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NUMERICAL EVALUATION OF SPECIAL FUNCTIONS

D. W. LOZIER AND F. W. J. OLVER

ABSTRACT. This document is an excerpt from the current hypertext version of an article that appeared in Walter Gautschi (ed.), *Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics*, Proceedings of Symposia in Applied Mathematics 48, American Mathematical Society, Providence, RI 02940, 1994. The symposium was held at the University of British Columbia August 9–13, 1993, in honor of the fiftieth anniversary of the journal *Mathematics of Computation*.

The original abstract follows.

Higher transcendental functions continue to play varied and important roles in investigations by engineers, mathematicians, scientists and statisticians. The purpose of this paper is to assist in locating useful approximations and software for the numerical generation of these functions, and to offer some suggestions for future developments in this field.

5.1. Bessel Functions. All of the following subsections apply to the ordinary Bessel functions (J and Y) and the modified Bessel functions (I and K).

5.1.1. *Orders 0 and 1, Real Arguments.*

Interactive Systems: [**Bla74**], [**Cle62**], [**Luk69b**], [**Luk75**], [**WBR82**].

Software Packages: [**BS92**, Fortran], [**Hil81**, Fortran].

Intermediate Libraries: [**Bak92**], [**Mos89**].

Comprehensive Libraries: **IMSL**, **NAG**, **Numerical Recipes**, **Scientific Desk**, **SLATEC**.

5.1.2. *Integer or Half-Integer Orders, Real Arguments.* This subsection includes spherical Bessel functions.

Interactive Systems: [**AM61**], [**MM90**], [**PB82**].

Software Packages: [**AM78**, Fortran], [**Col80**, Fortran], [**Hil81**, Fortran], [**RF93**, Fortran].

Intermediate Libraries: [**Bak92**], [**Mos89**], [**ULI90**].

Comprehensive Libraries: **IMSL**, **Numerical Recipes**.

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[3] /nesf/Sect4-7/Sect4-7.html
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[6] ftp://gams.nist.gov/pub/lozier/nesf/Sect5-1.ps.Z

1991 *Mathematics Subject Classification.* Primary 65D20; Secondary 33-00.

The research of the second author has been supported by NSF Grant CCR 89-14933.

5.1.3. *Real Orders, Real Arguments.*
Interactive Systems: [**CP66**], [**Luk69b**], [**Luk71a**], [**Luk71b**], [**Luk72a**], [**Luk75**].
Software Packages: [**ADW77a**, Fortran], [**Bar82b**, Fortran], [**Cam79**, Fortran], [**Cod83**, Fortran], [**Mat93b**, Fortran], [**Pie84b**, Fortran], [**Tem75**, Algol], [**Tem76**, Algol].
Intermediate Libraries: [**Mos89**], [**ULI90**].
Comprehensive Libraries: **IMSL**, **Numerical Recipes**, **Scientific Desk**, **SLATEC**.
Interactive Systems: **Maple**.
5.1.4. *Integer or Half-Integer Orders, Complex Arguments.* This subsection includes Kelvin functions.
Interactive Systems: [**Bur63**], [**CM83**].
Software Packages: [**BKN88a**, Fortran], [**BKN88b**, Fortran],
Intermediate Libraries: [**Bak92**], [**ULI90**].
Comprehensive Libraries: **IMSL**, **NAG**.
5.1.5. *Real Orders, Complex Arguments.*
Interactive Systems: [**Luk69b**], [**Luk75**].
Software Packages: [**Amo86**, Fortran], [**Cam81**, Fortran], [**TB87**, Fortran]. **IMSL**,
Comprehensive Libraries: **NAG**, **Scientific Desk**, **SLATEC**.
5.1.6. *Complex Orders, Complex Arguments.*
Software Packages: [**TB85**, Fortran].
Interactive Systems: **Mathematica**.
5.1.7. *Integrals of Bessel Functions.*
Interactive Systems: [**BEJ78**], [**GP64**].
Software Packages: [**Amo83a**, Fortran], [**And82a**, Fortran], [**Cha83**, Fortran], [**Feu91**, Fortran], [**PB84**, Fortran], [**Pie82**, Fortran], [**SZ79**, Fortran], [**Tal83**, Fortran].
Intermediate Libraries: [**Bak92**].
Comprehensive Libraries: **SLATEC**.
5.1.8. *Zeros of Bessel Functions.*
Interactive Systems: [**Pie84a**].
Software Packages: [**Cam84**, Fortran], [**Pie90**, Fortran], [**Tem79**, Algol].
Intermediate Libraries: [**Bak92**].
5.1.9. *Articles—Functions.* [**Ach86**], [**ADW77b**], [**Amo74**], [**Bar81a**], [**BGV93**], [**Cam80**], [**CF87**], [**CMF77**], [**Cod80**, includes survey], [**Col87b**], [**CS89**, includes survey], [**Gau91b**], [**GB87**], [**GS78**], [**Hit68**], [**KS84b**], [**Luk72b**], [**Luk77b**], [**Nes84**], [**OS72**], [**Rem73**], [**TB86**], [**VGK+91**], [**Wal84**], [**WC90**], [**YN74**], [**Yos92**].
5.1.10. *Articles—Integrals.* [**Amo83c**], [**And82b**], [**BFST86**], [**BGV93**], [**Can81**], [**Chr90**], [**Cof91**], [**Cor72**], [**DK90**], [**Gab79**], [**Gab80**], [**GM81**], [**Han85**], [**Joh75**], [**Lew91**], [**Lin72**], [**LK73**], [**LPM81**], [**Lun85**], [**MDS92**], [**Moo83**], [**OFM78**], [**PB82**], [**PB83**], [**PB85**, includes survey], [**Puo88**], [**SBK92**], [**Sie77**].
5.1.11. *Articles—Zeros.* [**CH70a**], [**IKF91**], [**KS84a**], [**KS84c**], [**KS85a**], [**KS85b**], [**KS85c**], [**KS87**], [**MF86**], [**Sko85**].

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Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, Md 20899

E-mail address: dlozier@nist.gov

Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742

E-mail address: olver@bessel.umd.edu