_3' \end{Bmatrix} = \sum_j (-1)^{l_3+l_3'+j}(2j+1)\begin{Bmatrix} j_1 & j_1' & j \\ j_2' & j_2 & j_3 \end{Bmatrix}\begin{Bmatrix} l_3 & l_3' & j \\ j_1' & j_1 & l_2 \end{Bmatrix}\begin{Bmatrix} l_3 & l_3' & j \\ j_2' & j_2 & l_1 \end{Bmatrix}.$$

Equations (34.5.15) and (34.5.16) are the *sum rules*. They constitute addition theorems for the 6j symbol.

34.5.17
$$\sum_j (2j+1)\begin{Bmatrix} j_1 & j_2 & j \\ j_1 & j_2 & j' \end{Bmatrix} = (-1)^{2(j_1+j_2)},$$

34.5.18
$$\sum_j (-1)^{j_1+j_2+j}(2j+1)\begin{Bmatrix} j_1 & j_2 & j \\ j_2 & j_1 & j' \end{Bmatrix} = \sqrt{(2j_1+1)(2j_2+1)}\,\delta_{j',0},$$

34.5.19
$$\sum_l \begin{Bmatrix} j_1 & j_2 & l \\ j_2 & j_1 & j \end{Bmatrix} = 0,$$

$$2\mu - j \text{ odd}, \quad \mu = \min(j_1, j_2),$$

Ch.18 Orthogonal Polynomials
Askey Scheme

§18.24 Hahn Class: Asymptotic Approximations §18.26 Wilson Class: Continued

§ 18.25. Wilson Class: Definitions

Contents

§ 18.25(i). Preliminaries

For the Wilson class OP's $p_n(x)$ with $x = \lambda(y)$: if the y-orthogonality set is $\{0, 1, \dots, N\}$, then the role of the differentiation operator d/dx in the Jacobi, Laguerre, and Hermite cases is played by the operator Δ_y followed by division by $\Delta_y(\lambda(y))$, or by the operator ∇_y followed by division by $\nabla_y(\lambda(y))$. Alternatively if the y-orthogonality interval is $(0, \infty)$, then the role of d/dx is played by the operator δ_y followed by division by $\delta_y(\lambda(y))$.

Table 18.25.1 lists the transformations of variable, orthogonality ranges, and parameter constraints that are needed in §18.2(i) for the Wilson polynomials $W_n(x; a, b, c, d)$, continuous dual Hahn polynomials $S_n(x; a, b, c)$, Racah polynomials $R_n(x; \alpha, \beta, \gamma, \delta)$, and dual Hahn polynomials $R_n(x; \gamma, \delta, N)$.

Table 18.25.1. Wilson class OP's: transformations of variable, orthogonality ranges, and parameter constraints.

$p_n(x)$	$x = \lambda(y)$	Orthogonality range for y	Constraints
$W_n(x; a, b, c, d)$	y^2	$(0, \infty)$	$\Re(a, b, c, d) > 0$; nonreal parameters in conjugate pairs
$S_n(x; a, b, c)$	y^2	$(0, \infty)$	$\Re(a, b, c) > 0$; nonreal parameters in conjugate pairs
$R_n(x; \alpha, \beta, \gamma, \delta)$	$y(y+\gamma+\delta+1)$	$\{0, 1, \dots, N\}$	$\alpha+1$ or $\beta+\delta+1$ or $\gamma+1 = -N$; for further constraints see (18.25.1)
$R_n(x; \gamma, \delta, N)$	$y(y+\gamma+\delta+1)$	$\{0, 1, \dots, N\}$	$\gamma, \delta > -1$ or $< -N$

¶ Further Constraints for Racah Polynomials

If $\alpha + 1 = -N$, then the weights will be positive iff one of the following eight sets of inequalities holds:

18.25.1
$$-\delta - 1 < \beta < \gamma + 1 < -N + 1. \quad N - 1 < -\delta - 1$$

$< \beta$
$< \gamma + 1.$
$\gamma, \delta > -1, \beta$
$> N + \gamma.$
$\gamma, \delta > -1, \beta$
$< -N - \delta.$
$N - 1 < N + \gamma$
$< \beta$
$< -N - \delta.$
$N + \gamma < \beta$
$< -N - \delta$
$< -N - 1.$
$\gamma, \delta < -N, \beta$
$> -1 - \delta.$
$\gamma, \delta < -N, \beta$
$< \gamma + 1.$

The first four sets imply $\gamma + \delta > -2$, and the last four imply $\gamma + \delta < -2N$.

§ 18.25(ii). Weights and Normalizations: Continuous Cases

18.25.2 $$\int_0^\infty p_n(x) p_m(x) w(x) dx = h_n \delta_{n,m}.$$

¶ Wilson

18.25.3 $$p_n(x) = W_n(x; a_1, a_2, a_3, a_4),$$

18.25.4 $$w(y^2) = \frac{1}{2y} \left| \frac{\prod_j \Gamma(a_j + i y)}{\Gamma(2 i y)} \right|^2,$$

18.25.5 $$h_n = \frac{n!\, 2\pi \prod_{j<\ell} \Gamma(n + a_j + a_\ell)}{\left(2n - 1 + \sum_j a_j\right) \Gamma\left(n - 1 + \sum_j a_j\right)}.$$

¶ Continuous Dual Hahn

18.25.6 $$p_n(x) = S_n(x; a_1, a_2, a_3),$$

18.25.7 $$w(y^2) = \frac{1}{2y} \left| \frac{\prod_j \Gamma(a_j + i y)}{\Gamma(2 i y)} \right|^2,$$

18.25.8 $$h_n = n!\, 2\pi \prod_{j<\ell} \Gamma(n + a_j + a_\ell).$$

§ 18.25(iii). Weights and Normalizations: Discrete Cases

18.25.9 $$\sum_{y=0}^{N} p_n(y(y + \gamma + \delta + 1)) p_m(y(y + \gamma + \delta + 1)) \frac{\gamma + \delta + 1 + 2y}{\gamma + \delta + 1 + y} \omega_y = h_n \delta_{n,m}.$$

¶ Racah

18.25.10 $$p_n(x) = R_n(x; \alpha, \beta, \gamma, \delta),$$

$$\alpha + 1 = -N,$$

18.25.11 $$\omega_y = \frac{(\alpha+1)_y(\beta+\delta+1)_y(\gamma+1)_y(\gamma+\delta+2)_y}{(-\alpha+\gamma+\delta+1)_y(-\beta+\gamma+1)_y(\delta+1)_y y!},$$

18.25.12 $$h_n = \frac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N}\frac{(n+\alpha+\beta+1)_n n!}{(\alpha+\beta+2)_{2n}}\frac{(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n(\beta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}.$$

¶ Dual Hahn

18.25.13 $$p_n(x) = R_n(x; \gamma, \delta, N),$$

18.25.14 $$\omega_y = \frac{(-1)^y(-N)_y(\gamma+1)_y(\gamma+\delta+1)_2}{(N+\gamma+\delta+2)_y(\delta+1)_y y!},$$

18.25.15 $$h_n = \frac{n!(N-n)!(\gamma+\delta+2)_N}{N!(\gamma+1)_n(\delta+1)_{N-n}}.$$

§ 18.25(iv). Leading Coefficients

Table 18.25.2 provides the leading coefficients k_n (§18.2(iii)) for the Wilson, continuous dual Hahn, Racah, and dual Hahn polynomials.

Table 18.25.2. Wilson class OP's: leading coefficients.

$p_n(x)$	k_n
$W_n(x; a, b, c, d)$	$(-1)^n(n+a+b+c+d-1)_n$
$S_n(x; a, b, c)$	$(-1)^n$
$R_n(x; \alpha, \beta, \gamma, \delta)$	$\frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}$
$R_n(x; \gamma, \delta, N)$	$\frac{1}{(\gamma+1)_n(-N)_n}$

§18.24 Hahn Class: Asymptotic Approximations §18.26 Wilson Class: Continued

Ch.18 Orthogonal Polynomials
Other Orthogonal Polynomials

§18.34 Bessel Polynomials §18.36 Miscellaneous Polynomials

§ 18.35. Pollaczek Polynomials

Contents

§ 18.35(i). Definition and Hypergeometric Representation

18.35.1
$$P^{(\lambda)}_{-1}(x;a,b)=0,$$
$$P^{(\lambda)}_{0}(x;a,b)=1,$$

and

18.35.2
$$(n+1)P^{(\lambda)}_{n+1}(x;a,b)=2((n+\lambda+a)x+b)P^{(\lambda)}_{n}(x;a,b)-(n+2\lambda-1)P^{(\lambda)}_{n-1}(x;a,b),$$
$$n=0,1,\dots.$$

Next, let

18.35.3
$$\tau_{a,b}(\theta)=\frac{a\cos\theta+b}{\sin\theta},$$
$$0<\theta<\pi.$$

Then

18.35.4
$$P^{(\lambda)}_{n}(\cos\theta;a,b)=\frac{(\lambda-i\tau_{a,b}(\theta))_n}{n!}e^{in\theta}\,{}_2F_1\left({-n,\lambda+i\tau_{a,b}(\theta)\atop -n-\lambda+1+i\tau_{a,b}(\theta)};e^{-2i\theta}\right)$$
$$=\sum_{\ell=0}^{n}\frac{(\lambda+i\tau_{a,b}(\theta))_\ell}{\ell!}\frac{(\lambda-i\tau_{a,b}(\theta))_{n-\ell}}{(n-\ell)!}e^{i(n-2\ell)\theta}.$$

For the hypergeometric function ${}_2F_1$ see §§15.1, 15.2(i).

§ 18.35(ii). Orthogonality

18.35.5
$$\int_{-1}^{1}P^{(\lambda)}_{n}(x;a,b)P^{(\lambda)}_{m}(x;a,b)w^{(\lambda)}(x;a,b)\,dx=0,$$
$$n\neq m,$$

where

18.35.6
$$w^{(\lambda)}(\cos\theta;a,b)=\pi^{-1}2^{2\lambda-1}e^{(2\theta-\pi)\tau_{a,b}(\theta)}(\sin\theta)^{2\lambda-1}|\Gamma(\lambda+i\tau_{a,b}(\theta))|^2,$$

§ 5.18. q-Gamma and Beta Functions

Contents

§ 5.18(i). q-Factorials

5.18.1
$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$
$$n = 0, 1, 2, \ldots,$$

5.18.2
$$n!_q = 1(1+q)\cdots(1+q+\ldots+q^{n-1}) = (q;q)_n(1-q)^{-n}.$$

When $|q| < 1$,

5.18.3
$$(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

See also §17.2(i).

§ 5.18(ii). q-Gamma Function

When $0 < q < 1$,

5.18.4
$$\Gamma_q(z) = (q;q)_\infty (1-q)^{1-z} / (q^z;q)_\infty,$$

5.18.5
$$\Gamma_q(1) = \Gamma_q(2) = 1,$$

5.18.6
$$n!_q = \Gamma_q(n+1),$$

5.18.7
$$\Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z).$$

Also, $\ln \Gamma_q(x)$ is convex for $x > 0$, and the analog of the Bohr-Mollerup theorem (§5.5(iv)) holds.

If $0 < q < r < 1$, then

5.18.8
$$\Gamma_q(x) < \Gamma_r(x),$$

when $0 < x < 1$ or when $x > 2$, and

5.18.9
$$\Gamma_q(x) > \Gamma_r(x),$$

when $1 < x < 2$.

5.18.10 $$\lim_{q\to 1-} \Gamma_q(z) = \Gamma(z).$$

For generalized asymptotic expansions of $\ln \Gamma_q(z)$ as $|z| \to \infty$ see Olde Daalhuis (1994) and Moak (1984).

§ 5.18(iii). q-Beta Function

5.18.11 $$\mathrm{B}_q(a, b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}.$$

5.18.12 $$\mathrm{B}_q(a, b) = \int_0^1 \frac{t^{a-1}(tq; q)_\infty}{(tq^b; q)_\infty} d_q t,$$

$$0 < q < 1, \Re a > 0, \Re b > 0.$$

For q-integrals see §17.2(v).

§ 17.13. Integrals

In this section, for the function Γ_q see §5.18(ii).

17.13.1 $$\int_{-c}^{d} \frac{(-qx/c;q)_\infty (qx/d;q)_\infty}{(-ax/c;q)_\infty (bx/d;q)_\infty} d_q x = \frac{(1-q)(q;q)_\infty (ab;q)_\infty cd(-c/d;q)_\infty (-d/c;q)_\infty}{(a;q)_\infty (b;q)_\infty (c+d)(-bc/d;q)_\infty (-ad/c;q)_\infty},$$

or, when $0 < q < 1$,

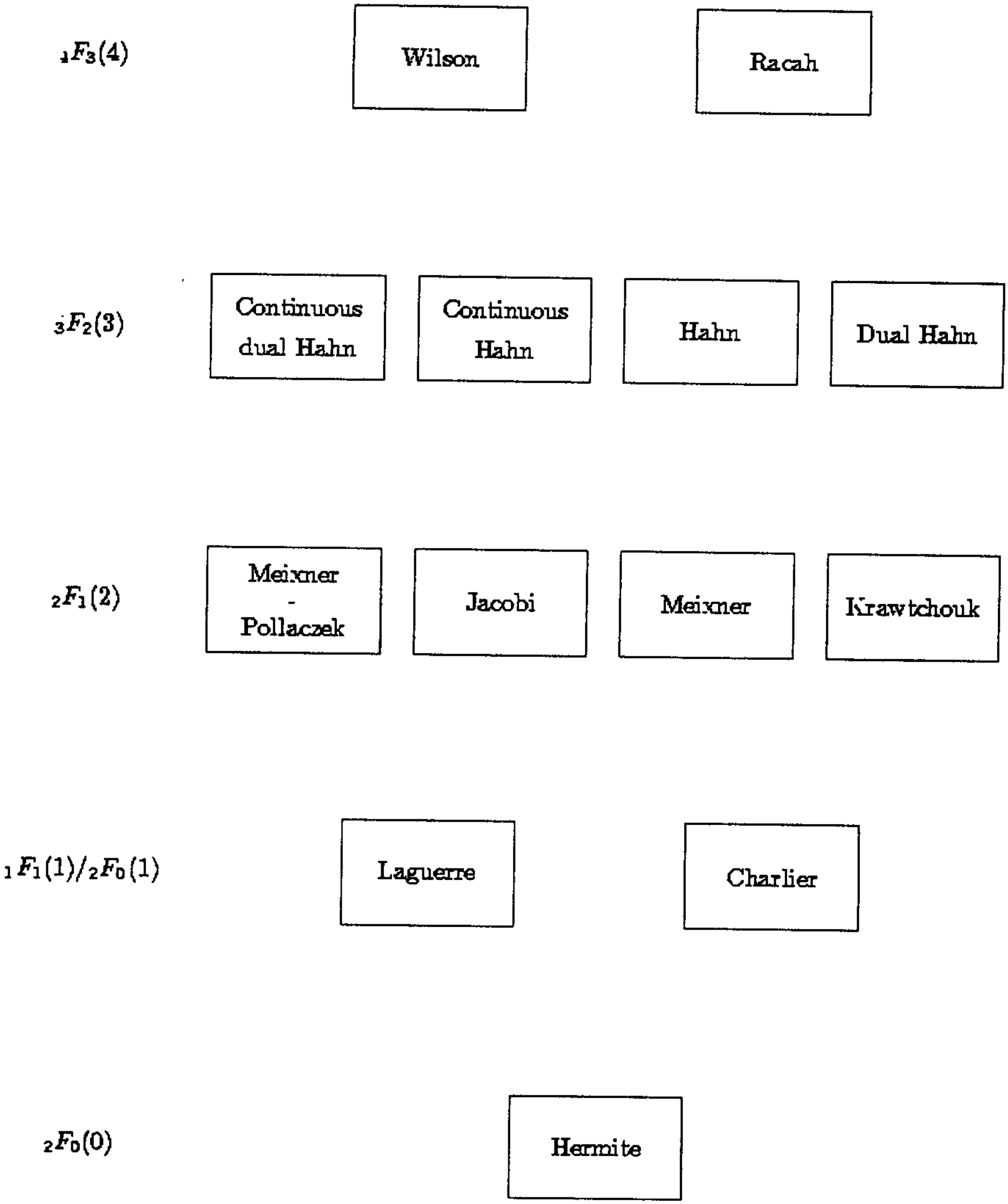
17.13.2 $$\int_{-c}^{d} \frac{(-qx/c;q)_\infty (qx/d;q)_\infty}{(-xq^\alpha/c;q)_\infty (xq^\beta/d;q)_\infty} d_q x = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \frac{(-c/d;q)_\infty (-d/c;q)_\infty}{(-q^\beta c/d;q)_\infty (-q^\alpha d/c;q)_\infty}.$$

¶ Ramanujan's Integrals

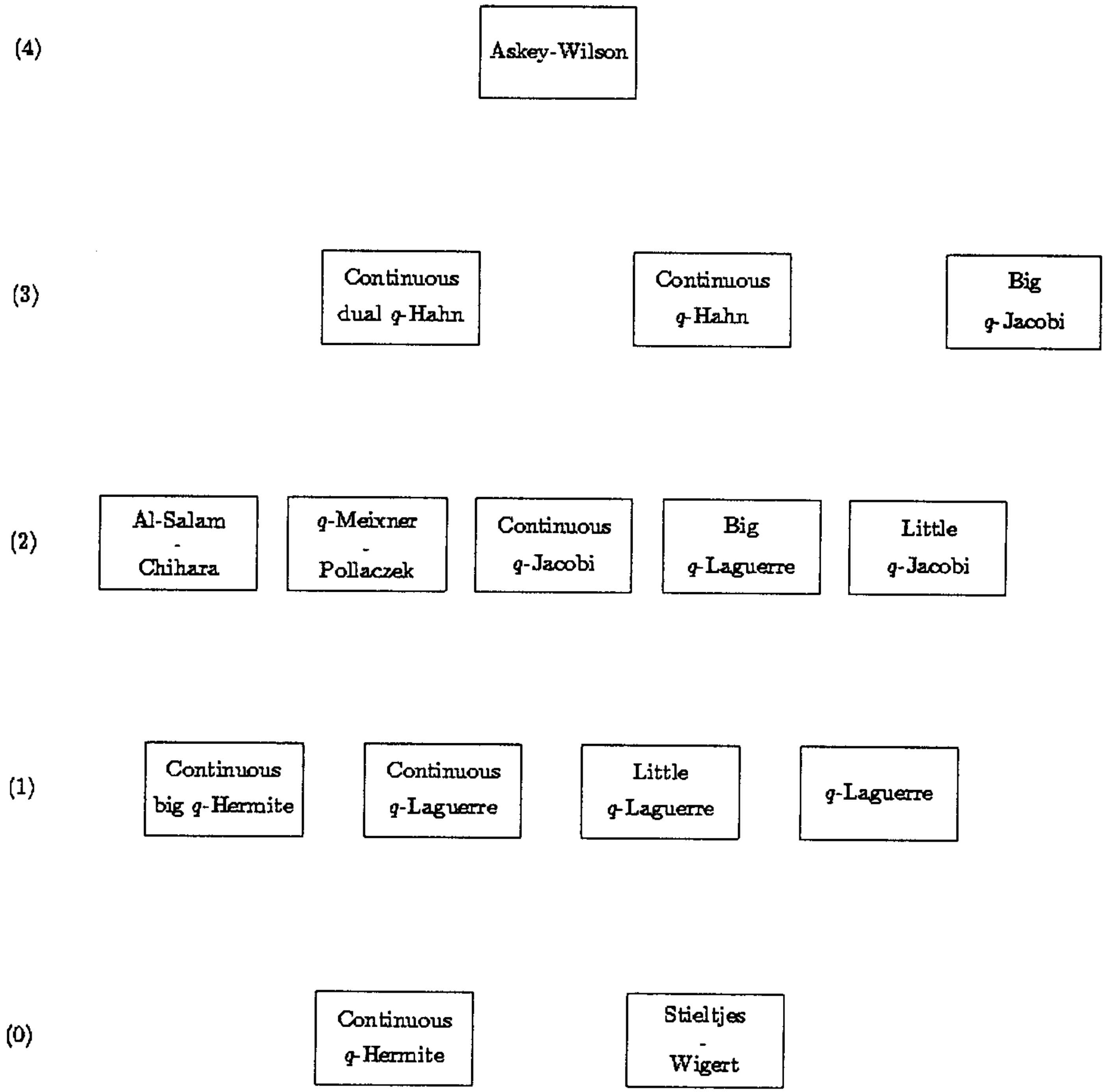
17.13.3 $$\int_0^\infty t^{\alpha-1} \frac{(-tq^{\alpha+\beta};q)_\infty}{(-t;q)_\infty} d_q t = \frac{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma_q(\beta)}{\Gamma_q(1-\alpha)\Gamma_q(\alpha+\beta)},$$

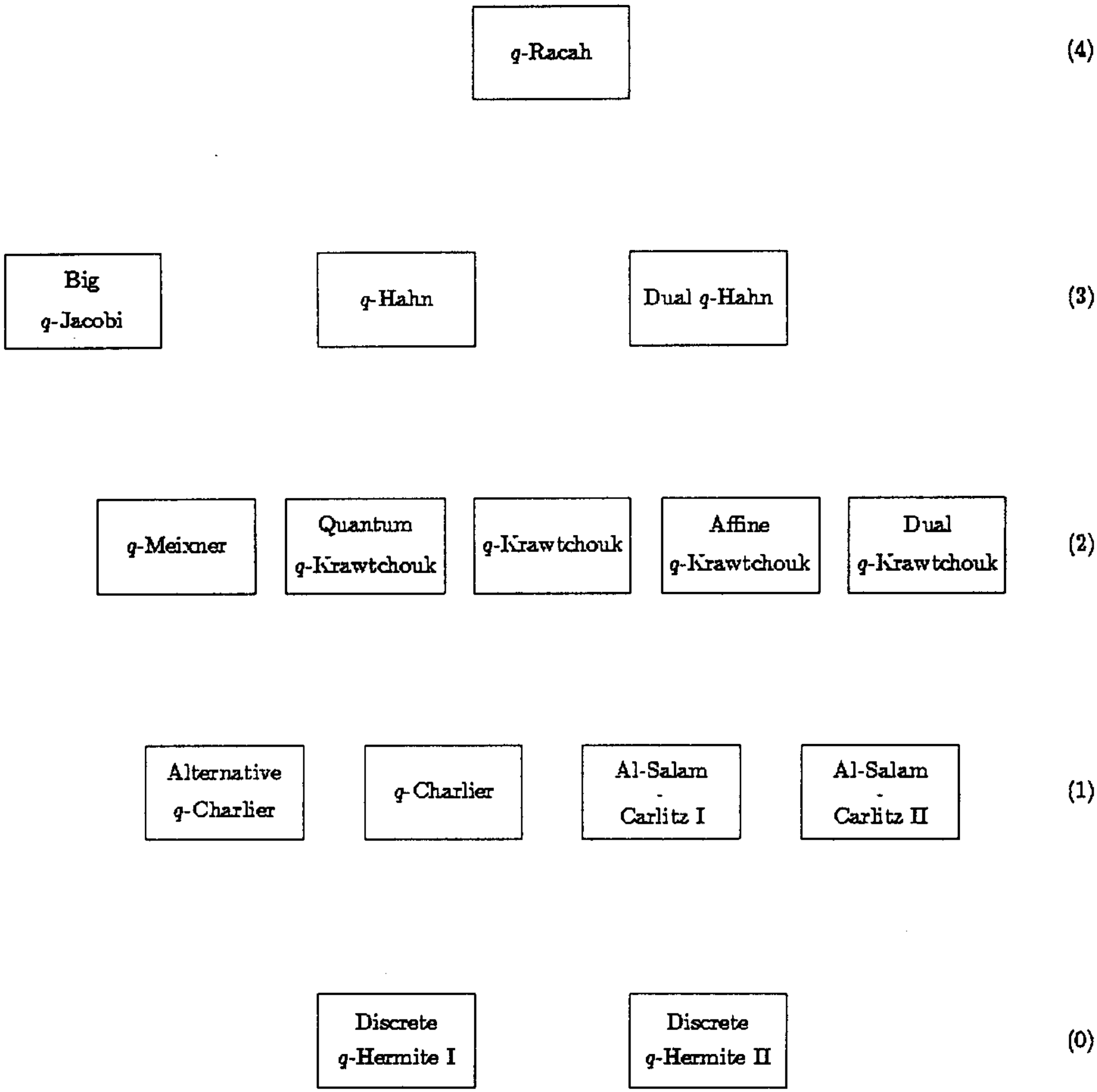
17.13.4 $$\int_0^\infty t^{\alpha-1} \frac{(-ctq^{\alpha+\beta};q)_\infty}{(-ct;q)_\infty} d_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)(-cq^\alpha;q)_\infty (-q^{1-\alpha}/c;q)_\infty}{\Gamma_q(\alpha+\beta)(-c;q)_\infty (-q/c;q)_\infty}.$$

Askey (1980) conjectured extensions of the foregoing integrals that are closely related to Macdonald (1982). These conjectures are proved independently in Habsieger (1988) and Kadell (1988).



$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner - Pollaczek
Jacobi
Meixner
Krawtchouk
$_1F_1(1)/_2F_0(1)$
Laguerre
Charlier
$_2F_0(0)$
Hermite


(4)
Askey-Wilson
(3)
Continuous dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
(2)
Al-Salam - Chihara
q-Meixner - Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
(1)
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
(0)
Continuous q-Hermite
Stieltjes - Wigert


q-Racah
(4)
Big q-Jacobi
q-Hahn
Dual q-Hahn
(3)
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
(2)
Alternative q-Charlier
q-Charlier
Al-Salam - Carlitz I
Al-Salam - Carlitz II
(1)
Discrete q-Hermite I
Discrete q-Hermite II
(0)

§ 18.27(vi). Stieltjes–Wigert Polynomials

18.27.18 $$S_n(x;q)=\sum_{\ell=0}^{n}\frac{q^{\ell^2}(-x)^{\ell}}{(q;q)_\ell(q;q)_{n-\ell}}=\frac{1}{(q;q)_n}\,{}_1\phi_1\left({q^{-n}\atop 0};q,-q^{n+1}x\right).$$

(Sometimes in the literature x is replaced by $q^{\frac{1}{2}}x$.)

The measure is not uniquely determined:

18.27.19 $$\int_0^\infty\frac{S_n(x;q)S_m(x;q)}{(-x,-qx^{-1};q)_\infty}\,dx=\frac{\ln(q^{-1})}{q^n}\frac{(q;q)_\infty}{(q;q)_n}\delta_{n,m},$$

and

18.27.20 $$\int_0^\infty S_n\left(q^{\frac{1}{2}}x;q\right)S_m\left(q^{\frac{1}{2}}x;q\right)\exp\left(-\frac{(\ln x)^2}{2\ln(q^{-1})}\right)dx=\frac{\sqrt{2\pi q^{-1}\ln(q^{-1})}}{q^n(q;q)_n}\delta_{n,m}.$$

§ 18.28(v). Continuous q-Ultraspherical Polynomials

18.28.13
$$C_n(\cos\theta;\beta|q) = \sum_{\ell=0}^{n} \frac{(\beta;q)_\ell(\beta;q)_{n-\ell}}{(q;q)_\ell(q;q)_{n-\ell}} e^{i(n-2\ell)\theta} = \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta}\,{}_2\phi_1\left(\begin{matrix} q^{-n},\beta \\ \beta^{-1}q^{1-n}\end{matrix};q,\beta^{-1}q\,e^{-2i\theta}\right).$$

18.28.14
$$C_n(\cos\theta;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n\beta^{\frac{1}{2}n}}\,{}_4\phi_3\left(\begin{matrix} q^{-n},\beta^2q^n,\beta^{\frac{1}{2}}e^{i\theta},\beta^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}},-\beta,-\beta q^{\frac{1}{2}}\end{matrix};q,q\right).$$

18.28.15
$$\frac{1}{2\pi}\int_0^\pi C_n(\cos\theta;\beta|q)C_m(\cos\theta;\beta|q)\left|\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty}\right|^2 d\theta = \frac{(\beta,\beta q;q)_\infty}{(\beta^2,q;q)_\infty}\frac{(1-\beta)(\beta^2;q)_n}{(1-\beta q^n)(q;q)_n}\delta_{n,m},$$
$$-1<\beta<1.$$

These polynomials are also called *Rogers polynomials.*

§ 18.28(vi). Continuous q-Hermite Polynomials

18.28.16
$$H_n(\cos\theta|q) = \sum_{\ell=0}^{n}\frac{(q;q)_n\,e^{i(n-2\ell)\theta}}{(q;q)_\ell(q;q)_{n-\ell}} = e^{in\theta}\,{}_2\phi_0\left(\begin{matrix} q^{-n},0 \\ -\end{matrix};q,q^n e^{-2i\theta}\right).$$

18.28.17
$$\frac{1}{2\pi}\int_0^\pi H_n(\cos\theta|q)H_m(\cos\theta|q)\left|(e^{2i\theta};q)_\infty\right|^2 d\theta = \frac{\delta_{n,m}}{(q^{n+1};q)_\infty}.$$

§ 18.28(vii). Continuous q^{-1}-Hermite Polynomials

18.28.18
$$h_n(\sinh t|q) = \sum_{\ell=0}^{n} q^{\frac{1}{2}\ell(\ell+1)} \frac{(q^{-n};q)_\ell}{(q;q)_\ell} e^{(n-2\ell)t} = e^{nt}\,{}_1\phi_1\left({q^{-n} \atop 0}; q, -q e^{-2t}\right) = i^{-n} H_n(i \sinh t|q^{-1}).$$

For continuous q^{-1}-Hermite polynomials the orthogonality measure is not unique. See Askey (1989) and Ismail and Masson (1994) for examples.

§ 18.28(viii). q-Racah Polynomials

With $x = q^{-y} + \gamma\delta q^{y+1}$,

18.28.19
$$R_n(x) = R_n(x;\alpha,\beta,\gamma,\delta|q) = \sum_{\ell=0}^{n} \frac{q^\ell (q^{-n}, \alpha\beta q^{n+1}; q)_\ell}{(\alpha q, \beta\delta q, \gamma q, q; q)_\ell} \prod_{j=0}^{\ell-1} (1 - q^j x + \gamma\delta q^{2j+1}) = {}_4\phi_3\left({q^{-n}, \alpha\beta q^{n+1}, q^{-y}, \gamma\delta q^{y+1} \atop \alpha q, \beta\delta q, \gamma q}; q, q\right),$$

αq, $\beta\delta q$, or $\gamma q = q^{-N}$; $n = 0, 1, \ldots, N$.

18.28.20
$$\sum_{y=0}^{N} R_n(q^{-y} + \gamma\delta q^{y+1}) R_m(q^{-y} + \gamma\delta q^{y+1}) \omega_y = h_n \delta_{n,m},$$

$n, m = 0, 1, \ldots, N$.

For ω_y and h_n see Koekoek and Swarttouw (1998, Eq. (3.2.2)).

§18.27 q-Hahn Class

§18.29 Asymptotic Approximations for q-Hahn and Askey–Wilson Classes

§ 5.14. Multidimensional Integrals

Let V_n be the simplex: $t_1 + t_2 + \dots + t_n \le 1$, $t_k \ge 0$. Then for $\Re z_k > 0$, $k = 1, 2, \dots, n+1$,

5.14.1 $$\int_{V_n} t_1^{z_1-1} t_2^{z_2-1} \cdots t_n^{z_n-1} \, dt_1 \, dt_2 \cdots dt_n = \frac{\Gamma(z_1)\Gamma(z_2)\cdots\Gamma(z_n)}{\Gamma(1+z_1+z_2+\dots+z_n)},$$

5.14.2 $$\int_{V_n} \left(1 - \sum_{k=1}^{n} t_k\right)^{z_{n+1}-1} \prod_{k=1}^{n} t_k^{z_k-1} \, dt_k = \frac{\Gamma(z_1)\Gamma(z_2)\cdots\Gamma(z_{n+1})}{\Gamma(z_1+z_2+\dots+z_{n+1})}.$$

¶ Selberg-type Integrals

Let

5.14.3 $$\Delta(t_1, t_2, \dots, t_n) = \prod_{1 \le j < k \le n} (t_j - t_k).$$

Then

5.14.4 $$\int_{[0,1]^n} t_1 t_2 \cdots t_m |\Delta(t_1, \dots, t_n)|^{2c} \prod_{k=1}^{n} t_k^{a-1}(1-t_k)^{b-1} \, dt_k = \frac{1}{(\Gamma(1+c))^n} \prod_{k=1}^{m} \frac{a+(n-k)c}{a+b+(2n-k-1)c} \prod_{k=1}^{n} \frac{\Gamma(a+(n-k)c)\Gamma(b+(n-k)c)\Gamma(1+kc)}{\Gamma(a+b+(2n-k-1)c)},$$

provided that $\Re a, \Re b > 0$, $\Re c > -\min(1/n, \Re a/(n-1), \Re b/(n-1))$.

Secondly,

5.14.5 $$\int_{[0,\infty)^n} t_1 t_2 \cdots t_m |\Delta(t_1, \dots, t_n)|^{2c} \prod_{k=1}^{n} t_k^{a-1} e^{-t_k} \, dt_k = \prod_{k=1}^{m} (a+(n-k)c) \frac{\prod_{k=1}^{n} \Gamma(a+(n-k)c)\Gamma(1+kc)}{(\Gamma(1+c))^n},$$

when $\Re a > 0$, $\Re c > -\min(1/n, \Re a/(n-1))$.

Thirdly,

5.14.6 $$\frac{1}{(2\pi)^{n/2}} \int_{(-\infty,\infty)^n} |\Delta(t_1, \dots, t_n)|^{2c} \prod_{k=1}^{n} \exp(-\tfrac{1}{2}t_k^2) \, dt_k = \frac{\prod_{k=1}^{n} \Gamma(1+kc)}{(\Gamma(1+c))^n},$$

$\Re c > -1/n$.

¶ Dyson's Integral

5.14.7 $$\frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} \prod_{1 \le j < k \le n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^{2b} d\theta_1 \cdots d\theta_n = \frac{\Gamma(1+bn)}{(\Gamma(1+b))^n},$$

$\Re b > -1/n$.

§ 17.14. Constant Term Identities

¶ Zeilberger–Bressoud Theorem (Andrews' q-Dyson Conjecture)

17.14.1
$$\frac{(q;q)_{a_1+a_2+\cdots+a_n}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}} = \text{coeff. of } x_1^0x_2^0\cdots x_n^0 \text{ in } \prod_{1\le j<k\le n}\left(\frac{x_j}{x_k};q\right)_{a_j}\left(\frac{qx_k}{x_j};q\right)_{a_k}.$$

¶ Rogers–Ramanujan Constant Term Identities

In the following, $G(q)$ and $H(q)$ denote the left-hand sides of (17.2.49) and (17.2.50), respectively.

17.14.2
$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n(-q;q^2)_{n+1}} = \text{coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(z^{-1}q^2;q^2)_\infty(-q;q^2)_\infty(z^{-1}q;q^2)_\infty}$$
$$= \frac{1}{(-q;q^2)_\infty}\text{ coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(z^{-1}q;q)_\infty} = \frac{H(q)}{(-q;q^2)_\infty},$$

17.14.3
$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n(-q;q^2)_{n+1}} = \text{coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(z^{-1};q^2)_\infty(-q;q^2)_\infty(z^{-1}q;q^2)_\infty}$$
$$= \frac{1}{(-q;q^2)_\infty}\text{ coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(z^{-1};q)_\infty} = \frac{G(q)}{(-q;q^2)_\infty},$$

17.14.4
$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n(q;q^2)_n} = \text{coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(-z^{-1};q^2)_\infty(q;q^2)_\infty(z^{-1};q^2)_\infty}$$
$$= \frac{1}{(q;q^2)_\infty}\text{ coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(z^{-2};q^4)_\infty} = \frac{G(q^4)}{(q;q^2)_\infty},$$

17.14.5
$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n(q;q^2)_{n+1}} = \text{coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2z^{-1};q^2)_\infty(q;q^2)_\infty(z^{-1}q^2;q^2)_\infty}$$
$$= \frac{1}{(q;q^2)_\infty}\text{ coeff. of } z^0 \text{ in } \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(q^4z^{-2};q^4)_\infty} = \frac{H(q^4)}{(q;q^2)_\infty}.$$

Macdonald (1982) includes extensive conjectures on generalizations of (17.14.1) to root systems. These conjectures were proved in Cherednik (1995), Habsieger (1986), and Kadell (1994); see also Macdonald (1998). For additional results of the type (17.14.2)–(17.14.5) see Andrews (1986, Chapter 4).