A Class of Bases in L^2 for the Sparse Representation of Integral Operators

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Abstract

A class of *multi-wavelet* bases for L^2 is constructed with the property that a variety of integral operators is represented in these bases as sparse matrices, to high precision. In particular, an integral operator $\mathcal{K}$ whose kernel is smooth except along a finite number of singular bands has a sparse representation. In addition, the inverse operator $(I-\mathcal{K})^{-1}$ appearing in the solution of a second-kind integral equation involving $\mathcal{K}$ is often sparse in the new bases. The result is an order $O(n \log^2 n)$ algorithm for numerical solution of a large class of second-kind integral equations.

Key Words. wavelets, integral equations, sparse matrices

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Families of functions $h_{a,b}$,

$$h_{a,b}(x) = |a|^{-1/2}\, h\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbf{R},\ a \neq 0,$$

derived from a single function h by dilation and translation, which form a basis for $L^2(\mathbf{R})$, are known as *wavelets* (Grossman and Morlet [9]). In recent years, these families have received study by many authors, resulting in constructions with a variety of properties. Meyer [11] constructed orthonormal wavelets for which $h \in C^\infty(\mathbf{R})$. Daubechies [6] constructed compactly supported wavelets with $h \in C^k(\mathbf{R})$ for arbitrary k, and [6] gives an overview and synthesis of the field.

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Beylkin, Coifman, and Rokhlin [4] develop the connection between wavelets and recent fast numerical algorithms devised by Rokhlin and other authors ([3], [8], [14], [15]). These algorithms exploit analytical properties of specific linear operators to achieve, in each case, fast application of an operator to an arbitrary function. The operator and function are discretized to a matrix and vector; in the discrete representation a full $n \times n$-matrix is applied to a vector of length n in order $O(n)$ operations, as opposed to order $O(n^2)$ operations for naive matrix-vector multiplication. Each algorithm depends on "local smoothness" of the underlying operator. In particular, each algorithm may be viewed as the division of the operator matrix into order $O(n)$ square submatrices, each approximated by a matrix of low rank, followed by the fast application of the submatrices to the function vector.

In [4] it is observed that these numerical algorithms can be generalized by a technique in which the underlying operator is represented in a basis of wavelets. Discretization (*i.e.*, truncation of the operator expansion) then results in an operator matrix that is approximated by a sparse matrix. The characteristics of the wavelets bases which lead to a sparse matrix representation are that

1. the basis functions are orthogonal to low-order polynomials (have vanishing moments), and

2. most basis functions have small intervals of support.

An integral operator whose kernel is a smooth, non-oscillatory, function of its arguments over most of their range (and therefore can be approximated locally by low-order polynomials) will have negligible projection on most basis functions.

One difficulty of using wavelets bases for the representation of integral operators is that they do not form a basis for functions on a finite interval. Wavelet basis functions overlap in such a way that either the interval must be extended, a periodization must be performed, or the basis functions at the interval ends must be modified. In [4] the integrand is treated as periodic, with some loss of sparsity. In [13] Meyer showed how the basis functions overlapping the interval ends can be truncated and reorthogonalized to obtain a basis on the finite interval.

A second difficulty of using wavelets for the representation of integral operators is that projection onto the basis functions requires appropriate integration quadratures (as is true with other bases). The order of convergence of the quadratures determines the order of the numerical method as a whole. The difficulty is that quadratures must be employed for each element of the resulting matrix, leading to potentially high cost. On the other hand, use of the Nyström method, in which the interval is discretized into n points and the integral at each point is approximated by a quadrature, requires the application of quadratures only n times.

In this paper we construct a class of wavelet-like bases, which we call *multi-wavelet* bases, which lead to the sparse representation of smooth integral operators on a finite interval. For each basis, the interval is recursively bisected; the basis functions on a given scale are supported on the dyadic subintervals of a particular size. Out of this class of bases, different bases differ in the number of basis functions supported on each subinterval, and this number corresponds to the order of convergence of expansions of C^∞ functions. The lack of overlap of the basis functions on a single scale eliminates the first difficulty (mentioned above) of using wavelets for the representation of integral operators. The second difficulty is eliminated by the construction of a discrete counterpart to the bases developed here. The latter construction is described in [2]. A principal advantage of the present construction is its simplicity.

In §1, we construct multi-wavelet bases and in §2, we prove that the representations in these bases of certain integral operators are sparse, to high precision. In §3 we give several numerical examples of the bases and the solution of second-kind integral equations and conclude with a discussion.

1 Multi-Wavelet Bases

1.1 The One-Dimensional Construction

We construct a class of bases for $L^2(\mathbf{R})$ that can be readily revised to bases for $L^2[0,1]$. Each basis is comprised of dilates and translates of a finite set of functions $h_1, \ldots, h_k$. In particular, these bases consist of orthonormal systems

$$h^n_{j,m}(x) = 2^{m/2}\, h_j(2^m x - n), \qquad j = 1, \ldots, k;\; m, n \in \mathbf{Z}, \tag{1}$$

where the functions $h_1, \ldots, h_k$ are piecewise polynomial, vanish outside the interval $[0,1]$, and are orthogonal to low-order polynomials (have vanishing moments),

$$\int_0^1 h_j(x)\, x^i \, dx = 0, \qquad i = 0, 1, \ldots, k-1. \tag{2}$$

We first restrict our attention to the finite interval $[0,1] \subset \mathbf{R}$ and we construct a basis for $L^2[0,1]$. We employ the multi-resolution analysis framework developed by Mallat [10] and Meyer [12], and discussed at length by Daubechies [6]. We suppose that k is a positive integer and for $m = 0, 1, 2, \ldots$ we define a space S^k_m of piecewise polynomial functions,

$$\begin{aligned} S^k_m = \{f:\ & \text{the restriction of } f \text{ to the interval } (2^{-m}n, 2^{-m}(n+1)) \text{ is} \\ & \text{a polynomial of degree less than } k, \text{ for } n = 0, \ldots, 2^m - 1, \\ & \text{and } f \text{ vanishes elsewhere}\}. \end{aligned} \tag{3}$$

It is apparent that the space S^k_m has dimension $2^m k$ and

$$S^k_0 \subset S^k_1 \subset \cdots \subset S^k_m \subset \cdots.$$

For $m = 0, 1, 2, \ldots$ we define the $2^m k$-dimensional space R_m^k to be the orthogonal complement of S_m^k in S_{m+1}^k,

$$S_m^k \oplus R_m^k = S_{m+1}^k, \qquad R_m^k \perp S_m^k,$$

so we inductively obtain the decomposition

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k. \tag{4}$$

Suppose that functions $h_1, \ldots, h_k : \mathbf{R} \to \mathbf{R}$ form an orthogonal basis for R_0^k. Since R_0^k is orthogonal to S_0^k, the first k moments of $h_1, \ldots, h_k$ vanish,

$$\int_0^1 h_j(x)\, x^i\, dx = 0, \qquad i = 0, 1, \ldots, k-1.$$

The $2k$-dimensional space R_1^k is spanned by the $2k$ orthogonal functions $h_1(2x)$, $\ldots, h_k(2x)$, $h_1(2x-1), \ldots, h_k(2x-1)$, of which k are supported on the interval $[0, \frac{1}{2}]$ and k on $[\frac{1}{2}, 1]$. In general, the space R_m^k is spanned by $2^m k$ functions obtained from $h_1, \ldots, h_k$ by translation and dilation. There is some freedom in choosing the functions $h_1, \ldots, h_k$ within the constraint that they be orthogonal; by requiring normality and additional vanishing moments, we specify them uniquely, up to sign. The remainder of this subsection is devoted to the explicit construction of $h_1, \ldots, h_k$; in the following sections we exploit only the property that $h_1, \ldots, h_k$ form an orthonormal basis for R_0^k.

In preparation for the definition of $h_1, \ldots, h_k$, we construct the k functions $f_1, \ldots, f_k : \mathbf{R} \to \mathbf{R}$, supported on the interval $[-1, 1]$, with the following properties:

1. The restriction of f_i to the interval $(0, 1)$ is a polynomial of degree $k-1$.

2. The function f_i is extended to the interval $(-1, 0)$ as an even or odd function according to the parity of $i + k - 1$.

3. The functions $f_1, \ldots, f_k$ satisfy the following orthogonality and normality conditions:

$$\int_{-1}^1 f_i(x)\, f_j(x)\, dx \equiv \langle f_i, f_j \rangle = \delta_{ij}, \qquad i, j = 1, \ldots, k.$$

4. The function f_j has vanishing moments,

$$\int_{-1}^1 f_j(x)\, x^i\, dx = 0, \qquad i = 0, 1, \ldots, j + k - 2.$$

Properties 1 and 2 imply that there are k^2 polynomial coefficients that determine the functions $f_1, \ldots, f_k$, while properties 3 and 4 provide k^2 (non-trivial) constraints. It turns out that the equations uncouple to give k nonsingular linear systems that may be solved to obtain the coefficients, yielding the functions uniquely (up to sign). Rather than prove that these systems are nonsingular, however, we now determine $f_1, \ldots, f_k$ constructively.

We start with $2k$ functions which span the space of functions that are polynomials of degree less than k on the interval $(0,1)$ and on $(-1,0)$, then orthogonalize k of them, first to the functions $1, x, \ldots, x^{k-1}$, then to the functions $x^k, x^{k+1}, \ldots, x^{2k-1}$, and finally among themselves. We define $f_1^1, f_2^1, \ldots, f_k^1$ by the formula

$$f_j^1(x) = \begin{cases} x^{j-1}, & x \in (0,1), \\ -x^{j-1}, & x \in (-1,0), \\ 0, & \text{otherwise}, \end{cases}$$

and note that the $2k$ functions $1, x, \ldots, x^{k-1}, f_1^1, f_2^1, \ldots, f_k^1$ are linearly independent, hence span the space of functions that are polynomials of degree less than k on $(0,1)$ and on $(-1,0)$.

1. By the Gram-Schmidt process we orthogonalize f_j^1 with respect to $1, x, \ldots, x^{k-1}$, to obtain f_j^2, for $j = 1, \ldots, k$. This orthogonality is preserved by the remaining orthogonalizations, which only produce linear combinations of the f_j^2.

2. The next sequence of steps yields $k-1$ functions orthogonal to x^k, of which $k-2$ functions are orthogonal to x^{k+1}, and so forth, down to 1 function which is orthogonal to x^{2k-2}. First, if at least one of f_j^2 is not orthogonal to x^k, we reorder the functions so that it appears first, $\langle f_1^2, x^k \rangle \neq 0$. We then define $f_j^3 = f_j^2 - a_j \cdot f_0^2$ where a_j is chosen so $\langle f_j^3, x^k \rangle = 0$ for $j = 2, \ldots, k$, achieving the desired orthogonality to x^k. Similarly, we orthogonalize to $x^{k+1}, \ldots, x^{2k-2}$, each in turn, to obtain $f_1^2, f_2^3, f_3^4, \ldots, f_k^{k+1}$ such that $\langle f_j^{j+1}, x^i \rangle = 0$ for $i \leq j+k-2$.

3. Finally, we do Gram-Schmidt orthogonalization on $f_k^{k+1}, f_{k-1}^k, \ldots, f_1^2$, in that order, and normalize to obtain $f_k, f_{k-1}, \ldots, f_1$.

It is readily seen that the f_j satisfy properties 1-4 of the previous paragraph. Defining $h_1, \ldots, h_k : \mathbf{R} \to \mathbf{R}$ by the formula

$$h_i(x) = 2^{1/2} f_i(2x-1), \qquad i = 1, \ldots, k,$$

we obtain the equality

$$R_0^k = \text{linear span } \{h_i : \ i = 1, \ldots, k\},$$

and, more generally,

$$R_m^k = \text{linear span}\ \{h_{j,m}^n: \quad h_{j,m}^n(x) = 2^{m/2}\, h_j(2^m x - n), \quad j=1,\ldots,k;\ n=0,\ldots,2^m-1\}. \tag{5}$$

We will show next that dilates and translates of the piecewise polynomial functions $h_1, \ldots, h_k$ form an orthonormal basis for $L^2(\mathbf{R})$. Furthermore, a subset of these dilates and translates, combined with a basis for S_0^k, forms a basis for $L^2[0,1]$.

1.2 Completeness of One-Dimensional Construction

We define the space S^k to be the union of the S_m^k, given by the formula

$$S^k = \bigcup_{m=0}^{\infty} S_m^k, \tag{6}$$

and observe that $\overline{S^k} = L^2[0,1]$. In particular, S^k contains the Haar basis for $L^2[0,1]$, consisting of functions piecewise constant on each of the subintervals $(2^{-m}n, 2^{-m}(n+1))$. Here the closure $\overline{S^k}$ is defined with respect to the L^2-norm,

$$\|f\| = \langle f, f\rangle^{1/2},$$

where the inner product $\langle f, g\rangle$ is defined by the formula

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx.$$

We let $\{u_1, \ldots, u_k\}$ denote an orthonormal basis for S_0^k; in view of Eqs. (4), (5), and (6), the orthonormal system

$$\begin{aligned} B_k = \quad & \{u_j: \quad j=1,\ldots,k\} \\ & \cup \{h_{j,m}^n: \quad j=1,\ldots,k;\ m=0,1,2,\ldots;\ n=0,\ldots,2^m-1\} \end{aligned}$$

spans $L^2[0,1]$; we refer to B_k as the *multi-wavelet basis of order* k for $L^2[0,1]$.

Now we construct a basis for $L^2(\mathbf{R})$ by defining, for $m \in \mathbf{Z}$, the space $\tilde{S}_m^k$ by the formula

$$\begin{aligned} \tilde{S}_m^k = \{f: \quad & \text{the restriction of } f \text{ to the interval } (2^{-m}n, 2^{-m}(n+1)) \text{ is} \\ & \text{a polynomial of degree less than } k, \text{ for } n \in \mathbf{Z}\} \end{aligned}$$

and observing that the space $\tilde{S}_{m+1}^k \backslash \tilde{S}_m^k$ is spanned by the orthonormal set

$$\{h_{j,m}^n: \quad h_{j,m}^n(x) = 2^{m/2}\, h_j(2^m x - n),\ j=1,\ldots,k;\ n \in \mathbf{Z}\}.$$

Thus $L^2(\mathbf{R})$, which is contained in $\overline{\bigcup_m \tilde{S}_m^k}$, has an orthonormal basis

$$\{h_{j,m}^n: \quad j=1,\ldots,k;\ m,n \in \mathbf{Z}\}.$$

1.3 Construction in Multiple Dimensions

The construction of bases for $L^2[0,1]$ and $L^2(\mathbf{R})$ can be extended to certain other function spaces, including $L^2[a,b]^d$ and $L^2(\mathbf{R}^d)$, for any positive integer d. We now outline this extension by giving the basis for $L^2[0,1]^2$, which is illustrative of the construction for any finite-dimensional space. We define the space $S_m^{k,2}$ by the formula

$$S_m^{k,2} = S_m^k \times S_m^k, \qquad m = 0, 1, 2, \ldots,$$

where S_m^k is defined by Eq. (3). We further define $R_m^{k,2}$ to be the orthogonal complement of $S_m^{k,2}$ in $S_{m+1}^{k,2}$,

$$S_m^{k,2} \oplus R_m^{k,2} = S_{m+1}^{k,2}, \qquad R_m^{k,2} \perp S_m^{k,2}.$$

Then $R_0^{k,2}$ is the space spanned by the orthonormal basis

$$\{u_i(x)h_j(y),\ h_i(x)u_j(y),\ h_i(x)h_j(y) :\ \ i,j = 1, \ldots, k\}.$$

Among these $3k^2$ basis elements each element $v(x,y)$ has no projection on low-order polynomials,

$$\int_0^1 \int_0^1 v(x,y)\, x^i\, y^j\, dx\, dy = 0, \qquad i,j = 0, 1, \ldots, k-1.$$

The space $R_m^{k,2}$ is spanned by dilations and translations of the $v(x,y)$ and the basis of $L^2[0,1]^2$ consists of these functions and the low-order polynomials $\{u_i(x)u_j(y) : i,j = 1, \ldots, k\}$.

1.4 Convergence of the Multi-Wavelet Bases

For a function $f \in L^2[0,1]$, a positive integer k, and $m = 0, 1, 2 \ldots$, we define the orthogonal projection $Q_m^k f$ of f onto S_m^k by the formula

$$(Q_m^k f)(x) = \sum_{j,n} \langle f, u_{j,m}^n \rangle \cdot u_{j,m}^n(x),$$

where $\{u_{j,m}^n\}$ is an orthonormal basis for S_m^k. The projections $Q_m^k f$ converge (in the mean) to f as $m \to \infty$. If the function f is several times differentiable, we can bound the error, as established by the following lemma.

Lemma 1.1 *Suppose that the function $f : [0,1] \to \mathbf{R}$ is k times continuously differentiable, $f \in C^k[0,1]$. Then $Q_m^k f$ approximates f with mean error bounded as follows:*

$$\|Q_m^k f - f\| \leq 2^{-mk}\, \frac{2}{4^k k!} \sup_{x \in [0,1]} |f^{(k)}(x)| \tag{7}$$

Proof. We divide the interval $[0,1]$ into subintervals on which $Q_m^k f$ is a polynomial; the restriction of $Q_m^k f$ to one such subinterval $I_{m,n}$ is the polynomial of degree less than k that approximates f with minimum mean error. We then use the maximum error estimate for the polynomial which interpolates f at Chebyshev nodes of order k on $I_{m,n}$.

We define $I_{m,n} = [2^{-m}n, 2^{-m}(n+1)]$ for $n = 0, 1, \ldots, 2^m - 1$, and obtain

$$\begin{aligned}
\|Q_m^k f - f\|^2 &= \int_0^1 \left[(Q_m^k f)(x) - f(x)\right]^2 dx \\
&= \sum_n \int_{I_{m,n}} \left[(Q_m^k f)(x) - f(x)\right]^2 dx \\
&\leq \sum_n \int_{I_{m,n}} \left[(C_{m,n}^k f)(x) - f(x)\right]^2 dx \\
&\leq \sum_n \int_{I_{m,n}} \left(\frac{2^{1-mk}}{4^k k!} \sup_{x \in I_{m,n}} |f^{(k)}(x)|\right)^2 dx \\
&\leq \left(\frac{2^{1-mk}}{4^k k!} \sup_{x \in [0,1]} |f^{(k)}(x)|\right)^2,
\end{aligned}$$

and by taking square roots we have bound (7). Here $C_{m,n}^k f$ denotes the polynomial of degree k which agrees with f at the Chebyshev nodes of order k on $I_{m,n}$, and we have used the well-known maximum error bound for Chebyshev interpolation (see, *e.g.*, [5]). □

The error of the approximation $Q_m^k f$ of f therefore decays like 2^{-mk} and, since S_m^k has a basis of $2^m k$ elements, we have convergence of order k. For the generalization to d dimensions, a similar argument shows that the rate of convergence is of order k/d.

2 Sparse Representation of Integral Operators

The matrix representations of integral operators in multi-wavelet bases are sparse (to finite precision) for the same class of integral operators as is treated in [4], namely, all Calderon-Zygmund and pseudo-differential operators. In applications, an operator kernel commonly has the form

$$K(x,t) = f(x,t)\, s(|x-t|) + g(x,t), \tag{8}$$

where f and g are analytic functions of x, t and s is analytic except at the origin where it is singular. In the following development we initially restrict ourselves to a simple example of this latter class of kernels, with $K(x,t) = \log|x-t|$. Although this kernel is symmetric and convolutional, neither of these properties is related to the sparsity. Instead, a proof of sparsity (presented in Lemma 2.2 below) relies solely on derivative estimates provided by the Cauchy integral formula for intervals separated from the singularity. Later we treat the more general situation of Eq. (8) with $s(x) = \log(x)$.

We begin this section by introducing some notation for integral equations.

2.1 Second-Kind Integral Equations

A linear Fredholm integral equation of the second kind is an expression of the form

$$f(x) - \int_a^b K(x,t)\, f(t)\, dt = g(x), \tag{9}$$

where we assume that the kernel K is in $L^2[a,b]^2$ and the unknown f and right-hand-side g are in $L^2[a,b]$. For notational simplicity, we restrict our attention to the interval $[a,b] = [0,1]$. We use the symbol $\mathcal{K}$ to denote the integral operator of Eq. (9), given by the formula

$$(\mathcal{K}f)(x) = \int_0^1 K(x,t)\, f(t)\, dt,$$

for all $f \in L^2[0,1]$ and $x \in [0,1]$. Suppose that $\{b_1, b_2, \ldots\}$ is an orthonormal basis for $L^2[0,1]$; the expansion of K in this basis is given by the formula

$$K(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} K_{ij}\, b_i(x)\, b_j(t), \tag{10}$$

where the coefficient K_{ij} is given by the expression

$$K_{ij} = \int_0^1 \int_0^1 K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt, \qquad i,j = 1,2,\ldots. \tag{11}$$

Similarly, the functions f and g have expansions

$$f(x) = \sum_{i=1}^{\infty} f_i\, b_i(x), \qquad g(x) = \sum_{i=1}^{\infty} g_i\, b_i(x),$$

where the coefficients f_i and g_i are given by the formulae

$$f_i = \int_0^1 f(x)\, b_i(x)\, dx, \qquad g_i = \int_0^1 g(x)\, b_i(x)\, dx, \qquad i = 1,2,\ldots.$$

The integral equation (9) then corresponds to the infinite system of equations

$$f_i - \sum_{j=1}^{\infty} K_{ij}\, f_j = g_i, \qquad i = 1,2,\ldots.$$

The expansion for K may be truncated at a finite number of terms, yielding the integral operator R defined by the formula

$$(Rf)(x) = \int_0^1 \sum_{i=1}^{n} \sum_{j=1}^{n} (K_{ij}\, b_i(x)\, b_j(t))\, f(t)\, dt, \qquad f \in L^2[0,1],\ x \in [0,1],$$

which approximates $\mathcal{K}$. Integral equation (9) is thereby approximated by the system

$$f_i - \sum_{j=1}^{n} K_{ij}\, f_j = g_i, \qquad i = 1, \ldots, n, \tag{12}$$

which is a system of n equations in n unknowns. Eqs. (12) may be solved numerically to yield an approximate solution to Eq. (9), given by the expression

$$f_R(x) = \sum_{i=1}^{n} f_i\, b_i(x).$$

How large is the error $e_R = f - f_R$ of the approximate solution? We follow the derivation by Delves and Mohamed in [7]. Defining g_R by the formula

$$g_R(x) = \sum_{i=1}^{n} g_i\, b_i(x),$$

we rewrite Eqs. (9) and (12) in terms of operators $\mathcal{K}$ and R to obtain

$$\begin{aligned} (I - \mathcal{K})f &= g \\ (I - R)f_R &= g_R. \end{aligned}$$

Combining the latter equations yields

$$(I - \mathcal{K})e_R = (\mathcal{K} - R)f_R + (g - g_R).$$

Provided that $(I - \mathcal{K})^{-1}$ exists, we obtain the error bound

$$\|e_R\| \le \|(I - \mathcal{K})^{-1}\| \cdot \|(\mathcal{K} - R)f_R + (g - g_R)\|. \tag{13}$$

The error depends, therefore, on the conditioning of the original integral equation, as is apparent from the term $\|(I-\mathcal{K})^{-1}\|$, on the fidelity of the finite-dimensional operator R to the integral operator $\mathcal{K}$, and on the approximation of g_R to g.

2.2 Representation in Multi-Wavelet Bases

We consider integral operators $\mathcal{K}$ with kernels that are analytic, except at $x = t$, where they are singular. In particular, we analyze singularities of the form $\log|x - t|$. An operator with such a kernel K, expanded in one of the multi-wavelet bases defined above, is represented as a sparse matrix. This sparseness is due to the smoothness of K on rectangles separated from the "diagonal".

Definition 2.1 We say that a rectangular region oriented parallel to the coordinate axes x, t is *separated from the diagonal* if its distance in the horizontal or vertical direction from the line $x = t$ is at least the length of its longer side. In symbols, a region $[x, x + a] \times [t, t + b] \subset \mathbf{R}^2$ is separated from the diagonal if $a + \max\{a, b\} \le t - x$ or $b + \max\{a, b\} \le x - t$.

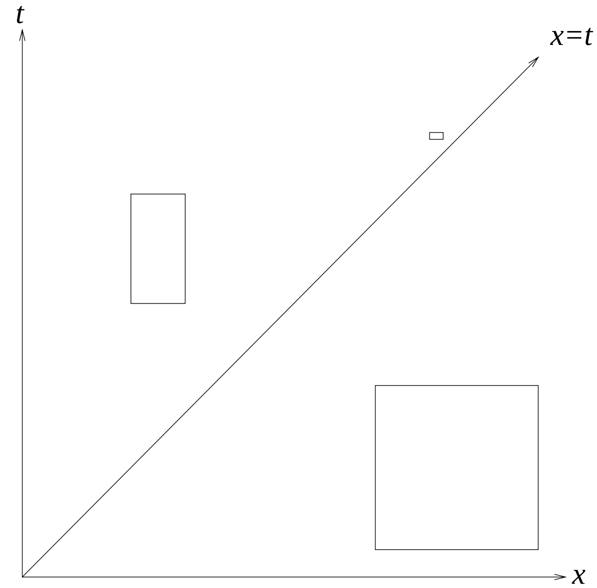

Figure 1: *Rectangular regions (just) separated from the diagonal.*

This definition is illustrated in Fig. 1.

Suppose that k is a positive integer and that $B_k = \{b_1, b_2, \ldots\}$ is the multi-wavelet basis for $L^2[0,1]$ of order k, defined in §1. We let I_j denote the interval of support of b_j, and we assume that the sequence of basis functions $b_1, b_2, \ldots$ is ordered so that $I_1, I_2, \ldots$ have non-increasing lengths. For large n, the matrix $\{K_{ij}\}_{i,j=1,\ldots,n}$ is sparse, to high precision, as is proved in the following propositions.

Lemma 2.2 *Suppose that the function $K : [0,1] \times [0,1] \to \mathbf{R}$ is given by the formula $K(x,t) = \log|x-t|$. The expansion (Eq. 10) of K in the multi-wavelet basis B_k of order k has coefficients K_{ij} which satisfy the bound*

$$|K_{ij}| \leq \frac{1}{8k \cdot 3^{k-1}} \tag{14}$$

whenever the rectangular region $I_i \times I_j$ is separated from the diagonal.

Proof. Suppose that the intervals I_i and I_j are given by the expressions $I_i = [x_0, x_0 + a]$ and $I_j = [t_0, t_0 + b]$; without loss of generality we assume (as one of two equivalent cases) that $b + \max\{a, b\} \leq x_0 - t_0$. It is immediate from this inequality that

$$\left| \frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right| \leq \frac{1}{3} \tag{15}$$

for $(x,t) \in I_i \times I_j$.

We use the Taylor expansion for the natural logarithm about $c > 0$,

$$\log(c+y) = \log(c) + (y/c) - (y/c)^2/2 + (y/c)^3/3 - (y/c)^4/4 + \cdots,$$

for $|y| < c$. We now let $c = x_0 + a/2 - t$ and $y = x - x_0 - a/2$ and for $(x,t) \in I_i \times I_j$ we obtain the formula

$$\log|x-t| = \log(x_0 + a/2 - t) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right)^m. \tag{16}$$

We now apply Eqs. (11), (16), (2), and (15), each in turn, to obtain

$$\begin{aligned}
|K_{ij}| &= \left| \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt \right| \\
&\le \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \log|x-t|\, b_i(x)\, dx \right| |b_j(t)|\, dt \\
&= \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \left[\log(x_0 + \frac{a}{2} - t) \right.\right. \\
&\qquad \left.\left. - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right)^m \right] b_i(x)\, dx \right| |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} \sum_{m=k}^{\infty} \frac{1}{m} \left(\frac{x_0 + a/2 - x}{x_0 + a/2 - t} \right)^m b_i(x)\, dx \right| |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} \frac{1}{k} \sum_{m=k}^{\infty} \left(\frac{1}{3} \right)^m |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} \frac{1}{2k \cdot 3^{k-1}} |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\le \frac{1}{2k \cdot 3^{k-1}} \int_{t_0}^{t_0+b} \sqrt{ \left(\int_{x_0}^{x_0+a} {b_i}^2(x)\, dx \right) \left(\int_{x_0}^{x_0+a} 1\, dx \right) }\, |b_j(t)|\, dt \\
&\le \frac{\sqrt{ab}}{2k \cdot 3^{k-1}} \le \frac{1}{8k \cdot 3^{k-1}},
\end{aligned}$$

as was to be proved. □

We now consider a somewhat more general kernel.

Lemma 2.3 *Suppose that the function $L:\ D \times D \to \mathbf{C}$ is given by the formula $L(z,w) = f(z,w) \log|z-w| + g(z,w)$, where D is the closed disk of radius $\frac{3}{2}$ centered at $z = \frac{1}{2}$ and f and g are analytic in a domain containing $D \times D \subset \mathbf{C}^2$. Suppose further that the function K is the restriction of L to $[0,1] \times [0,1]$. The expansion of K in the basis B_k has coefficients K_{ij} which satisfy the bound*

$$|K_{ij}| \le \left(\frac{k}{8} + \frac{3}{16} \right) \frac{1}{3^{k-1}} \sup_{z,w \in \partial D} |f(z,w)| + \frac{2}{7 \cdot 8^k} \sup_{z,w \in \partial D} |g(z,w)|, \tag{17}$$

whenever the rectangular region $I_i \times I_j$ is separated from the diagonal.

Proof. We treat the parts of K separately by defining K' to be the restriction of $f(z,w)\log|z-w|$ to $[0,1]\times[0,1]$ and g' to be the restriction of g, so $K = K'+g'$.

We combine the method of proof used in Lemma 2.2 with the formula for the derivative of a product,

$$\frac{\partial^m K'(x,t)}{\partial x^m} = \sum_{r=0}^{m} \binom{m}{r} \frac{\partial^r f(x,t)}{\partial x^r} \cdot \frac{\partial^{m-r} \log|x-t|}{\partial x^{m-r}}. \tag{18}$$

By the Cauchy integral formula we obtain

$$\left|\frac{\partial^r f(x,t)}{\partial x^r}\right| \leq r! \sup_{z,w\in\partial D} |f(z,w)|, \qquad \left|\frac{\partial^r g(x,t)}{\partial x^r}\right| \leq r! \sup_{z,w\in\partial D} |g(z,w)| \tag{19}$$

for $(x,t)\in[0,1]\times[0,1]$. For the logarithm, differentiation yields the formula

$$\frac{\partial^{m-r}\log|x-t|}{\partial x^{m-r}} = \frac{(-1)^{m-r-1}(m-r-1)!}{(x-t)^{m-r}}, \tag{20}$$

for $r<m$. Combining (18), (19), and (20), we obtain

$$\begin{aligned}
\left|\frac{\partial^m K'(x,t)}{\partial x^m}\right| &\leq \sum_{r=0}^{m} \binom{m}{r} \left|\frac{\partial^r f(x,t)}{\partial x^r}\right| \cdot \left|\frac{\partial^{m-r}\log|x-t|}{\partial x^{m-r}}\right| \\
&\leq \sup_{z,w\in\partial D} |f(z,w)| \left(\sum_{r=0}^{m-1} \binom{m}{r} r! \frac{(m-r-1)!}{|x-t|^{m-r}} + m!\,|\log|x-t||\right) \\
&\leq S_f \cdot \left(m!\,\frac{2+\log m}{|x-t|^m}\right)
\end{aligned} \tag{21}$$

for $|x-t|\leq 1$ and $m\geq 1$, where $S_f = \sup_{z,w\in\partial D}|f(z,w)|$.

Suppose that the intervals I_i and I_j are given by the expressions $I_i = [x_0, x_0 + a]$ and $I_j = [t_0, t_0+b]$; we assume without loss of generality that $b + \max\{a,b\} \leq x_0 - t_0$. It follows directly from this inequality that

$$\left|\frac{x_0 + a/2 - x}{x_0 + a/2 - t}\right| \leq \frac{1}{3} \tag{22}$$

for $(x,t)\in I_i \times I_j$. We now apply Eqs. (11), (2), (21), and (22), to obtain

$$\begin{aligned}
|K'_{ij}| &= \left|\int_{t_0}^{t_0+b}\int_{x_0}^{x_0+a} K'(x,t)\, b_i(x)\, b_j(t)\, dx\, dt\right| \\
&\leq \int_{t_0}^{t_0+b}\left|\int_{x_0}^{x_0+a} \sum_{m=0}^{\infty} \frac{(x_0+a/2-x)^m}{m!}\,\frac{\partial^m K'(x_0+a/2,t)}{\partial {x_0}^m}\, b_i(x)\, dx\right| |b_j(t)|\, dt \\
&\leq \int_{t_0}^{t_0+b}\int_{x_0}^{x_0+a} \sum_{m=k}^{\infty} \left|\frac{x_0+a/2-x}{x_0+a/2-t}\right|^m S_f\,(2+\log m)\,|b_i(x)|\, dx\, |b_j(t)|\, dt
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} S_f \sum_{m=k}^{\infty} \left(\frac{1}{3}\right)^m (m+1)\, |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} S_f \left(\frac{k}{2}+\frac{3}{4}\right) \frac{1}{3^{k-1}}\, |b_i(x)|\, dx\, |b_j(t)|\, dt \\
&\le S_f \left(\frac{k}{2}+\frac{3}{4}\right) \frac{1}{3^{k-1}} \int_{t_0}^{t_0+b} \sqrt{\left(\int_{x_0}^{x_0+a} {b_i}^2(x)\, dx\right)\left(\int_{x_0}^{x_0+a} 1\, dx\right)}\, |b_j(t)|\, dt \\
&\le S_f \left(\frac{k}{2}+\frac{3}{4}\right) \frac{\sqrt{ab}}{3^{k-1}} \\
&\le S_f \left(\frac{k}{8}+\frac{3}{16}\right) \frac{1}{3^{k-1}}.
\end{aligned}
$$

For the second term of $K_{ij} = K'_{ij} + g'_{ij}$ we obtain

$$
\begin{aligned}
|g'_{ij}| &= \left| \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} g'(x,t)\, b_i(x)\, b_j(t)\, dx\, dt \right| \\
&\le \int_{t_0}^{t_0+b} \left| \int_{x_0}^{x_0+a} g'(x,t)\, b_i(x)\, dx \right| |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \int_{x_0}^{x_0+a} \sum_{m=k}^{\infty} \sup_{z,w\in\partial D} |g(z,w)|\, |x-x_0-a/2|^m\, |b_i(x)|\, dx |b_j(t)|\, dt \\
&\le \int_{t_0}^{t_0+b} \sum_{m=k}^{\infty} \sup_{z,w\in\partial D} |g(z,w)|\, \frac{1}{8^m} \int_{x_0}^{x_0+a} |b_i(x)|\, dx |b_j(t)|\, dt \\
&\le \frac{\sqrt{ab}}{7\cdot 8^{k-1}} \sup_{z,w\in\partial D} |g(z,w)| \\
&\le \frac{2}{7\cdot 8^{k}} \sup_{z,w\in\partial D} |g(z,w)|.
\end{aligned}
$$

Combining the estimates for K'_{ij} and g'_{ij} yields (17). □

The preceding lemma shows that for a smooth kernel K with logarithm singularity at $x = t$, the order k of the multi-wavelet basis B_k in which K is expanded may be chosen large enough that the expansion coefficient K_{ij} is negligible, provided $I_i \times I_j$ is separated from the diagonal. As mentioned above, a similar statement can be proven for any kernel of the form $K(x,t) = f(x,t)\, s(|x-t|) + g(x,t)$, where f, g are entire analytic functions of two variables and s is an analytic function except at the origin (where it has a singularity), provided that s is integrable. More generally, any Calderon-Zygmund or pseudo-differential operator can be similarly expressed (see [4]).

The next lemma establishes the fact that, asymptotically, most regions $I_i \times I_j$ *are* separated from the diagonal.

Lemma 2.4 *Suppose that $I_1, \ldots, I_n$ are the (non-increasing) intervals of support of the first n functions of the basis B_k. Of the n^2 rectangular regions $I_i \times I_j$, we* *denote the number separated from the diagonal by $S(n)$ and the number "near" the diagonal by $N(n) = n^2 - S(n)$. Then $N(n)$ grows as $\mathrm{O}(n \log n)$; in particular, for $n = 2^l k$ with $l > 0$, we have the formula*

$$N(n) = 6nlk - 15nk - 6lk^2 + 16k^2. \tag{23}$$

Proof. The restriction that $n = 2^l k$ ensures that the first n basis functions consist of those functions whose intervals of support have length at least 2^{1-l}. We define $S^=(p)$ to be the number of pairs (i, j) such that the rectangular region $I_i \times I_j$ is separated from the diagonal and $|I_i| = |I_j| = 2^{-p}$, and we observe that $S^=(p) = (2^p - 1)(2^p - 2)\, k^2$ for $p = 0, 1, 2, \ldots$. We further define $S^{\neq}(p, q)$ to be the number of pairs (i, j) such that $I_i \times I_j$ is separated from the diagonal and $|I_i| = 2^{-p}$, $|I_j| = 2^{-q}$, and we observe that $S^{\neq}(p, q) = S^=(\min\{p, q\})\, 2^{|p-q|}$ for $p, q = 0, 1, 2, \ldots$. Finally, we combine these formulae to obtain

$$\begin{aligned} S(n) &= \sum_{p=0}^{l-1} \left(S^=(p) + \sum_{q=p+1}^{l-1} \left(S^{\neq}(p,q) + S^{\neq}(q,p) \right) \right) \\ &= \sum_{p=0}^{l-1} S^=(p)\ \left(1 + 2(2^{l-p} - 2)\right) \\ &= \sum_{p=0}^{l-1} (2^p - 1)(2^p - 2)\, k^2\, (2^{l-p+1} - 3) \\ &= (4^l - 6 \cdot 2^l l + 15 \cdot 2^l + 6l - 16)\, k^2 \\ &= n^2 - 6nlk + 15nk + 6lk^2 - 16k^2, \end{aligned}$$

from which Eq. (23) follows directly. The assertion that the general growth of $N(n)$ is $O(n \log n)$ follows from Eq. (23) and the fact that N is a monotonic function of n. □

3 Numerical Examples and Discussion

3.1 Basis Functions

In this section we give numerical expressions for the multi-wavelet functions $f_0, f_1, \ldots, f_{k-1}$ and show their graphs for several values of k. These functions were obtained using the procedure of §1, implemented in a simple Maple program (available from the author). Table 1 contains, for small k, the polynomials which represent the f_i on the interval $(0, 1)$, together with the reflection formula to extend the functions to $(-1, 1)$, which is their interval of support. Fig. 2 shows the graphs of the functions for $k = 4$ and $k = 5$.

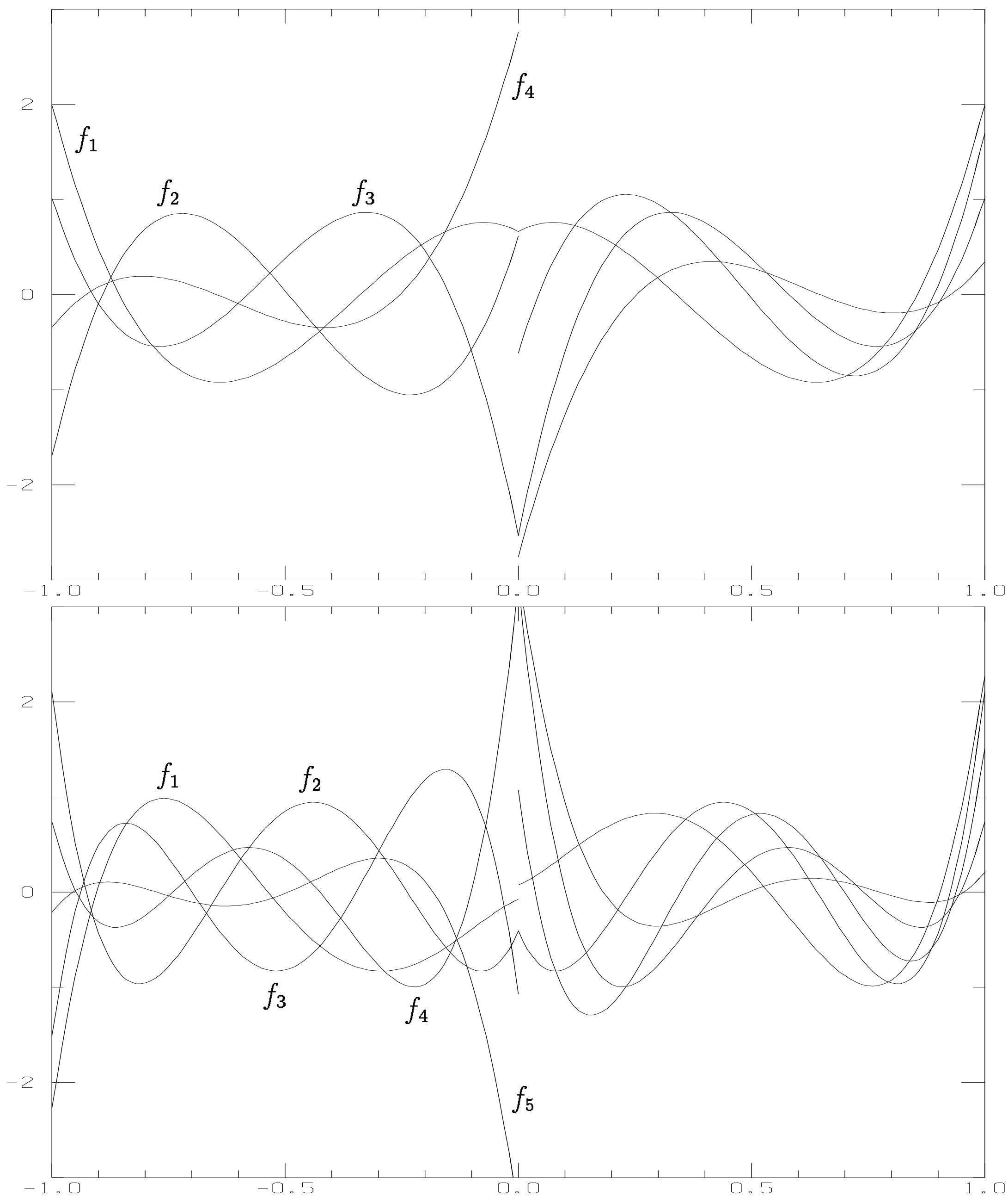

Figure 2: *Functions $f_1, \ldots, f_k$ are graphed for $k = 4$ (top graph) and $k = 5$ (bottom). Each function (given in Table 1) is a polynomial on the interval $(0, 1)$, is an odd or even function on $(-1, 1)$, and is zero elsewhere.*

Table 1: *Expressions for the orthonormal, vanishing-moment functions* $f_1, \ldots, f_k$, *for various* k, *for argument* x *in the interval* $(0,1)$. *The function* f_i *is extended to the interval* $(-1,1)$ *as an odd or even function, according to the formula* $f_i(x) = (-1)^{i+k-1} f_i(-x)$ *for* $x \in (-1,0)$, *and is zero outside* $(-1,1)$.

	$k=1$	
$f_1(x) =$	$\sqrt{\frac{1}{2}}$	
	$k=2$	
$f_1(x) =$	$\sqrt{\frac{3}{2}}$	$(-1+2x)$
$f_2(x) =$	$\sqrt{\frac{1}{2}}$	$(-2+3x)$
	$k=3$	
$f_1(x) =$	$\frac{1}{3}\sqrt{\frac{1}{2}}$	$(1-24x+30x^2)$
$f_2(x) =$	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$(3-16x+15x^2)$
$f_3(x) =$	$\frac{1}{3}\sqrt{\frac{5}{2}}$	$(4-15x+12x^2)$
	$k=4$	
$f_1(x) =$	$\sqrt{\frac{15}{34}}$	$(1+4x-30x^2+28x^3)$
$f_2(x) =$	$\sqrt{\frac{1}{42}}$	$(-4+105x-300x^2+210x^3)$
$f_3(x) =$	$\frac{1}{2}\sqrt{\frac{35}{34}}$	$(-5+48x-105x^2+64x^3)$
$f_4(x) =$	$\frac{1}{2}\sqrt{\frac{5}{42}}$	$(-16+105x-192x^2+105x^3)$
	$k=5$	
$f_1(x) =$	$\sqrt{\frac{1}{186}}$	$(1+30x+210x^2-840x^3+630x^4)$
$f_2(x) =$	$\frac{1}{2}\sqrt{\frac{1}{38}}$	$(-5-144x+1155x^2-2240x^3+1260x^4)$
$f_3(x) =$	$\sqrt{\frac{35}{14694}}$	$(22-735x+3504x^2-5460x^3+2700x^4)$
$f_4(x) =$	$\frac{1}{8}\sqrt{\frac{21}{38}}$	$(35-512x+1890x^2-2560x^3+1155x^4)$
$f_5(x) =$	$\frac{1}{2}\sqrt{\frac{7}{158}}$	$(32-315x+960x^2-1155x^3+480x^4)$

3.2 Integral Operators and Their Inverses

We compute the expansion in multi-wavelet bases of the integral operator $\mathcal{K}$ defined by the formula

$$(\mathcal{K}f)(x) = \int_0^1 \log|x-t|\, f(t)\, dt, \tag{24}$$

which yields the matrix

$$K^{(n)} = \{K_{ij}\}_{i,j=1,\ldots,n},$$

where

$$K_{ij} = \int_0^1 \int_0^1 K(x,t)\, b_i(x)\, b_j(t)\, dx\, dt$$

and $\{b_1, b_2, \ldots\}$ is a multi-wavelet basis of $L^2[0,1]$. We approximate $K^{(n)}$ with a matrix $T^{(n)}$ whose elements are defined by the formula

$$T_{ij}^{(n)} = \begin{cases} K_{ij}, & \text{if } |K_{ij}| \geq \tau, \\ 0, & \text{otherwise}, \end{cases} \tag{25}$$

where the threshold τ is chosen so that a desired precision ϵ is maintained: $\|T^{(n)} - K^{(n)}\| \leq \epsilon \|K^{(n)}\|$. Here the norm $\|\cdot\|$ is the row-sum norm, $\|A\| = \max_i \sum_{j=1}^{n} |A_{ij}|$. The threshold τ is given by $\tau = \epsilon \|K^{(n)}\|/n$. This computation was performed for the multi-wavelet basis of order $k=4$, for various sizes n, as shown in Table 2.

An interesting property of many operators of second-kind integral equations is that their inverses, when they exist, are also sparse in multi-wavelet coordinates (to high precision). The operator $(I-K)^{-1}$ has the Neumann expansion $\sum_{i=0}^{\infty} K^i$, which converges if $\|K\| < 1$; thus $(I-K)^{-1}$ may be approximated to arbitrary precision by a polynomial in K. More generally (regardless of $\|K\|$), $(I-K)^{-1} = A\sum_{i=0}^{\infty}(I-(I-K)A)^i$, where $A = (I-K^H)/\|(I-K^H)(I-K)\|$. The Schulz method [16] (see also [2]), a classical iterative matrix inversion technique, can be used to compute the first 2^m terms of this expansion with m iterations. Analogous to Newton iteration, the mth Schulz iterate X_m to invert a matrix M is given by $X_m = 2X_{m-1} - X_{m-1}\, M\, X_{m-1}$, where $X_0 = M^H/\|M^H M\|$. The iterates satisfy the equation $I - X_m M = (I - X_{m-1}M)^2$, which assures their quadratic convergence to M^{-1}.

The terms K^i and $((I-K^H)(I-K))^i$, of which these expansions are composed, have representations in multi-wavelets which are asymptotically sparse. Specifically, their $n \times n$-matrix representations, after thresholding, contain only order $O(n \log n)$ nonzero elements. This fact follows from arguments similar to those given in Lemmas 2.2 and 2.3. It is important to add, however, that the constants in these asymptotic estimates may not ensure useful sparsity for reasonable values of n.

Table 2: *The average number of elements per row of the matrices $S^{(n)} = I - T^{(n)}$ and $(S^{(n)})^{-1}$, where $T^{(n)}$ is defined in Eq. (25), is tabulated for various precisions ϵ and various sizes n. Here $k=4$.*

	$\epsilon = 10^{-2}$		$\epsilon = 10^{-3}$		$\epsilon = 10^{-4}$	
n	$S^{(n)}$	$(S^{(n)})^{-1}$	$S^{(n)}$	$(S^{(n)})^{-1}$	$S^{(n)}$	$(S^{(n)})^{-1}$
32	8.8	9.7	19.3	19.6	22.8	23.6
64	9.3	10.0	25.8	26.0	31.9	32.6
128	9.9	10.1	29.2	29.4	38.2	38.8
256	11.8	11.8	30.1	30.3	41.9	42.7

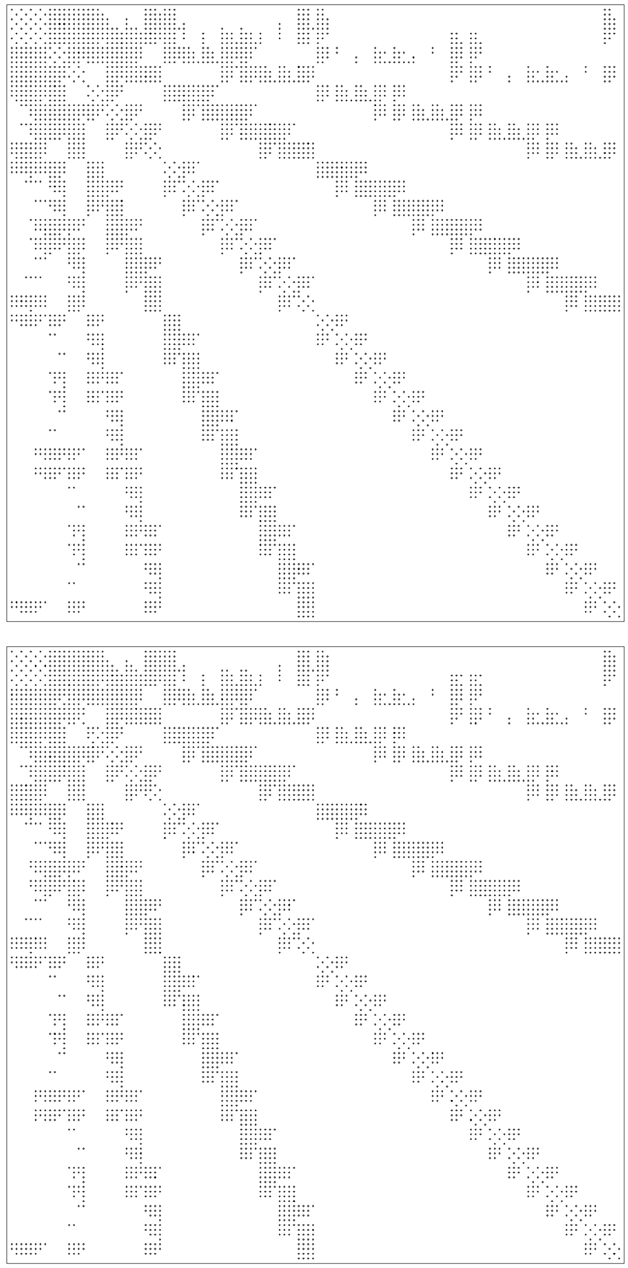

Figure 3: *Matrices representing the operators $I-\mathcal{K}$ (top) and $(I-\mathcal{K})^{-1}$ (bottom), with $\mathcal{K}$ defined by Eq. (24), expanded in the multi-wavelet basis of order $k=4$, for $n=128$. The dots represent elements above a threshold, which is determined so as to bound the relative truncation error at $\epsilon=10^{-3}$.*

For the operator $T^{(n)}$ defined above, the inverse $(I - T^{(n)})^{-1}$ is roughly as sparse as $I - T^{(n)}$. We have computed it by the Schulz method. Table 2 displays, for various precisions ϵ, the average number of elements per row in the matrices $I - T^{(n)}$ and $(I - T^{(n)})^{-1}$. Fig. 3 displays the matrices for $n = 128$ and $\epsilon = 10^{-3}$.

3.3 Discussion

The results of the previous subsection demonstrate, for a particular integral operator, that the multi-wavelet representations are sparse. The matrix has a peculiar structure in which the non-negligible elements are contained in blocks lying along rays emanating from one corner of the matrix. Furthermore, the inverse matrix shares that structure. This property is a general characteristic of integral operators with non-oscillatory kernels that possess diagonal singularities.

The kernel $K(x,t) = \log|x-t|$ of the previous subsection was chosen, however, because the projections K_{ij} could be computed analytically, thereby avoiding use of quadratures. The difficulty here with quadratures is that they would be required for each element K_{ij}, and would have to cope with the singularity of the logarithm. It was felt that the analytical computation would be more efficient. In fact, the analytical computation, which requires integrating monomials x^j $(0 \leq j < k)$ against the logarithm and combining the results with large coefficients, is a very poorly-conditioned procedure. The computations described above required quadruple-precision arithmetic to obtain single-precision accuracy for n as small as 64. This procedure is not recommended.

The fault lies, of course, not with the idea of projection to the multi-wavelet basis, but with the method of projection. The integration should be performed numerically, with quadratures. As mentioned above, such a procedure would require use of quadratures for each matrix element K_{ij}, or potentially order $O(n \log n)$ times. A more efficient procedure is to use the Nyström method, in which only n quadrature applications are required. Numerical quadratures and a vector-space analogue of the multi-wavelet bases are developed in [1],[2]; these tools enable efficient solution of second-kind integral equations using Nyström's method. We believe that the present paper, rather than directly providing numerical tools, offers a particularly simple framework in which to understand the ideas for sparse representation of integral operators.

Acknowledgement The bases constructed in this paper are the limiting case of the discrete construction in [2]; thanks to R. Coifman for prodding this author to consider the limit, which surprised us with its simplicity.

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