NUMERICAL EVALUATION OF SPECIAL FUNCTIONS

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ABSTRACT. This document is an excerpt from the current hypertext version of an article that appeared in Walter Gautschi (ed.), *Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics*, Proceedings of Symposia in Applied Mathematics 48, American Mathematical Society, Providence, RI 02940, 1994. The symposium was held at the University of British Columbia August 9–13, 1993, in honor of the fiftieth anniversary of the journal *Mathematics of Computation*.

The original abstract follows.

Higher transcendental functions continue to play varied and important roles in investigations by engineers, mathematicians, scientists and statisticians. The purpose of this paper is to assist in locating useful approximations and software for the numerical generation of these functions, and to offer some suggestions for future developments in this field.

7. Future Trends

Great progress has been made in recent years in the construction of software for generating the special functions, yet enormous gaps remain for functions having variable parameters in addition to the argument. This is especially true when the variables are complex. In this concluding section we offer some general suggestions concerning future work in this area.

First, because of the sheer magnitude of the effort required, there should be a perceived physical or other applied need before a decision is made to embark on the construction of extensive new software for functions of two or more variables. At present there are simply too many gaps to fill to be able to indulge in the luxury of arbitrary selection. Moreover, great care should be exercised in the choice of actual functions to be generated. For example, neither the Airy function $\mathrm{Bi}(z)$ nor the Bessel function of the second kind $Y_\nu(z)$ has a useful role when the argument z is not real; compare [**Olv74**, Chapters 7 and 11].

Second, coverage of a chosen region should be dictated by uniform accuracy requirements (in an appropriate measure), not by the limitations of the methods that happen to be used. At the very least it is frustrating for users to discover that the precision yielded by a package varies widely, or worse still disappears altogether, in parts of the claimed regions of coverage.

Third, the potential offered by the ongoing increase in power of computers should be exploited with a view to reducing the number and complexity of algorithms to be

[1] /nesf/Sect0/Sect0.html

[2] /nesf/Sect6/Sect6.html

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used. This includes, for example, the use of parallel or vector methods for summing series [**Kar91**] or solving differential or difference equations [**LO93**].

Fourth—and here we are looking further into the future—the use of systems of computer arithmetic other than floating-point should be considered. The floating-point system has two disadvantages which become especially annoying and time-consuming in the construction of special-function software. One is that the associated error measure, relative precision, is quite inappropriate in the neighborhoods of zeros. The other stems from failure due to overflow or underflow: here the usual remedy of rescaling can be difficult to apply, owing to the extremely varied asymptotic behavior of functions of several variables. A system of computer arithmetic that is capable of overcoming both problems in an elegant manner is the so-called level-index system [**COT89**].

Lastly, any new algorithm or package should be documented fully. It should also be subjected to exhaustive testing procedures, and these, too, need to be documented. Indeed, the proposed testing procedures should be considered at an early stage in the planning of the main algorithms[3]. There are so many pitfalls in the construction of algorithms for the special functions that the use of undocumented or insufficiently tested packages is a risky proposition[4].

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[3] For example, it is better to avoid the use of Wronskian and Casoratian relations in the main computing package, if possible, in order to reserve these identities for consistency checks.

[4] For a striking example see [**Olv91**].