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# NUMERICAL EVALUATION OF SPECIAL FUNCTIONS

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ABSTRACT. This document is an excerpt from the current hypertext version of an article that appeared in Walter Gautschi (ed.), *Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics*, Proceedings of Symposia in Applied Mathematics 48, American Mathematical Society, Providence, RI 02940, 1994. The symposium was held at the University of British Columbia August 9–13, 1993, in honor of the fiftieth anniversary of the journal *Mathematics of Computation*.

The original abstract follows.

Higher transcendental functions continue to play varied and important roles in investigations by engineers, mathematicians, scientists and statisticians. The purpose of this paper is to assist in locating useful approximations and software for the numerical generation of these functions, and to offer some suggestions for future developments in this field.

**5.3. Elliptic Integrals and Functions.** An important recent change in the old subject of elliptic integrals is a renormalization of the definitions of the integrals. This is due to B. C. Carlson: references will be found in §5.3.5.

5.3.1. *Complete Elliptic Integrals.*

Algorithms: [**Bel88**], [**Cod65a**], [**Cod65b**], [**Luk69b**].

Software Packages: [**Bul65a**, Algol], [**Bul65b**, Algol], [**Bul69b**, Algol], [**MH73**, Algol].

Intermediate Libraries: [**Bak92**], [**Mos89**], [**ULI90**].

Comprehensive Libraries: **IMSL**, **Numerical Recipes**.

Interactive Systems: **Mathematica**.

5.3.2. *Incomplete Elliptic Integrals.*

Algorithms: [**Luk69b**].

Software Packages: [**Bul65a**, Algol], [**Bul69b**, Algol], [**Car87**, Fortran], [**Car88**, Fortran], [**CN81**, Fortran], [**PT90**, Fortran].

Intermediate Libraries: [**Bak92**], [**Mos89**], [**ULI90**].

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[1] /nesf/Sect0/Sect0.html
[2] /nesf/Sect5/Sect5.html
[3] /nesf/Sect5-2/Sect5-2.html
[4] /nesf/Sect5-4/Sect5-4.html
[5] ftp://gams.nist.gov/pub/lozier/nesf/Paper.ps.Z
[6] ftp://gams.nist.gov/pub/lozier/nesf/Sect5-3.ps.Z

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Comprehensive Libraries: **IMSL**, **NAG**, **Numerical Recipes**, **Scientific Desk**, **SLATEC**.

Interactive Systems: **Mathematica**.

5.3.3. *Jacobi's Elliptic Functions.* This subsection includes the theta functions.

Software Packages: [**Bul65a**, Algol].

Intermediate Libraries: [**Bak92**], [**Mos89**], [**ULI90**].

Comprehensive Libraries: **IMSL**, **NAG**, **Numerical Recipes**.

Interactive Systems: **Mathematica** (includes inverse functions).

5.3.4. *Weierstrass' Elliptic Functions.*

Algorithms: [**Eck76**], [**Eck77**].

Software Packages: [**Eck80**, Fortran].

Intermediate Libraries: [**Bak92**], [**ULI90**].

Comprehensive Libraries: **IMSL**.

Interactive Systems: **Mathematica**.

5.3.5. *Articles.* [**ACJP85**, includes survey], [**Bul69a**], [**Car65**], [**Car77a**], [**Car77b**], [**Car79**], [**Car87**], [**Car88**], [**Car89**], [**Car91**], [**Car92**], [**CGL90**], [**Cri89**], [**FGG82**], [**FL67**], [**Lee90**], [**Lee92**], [**Luk68**], [**Luk70b**], [**LY88**], [**Mid75**], [**NC66**], [**Sal89**], [**War60**].

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