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# NUMERICAL EVALUATION OF SPECIAL FUNCTIONS


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ABSTRACT. This document is an excerpt from the current hypertext version of an article that appeared in Walter Gautschi (ed.), *Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics*, Proceedings of Symposia in Applied Mathematics 48, American Mathematical Society, Providence, RI 02940, 1994. The symposium was held at the University of British Columbia August 9–13, 1993, in honor of the fiftieth anniversary of the journal *Mathematics of Computation*.

The original abstract follows.

Higher transcendental functions continue to play varied and important roles in investigations by engineers, mathematicians, scientists and statisticians. The purpose of this paper is to assist in locating useful approximations and software for the numerical generation of these functions, and to offer some suggestions for future developments in this field.


## 2. Mathematical Developments

Comprehensive compendia of mathematical properties of the special functions are provided by the National Bureau of Standards' Handbook of Mathematical Functions [**AS64**], published originally in 1964, and the 3-volume set that resulted from the Bateman Manuscript Project [**EMOT53a,EMOT53b,EMOT55**], published in 1953 and 1955. These references employ the same notation for the special functions, and we shall follow them. The NBS Handbook has been reprinted many times by the U. S. Government Printing Office and has also been issued in whole, or in part, by other publishers including Dover Publications, Moscow Nauka, Verlag Harri Deutsch and Wiley-Interscience.

The forerunner of [**AS64**] is the book of Jahnke and Emde [**JE45**], published originally in 1909, and still in print[6]. It continues to be useful, especially for its collection of graphs. Other useful compendia include those of Magnus, Oberhettinger and Soni [**MOS66**], and (from the standpoint of hypergeometric functions) Luke [**Luk69a**]. For an introductory compendium, see the recent "atlas" of Spanier and Oldham [**SO87**].

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[1] /nesf/Sect0/Sect0.html
[2] /nesf/Sect1/Sect1.html
[3] /nesf/Sect3/Sect3.html
[4] ftp://gams.nist.gov/pub/lozier/nesf/Paper.ps.Z
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[6] A more recent edition, with F. Lösch added as author [**JEL60**], is no longer in print.

Books and articles that include descriptions or surveys of general methods for computing special functions include [**Bre78b,DKK81,Gau75,HCL+68,Luk69b,Luk77b,PT84,PTVF92,T**

Other books and articles that provide indepth coverage of pertinent topics include:

[**Ask89**, survey of compendia], [**BG81a,BG81b**, Padé approximations], [**BH75**, asymptotic approximations], [**Bre91**, continued fractions, Padé approximations], [**BvI93**, Padé approximations], [**Cod70**, polynomial and rational approximations], [**Fik68**, polynomial and rational approximations], [**FP68**, Chebyshev polynomials], [**JT80**, continued fractions], [**Kar91**, power series], [**KG80**, statistical computations], [**Luk75**, supplement to **AS64**—especially for functions of hypergeometric type], [**Mor80**, power series], [**Olv74**, asymptotic approximations], [**Riv90**, Chebyshev polynomials], [**Tem77**, integral representations], [**Tem85**, asymptotic approximations], [**Wim84**, recurrence relations], [**Won89**, asymptotic approximations].

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