AN EXPECTED LINEAR 3-DIMENSIONAL VORONOI DIAGRAM ALGORITHM

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ABSTRACT

Let S be a set of n sites chosen independently from a uniform distribution in a cube in $3-$dimensional Euclidean space. In this paper, an expected $O(n)$ algorithm for constructing the Voronoi diagram for S together with numerical results obtained from an implementation of the algorithm are presented.

1. INTRODUCTION

Consider a set $S = \{p_1, \ldots, p_n\}$ of n points (to be called *sites* in the following) in the Euclidean space E^3, and let $d(\cdot,\cdot)$ denote the Euclidean distance. The *Voronoi diagram for* S is a sequence $V(p_1), \ldots, V(p_n)$ of convex polyhedra covering E^3, where for each i, $i = 1, \ldots, n$, $V(p_i)$, the *Voronoi polyhedron of* p_i *relative to* S, is defined by

$$V(p_i) \equiv \cap_{j=1, j\neq i}^{n} \{p \in E^3 \ : \ d(p,p_i) \leq d(p,p_j)\}.$$

The Voronoi diagram has played an important role in computational geometry for a long time, and several algorithms have been devised and implemented for constructing it in two and higher dimensions (see Bentley, Weide and Yao (1980), Bowyer (1981), Brostow, Dussault and Fox (1978), Brown (1979), Dwyer (1988), Finney (1979), Green and Sibson (1978), Lee and Schachter (1980), Maus (1984), Ohya, Iri and Murota (1984), Seidel (1986), Shamos (1978), Shamos and Hoey (1975), Tanemura, Ogawa and Ogita (1983), Watson (1981), Witzgall (1973)).

Assume the sites in S have been chosen independently from a uniform distribution in a $3-$dimensional cube. In this paper we present an expected $O(n)$

algorithm for constructing the Voronoi diagram for S that is a consequence of proofs and results in the companion paper Bernal (1990). Numerical results obtained from a Fortran implementation of the algorithm are also presented.

2. TERMINOLOGY

Let $S = \{p_1, \ldots, p_n\}$ be a set of n sites in E^3 chosen independently from a uniform distribution in a cube R. With m defined as the *floor* of $n^{1/3}$, i. e. the largest integer less than or equal to $n^{1/3}$, assume as in Bentley, et al. (1980) that R has been divided into m^3 equal-sized cells. Given a site q, define *the* 1^{st} *layer of cells that surrounds* q as the collection of cells that contain q. Inductively, given $k \geq 1$, assume that the k^{th} layer of cells that surrounds q has been defined. Define *the* $(k+1)^{th}$ *layer of cells that surrounds* q as the collection, possibly empty, of cells that have one or more points in common with cells in the k^{th} layer, and that do not belong to the first k layers.

Let *lcell* and *vcell* represent, respectively, the length and volume of each cell.

Given numbers c, c', c'', $0 < c \leq c'$, $c'' \geq 1$, define $\mathrm{LG}(n)$ and $\mathrm{LG}'(n)$ as the floors of $c \cdot \log n$ and $c' \cdot \log n$, respectively, and assume n is large enough so that $\mathrm{LG}(n) > 2$ and $2^{3/2} \cdot c'' \cdot \mathrm{LG}'(n) \leq 2^{-1} \cdot n^{1/3}$.

Let $\hat{k}$ denote the largest integer k for which

$$2^{k/2} \cdot c'' \cdot \mathrm{LG}'(n) \leq 2^{-1} \cdot n^{1/3}.$$

It follows from the assumptions on n that $\hat{k} \geq 3$.

Set $\mathrm{LG}_0(n)$ equal to $\mathrm{LG}(n)$, and $\mathrm{LG}_k(n)$ equal to $\mathrm{LG}'(n)$ for each k, $k = 1, \ldots, \hat{k} - 2$.

Let f_i, $i = 1, \ldots, 6$, represent the facets of R, and let Π denote $\cup_{i=1}^{6} f_i$, i. e. the boundary of R.

Given a point x in E^3 and a closed subset W of E^3, define $\mathrm{dist}(x, W)$ as the minimum value of $||x - w||$ for w in W, where $||\cdot||$ represents the $3-$dimensional Euclidean norm.

From the assumptions on n, several nonempty subsets of R can be defined as follows:

$$\begin{aligned}
R_{-1} &\equiv \{x \in R \ : \ \mathrm{dist}(x, \Pi) \geq lcell \cdot \mathrm{LG}(n)\}. \\
R_0 &\equiv \{x \in R \ : \ lcell \cdot 2 \leq \mathrm{dist}(x, \Pi) < lcell \cdot \mathrm{LG}(n)\}. \\
R_{\hat{k}} &\equiv \{x \in R \ : \ \mathrm{dist}(x, \Pi) < lcell \cdot 2^{-\hat{k}+2}\}.
\end{aligned}$$

For each k, $k = 1, \ldots, \hat{k} - 1$,

$$R_k \ \equiv \ \{x \in R \ : \ lcell \cdot 2^{-k+1} \leq \mathrm{dist}(x, \Pi) < lcell \cdot 2^{-k+2}\}.$$

For each i, k, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$,

$$R_k^i \ \equiv \ \{x \in R_k \ : \ \mathrm{dist}(x, f_j) \geq lcell \cdot 2^{k/2} \cdot c'' \cdot \mathrm{LG}_k(n), \ j = 1, \ldots, 6, \ j \neq i\}.$$

It follows from these definitions that the sets R_k, $k = -1, \ldots, \hat{k}$, are pair-wise disjoint nested regions of the cube R, and

$$R = \cup_{k=-1}^{\hat{k}} R_k.$$

The significance of these regions for our purposes can be summarized as follows. R_{-1} is essentially that region of the cube R obtained by subtracting the outermost $\mathrm{LG}(n)$ layers of cells of R from R. From Bentley, et al. (1980), the Voronoi polyhedron of a site in R_{-1} can be constructed in expected constant time. R_0 is essentially that region of R obtained by subtracting from the outermost $\mathrm{LG}(n)$ layers of cells of R the outermost two layers. R_k, $k = 1, \ldots, \hat{k}$, are regions of R whose union is essentially that region of R composed of the outermost two layers of cells of R, and whose thicknesses correspond to the terms of the geometric series expanded to the first $\hat{k} - 1$ terms together with the remainder. R_k^i, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, are subsets of R_k, $k = 0, \ldots, \hat{k} - 2$, respectively, defined in such a way that as intimated in Bernal (1990), due to their positions relative to the boundary of R and the geometric series aspect of R_k, $k = 1, \ldots, \hat{k} - 2$, for a properly selected value of c'' the Voronoi polyhedra of sites in these regions can be constructed in expected linear time. They are also defined in such a way that due to the definitions of $\hat{k}$, $R_{\hat{k}-1}$ and $R_{\hat{k}}$, and the geometric series aspect of R_k, $k = 1, \ldots, \hat{k}$, the expected number of sites in $\cup_{k=0}^{\hat{k}} R_k \setminus \cup_{i=1}^{6} \cup_{k=0}^{\hat{k}-2} R_k^i$ is small enough that the Voronoi polyhedra of these sites can also be constructed in expected linear time even under the worst possible circumstances.

Given a site q in R_{-1}, let v, v', v'' and v''' be vertices of R for which $v' - v$, $v'' - v$ and $v''' - v$ are all perpendicular to one another, and for each j, $j = 0, \ldots, 8$, and each m, $m = 0, \ldots, 4$, define a point r_{jm} by

$$\begin{aligned}
r_{jm} \ \equiv \ & q + ((v' - v) \cdot \cos(j\pi/4) + (v'' - v) \cdot \sin(j\pi/4)) \cdot \sin(m\pi/4) \\
& + (v''' - v) \cdot \cos(m\pi/4).
\end{aligned}$$

In addition, for each j, $j = 1, \ldots, 8$, and each m, $m = 1, \ldots, 4$, let U_{jm} be the cone that is the convex hull of the rays $q\vec{r}_{j-1,m-1}$, $q\vec{r}_{j,m-1}$, $q\vec{r}_{j-1,m}$, and $q\vec{r}_{jm}$, and if within the first $\mathrm{LG}(n)$ layers of cells that surround q, for each j, $j = 1, \ldots, 8$, and each m, $m = 1, \ldots, 4$, there exists a site s_{jm}, $s_{jm} \neq q$, such that s_{jm} belongs to U_{jm}, say that q is *closed* and that s_{jm}, $j = 1, \ldots, 8$, $m = 1, \ldots, 4$, *render* q *closed.* As shown in Bentley, et al. (1980), the Voronoi polyhedron of a closed site can be constructed in expected constant time.

For each facet f of R, let $H(f)$ represent the plane that contains f, and for each site q, let $T^f(q)$ represent the point in f that is the perpendicular projection of q onto f.

Given i, k, $1 \leq i \leq 6$, $0 \leq k \leq \hat{k} - 2$, and a site q in R^i_k, let v, v' and v'' be vertices of R in f_i for which $v' - v$ is perpendicular to $v'' - v$, and for each j, $j = 0, \ldots, 8$, define a point t_j in $H(f_i)$ by

$$t_j \equiv T^{f_i}(q) + (v' - v) \cdot \cos(j\pi/4) + (v'' - v) \cdot \sin(j\pi/4).$$

In addition, for each j, $j = 1, \ldots, 8$, let O_j be the octant in $H(f_i)$ that is the convex hull of the rays $T^{f_i}(q)\vec{t}_{j-1}$ and $T^{f_i}(q)\vec{t}_j$, and if within the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, for each j, $j = 1, \ldots, 8$, there exists a site q_j such that $\mathrm{dist}(q_j, f_i) < lcell \cdot 2^{-k}$ and the ray $q\vec{q}_j$ intersects O_j, say that q is *octant-closed* and that q_j, $j = 1, \ldots, 8$, *render* q *octant-closed.*

Given i, k, q, v, v', v'' as above, let v''' be a vertex of R for which $v''' - v$ is perpendicular to $v' - v$ and $v'' - v$, and for each j, $j = 0, \ldots, 8$, and each m, $m = 2, 3$, define a point r_{jm} by

$$\begin{aligned} r_{jm} \;\; \equiv \;\; & q + ((v' - v) \cdot \cos(j\pi/4) + (v'' - v) \cdot \sin(j\pi/4)) \cdot \sin(m\pi/4) \\ & + (v''' - v) \cdot \cos(m\pi/4). \end{aligned}$$

In addition, for each j, $j = 1, \ldots, 8$, let U_j be the cone that is the convex hull of the rays $q\vec{r}_{j-1,2}$, $q\vec{r}_{j2}$, $q\vec{r}_{j-1,3}$, and $q\vec{r}_{j3}$, and if within the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, for each j, $j = 1, \ldots, 8$, there exists a site s_j, $s_j \neq q$, such that s_j belongs to U_j, say that q is *cone-semiclosed* and that s_j, $j = 1, \ldots, 8$, *render* q *cone-semiclosed.*

Given q as above, say that q is *semiclosed* if it is octant-closed and cone-semiclosed. As intimated in Bernal (1990), for a properly selected value of c'' the construction of Voronoi polyhedra of semiclosed sites is of expected complexity acceptable for our purposes.

Finally, given sites p and q, say that p is a *Voronoi neighbor of q relative to S* if $V(p)$ and $V(q)$ have a facet in common.

3. THE ALGORITHM

In this section we present the algorithm in the form of a procedure called VORNOI. The algorithm and its expected complexity follow from proofs and results in the companion paper Bernal (1990).

Essentially, the algorithm consists of three steps. Let n, S, R, R_{-1}, R_k^i, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, be as defined in the previous section. In the first step, the Voronoi polyhedra of sites in R_{-1} are constructed as suggested in Bentley, et al. (1980). Given a site in R_{-1}, a geometrical procedure is available for constructing in expected constant time the Voronoi polyhedron of the site. Thus, the first step of the algorithm has expected linear complexity. In the second step, the Voronoi polyhedra of sites in R_k^i, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, are constructed as intimated in Bernal (1990). Given a site in $\cup_{i=1}^{6} \cup_{k=0}^{\hat{k}-2} R_k^i$, a geometrical procedure that generalizes the one used in the first step is available for obtaining a subset of S that contains all of the Voronoi neighbors relative to S of the site. This is done in such a way that as implied in Bernal (1990), the expected time involved in obtaining all such subsets for all such sites is bounded above by $O(n^{2/3} \cdot (\log n)^4))$. Thus, since an $O(k \cdot \log k)$ procedure is also available for computing the intersection of k half-spaces in $3-$dimensional space (see Preparata and Muller (1979)), a computation can be carried out that shows that the second step of the algorithm has at most expected $O(n^{2/3} \cdot (\log n)^5)$ complexity. Finally, in the third step, the Voronoi polyhedra of sites in $R \setminus ((\cup_{i=1}^{6} \cup_{k=0}^{\hat{k}-2} R_k^i) \cup R_{-1})$ are constructed. As shown in Bernal (1990), a procedure is available for obtaining for each site in this region a subset of S that contains all of the Voronoi neighbors relative to S of the site. This is done in such a way that as implied in Bernal (1990), the expected time involved in obtaining all such subsets for all such sites is bounded above by $O(n^{2/3} \cdot (\log n)^4)$. Thus, since the $O(k \cdot \log k)$ procedure used in the second step for computing the intersection of k half-spaces is also available in this step, it can be shown in a manner similar to the one used for the second step that the third step of the algorithm has also at most expected $O(n^{2/3} \cdot (\log n)^5)$ complexity. Therefore, the entire algorithm has expected linear complexity.

In the following we list and describe, in the order of their first appearance in procedure VORNOI, functions and procedures used as primitives in that procedure.

FLOOR(x): For a positive real number x computes the largest integer less than or equal to x.

PRTION(R, R', m, P): Creates a partition of a cube R into m^3 equal-sized cells, and then reduces it to those cells that intersect a region R' of R. P will contain the data structure that describes the reduced partition.

CEASGN(S, P, A): Using data structure in P obtained from PRTION procedure, assigns each site in a set S to a cell that contains the site in the partition associated with P, and for each cell in the partition creates a list of those sites assigned to the cell. The corresponding data structure will be contained in A.

RGASGN(S, R', S', n', B'): Locates and orders those sites in a set S that are contained in a region R'. S' will be the set of ordered sites, n' will be the number of sites in S', and for each h, $h = 1, \ldots, n'$, $B'(h)$ will be the h^{th} site in S'.

CLTEST$(P, A, q, \mathrm{LG}(n), flag, Q)$: Using data structures in P and A obtained from procedures PRTION and CEASGN, tests whether a site q (assumed to be in R_{-1}) is closed. The test consists of searching at most the first LG(n) layers of cells that surround q in the partition associated with P and A for sites s_{jm}, $j = 1, \ldots, 8$, $m = 1, \ldots, 4$, assigned to cells in these layers that render q closed. As soon as q is found to be closed $flag$ is set equal to 1 and sites s_{jm}, $j = 1, \ldots, 8$, $m = 1, \ldots, 4$, that render q closed are placed in Q. Otherwise after LG(n) layers have been searched and q has not been found to be closed $flag$ is set equal to zero.

POLYHD(q, Q, V): Given a set Q of sites and a site q, constructs the Voronoi polyhedron V of q relative to $Q \cup \{q\}$ through an $O(k \cdot \log k)$ worst-case algorithm for constructing the intersection of k half-spaces (see e. g. Preparata and Muller (1979)).

BNDIST(q, V, d): Computes the maximum distance d, possibly infinite, from a site q to the boundary of a polyhedron V.

SEARCH(P, A, q, d, Q): Using data structures in P and A obtained from PRTION and CEASGN procedures, given a site q searches layers of cells that surround q in the partition associated with P and A for sites assigned to cells in these layers within a distance d from q. Q will contain the sites found during this search.

VNEISV(V, q, S, N): Given a site q in a set S and a polyhedron V such that V is the Voronoi polyhedron of q relative to S, identifies from V those sites in S that are Voronoi neighbors relative to S of q. On input N will contain for each site in S a list, possibly empty, of known Voronoi neighbors relative to S of the site obtained from previous executions of VNEISV. During the execution of VNEISV, N will be updated so that on output for each site that is a Voronoi neighbor relative to S of q, q will appear in the list of known Voronoi neighbors relative to S of the site.

VNEIGT(q, S, N, Q'): Given a site q in a set S, and N as described for VNEISV, produces from N a set Q' that will contain the known Voronoi neighbors relative to S of q, if any, since the last execution of VNEISV.

SCTEST$(P, A, q, \mathrm{LG}_k(n), H(f_i), flag, Q'', Q''')$: Using data structures in P and A obtained from procedures PRTION and CEASGN, tests whether a site q (assumed to be in R_k^i) is semiclosed. The test consists of searching at most the first $\mathrm{LG}_k(n)$ layers of cells that surround q in the partition associated with P and A for sites q_j, s_j, $j = 1, \ldots, 8$, assigned to cells in these layers that render q octant-closed and cone-semiclosed, respectively. As soon as q is found to be semiclosed $flag$ is set equal to 1, sites s_j, $j = 1, \ldots, 8$, that render q cone-semiclosed are placed in Q'', and points q_j', $j = 1, \ldots, 8$, are placed in Q''', where for each j, $j = 1, \ldots, 8$, q_j' is the intersection of $q\vec{q_j}$ and $H(f_i)$, where q_j, $j = 1, \ldots, 8$, are sites that render q octant-closed. Otherwise after $\mathrm{LG}_k(n)$ layers have been searched and q has not been found to be semiclosed $flag$ is set equal to zero.

HALFSP(q, H, C): For a site q and a plane H, $q \notin H$, computes the closed half-space C that contains q and that is determined by the plane parallel to H that contains $(T(q) + q)/2$, where $T(q)$ is the point in H that is the perpendicular projection of q onto H.

MAXDST(q, Q''', d''): Given a site q, and a finite set of points Q''', computes the maximum distance d'' between q and the points in Q'''.

MAXVAL(d', d''): Computes the maximum of two numbers d' and d''.

The outline of VORNOI follows. Here T is the output variable. For each h, $h = 1, \ldots, n$, if in some ordering of S, q_h is the h^{th} site in S then $T(q_h)$ will be the Voronoi polyhedron of q_h relative to S. All other arguments act as input variables and are as defined in the previous section.

procedure VORNOI$(S, R, n, \hat{k}, \mathrm{LG}(n), \mathrm{LG}_0(n), \ldots, \mathrm{LG}_{\hat{k}-2}(n),$
$R_{-1}, H(f_1), \ldots, H(f_6), R_0^1, \ldots, R_0^6, \ldots, R_{\hat{k}-2}^1, \ldots, R_{\hat{k}-2}^6, T)$

begin
$m := \text{FLOOR}(n^{1/3})$;
$\text{PRTION}(R, R, m, P)$;
$\text{CEASGN}(S, P, A)$;
$\text{RGASGN}(S, R_{-1}, S_{-1}, n_{-1}, B_{-1})$;
for $h := 1$ **until** n_{-1} **do**
    begin
    $q_h := B_{-1}(h)$
    $\text{CLTEST}(P, A, q_h, \text{LG}(n), flag, Q)$;
    if $(flag = 1)$ **then**
        begin
        $\text{POLYHD}(q_h, Q, V)$;
        $\text{BNDIST}(q_h, V, d)$;
        $d := 2 \cdot d$;
        $\text{SEARCH}(P, A, q_h, d, Q)$
        end
    else $Q := S \setminus \{q_h\}$
    $\text{POLYHD}(q_h, Q, V)$;
    $\text{VNEISV}(V, q_h, S, N)$;
    $T(q_h) := V$
    end
$S' := S \setminus S_{-1}$;
$R' := R \setminus R_{-1}$;
for $k := 0$ **until** $\hat{k} - 2$ **do**
    begin
    $m := \text{FLOOR}(2^{-k/2} \cdot n^{1/3})$;
    $\text{PRTION}(R, R', m, P)$;
    $\text{CEASGN}(S', P, A)$;
    for $i := 1$ **until** 6 **do**
        begin
        $\text{RGASGN}(S', R_k^i, S_k^i, n_k^i, B_k^i)$;
        for $h := 1$ **until** n_k^i **do**
            begin
            $q_h := B_k^i(h)$;
            $\text{VNEIGT}(q_h, S, N, Q')$;
            $\text{SCTEST}(P, A, q_h, \text{LG}_k(n), H(f_i), flag, Q'', Q''')$;
            if $(flag = 1)$ **then**
                begin
                $Q := Q' \cup Q''$;
                $\text{POLYHD}(q_h, Q, V)$;
                $\text{HALFSP}(q_h, H(f_i), C)$;
                $V := V \cap C$;

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            BNDIST(q_h, V, d');
            d' := 2 · d';
            MAXDST(q_h, Q''', d'');
            d'' := √2 · d'';
            d := MAXVAL(d', d'');
            SEARCH(P, A, q_h, d, Q);
            Q := Q ∪ Q'
            end
        else Q := (S' ∪ Q') \ {q_h}
        POLYHD(q_h, Q, V);
        VNEISV(V, q_h, S, N);
        T(q_h) := V
        end
    end
  S' := S' \ ∪_{i=1}^{6} S_k^i
  end
R' = R' \ ∪_{i=1}^{6} ∪_{k=0}^{k̂-2} R_k^i;
RGASGN(S', R', S', n', B');
for h := 1 until n' do
  begin
  q_h := B'(h);
  VNEIGT(q_h, S, N, Q');
  Q := (S' ∪ Q') \ {q_h};
  POLYHD(q_h, Q, V);
  VNEISV(V, q_h, S, N);
  T(q_h) := V
  end
end
```

4. NUMERICAL RESULTS

A Fortran implementation of the algorithm has been developed on a Control Data Cyber 205 at the National Institute of Standards and Technology. Table 1 shows the computing time per site in CPU seconds for the implementation when applied to eight randomly generated sets in a cube for 30 values of n. Table 2 shows the number of $0-$dimensional faces per site of the Voronoi diagrams that were obtained with the implementation for the same sets and values of n. We note that the numerical results in Table 1 and Table 2 seem to confirm our theoretical results. We note with interest from the results in Table 2 that the expected number of $0-$dimensional faces per site of a $3-$dimensional Voronoi diagram seems to be increasing very slowly as n increases but appears to be bounded above by the expected number of

0−dimensional faces per site of a 3−dimensional Poisson-Voronoi tessellation (approximately 6.768) (see Miles (1970)). Finally, we note that in the implementation of the algorithm the constants c, c', c'' used in the definitions of Section 2 were all set equal to 1. However, the implementation has been written so that it functions essentially as if they had been set equal to those values that render the implementation the most efficient. For example, the implementation has been written so that procedure CLTEST is also executed for sites in $R \setminus R_{-1}$ during the construction of their Voronoi polyhedra. Doing this is essentially equivalent to enlarging R_{-1} to a region that renders the implementation the most efficient which in turn is equivalent to setting c equal to that value that produces the same effect.

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n	*set 1*	*set 2*	*set 3*	*set 4*	*set 5*	*set 6*	*set 7*	*set 8*
8^3	0.2303	0.2452	0.2285	0.2457	0.2271	0.2275	0.2275	0.2049
9^3	0.2750	0.2770	0.2702	0.2835	0.2691	0.2751	0.2599	0.2524
10^3	0.3057	0.3175	0.2835	0.3177	0.3011	0.3034	0.2973	0.2876
11^3	0.3147	0.3243	0.2953	0.3158	0.3099	0.3098	0.3219	0.3218
12^3	0.3331	0.3347	0.3230	0.3347	0.3624	0.3028	0.3027	0.3262
13^3	0.3540	0.3393	0.3554	0.3558	0.3447	0.3663	0.3243	0.3324
14^3	0.3505	0.3543	0.3537	0.3739	0.3614	0.3904	0.3342	0.3481
15^3	0.3464	0.3569	0.3651	0.3622	0.3523	0.3773	0.3467	0.3592
16^3	0.3483	0.3550	0.3769	0.3636	0.3359	0.3760	0.3479	0.3650
17^3	0.3477	0.3523	0.3787	0.3608	0.3485	0.3736	0.3429	0.3654
18^3	0.3649	0.3569	0.3596	0.3711	0.3530	0.3528	0.3415	0.3650
19^3	0.3462	0.3521	0.3558	0.3586	0.3483	0.3447	0.3509	0.3473
20^3	0.3555	0.3437	0.3524	0.3623	0.3491	0.3378	0.3408	0.3481
21^3	0.3555	0.3544	0.3548	0.3531	0.3540	0.3430	0.3464	0.3480
22^3	0.3601	0.3553	0.3523	0.3447	0.3591	0.3388	0.3456	0.3432
23^3	0.3601	0.3526	0.3560	0.3438	0.3517	0.3297	0.3346	0.3432
24^3	0.3523	0.3466	0.3561	0.3409	0.3425	0.3290	0.3353	0.3442
25^3	0.3470	0.3391	0.3467	0.3346	0.3379	0.3206	0.3282	0.3368
26^3	0.3431	0.3430	0.3447	0.3270	0.3394	0.3105	0.3321	0.3308
27^3	0.3359	0.3428	0.3322	0.3306	0.3361	0.3189	0.3270	0.3263
28^3	0.3316	0.3377	0.3366	0.3225	0.3291	0.3150	0.3210	0.3272
29^3	0.3263	0.3339	0.3179	0.3173	0.3276	0.3116	0.3202	0.3208
30^3	0.3304	0.3224	0.3273	0.3184	0.3235	0.3107	0.3208	0.3220
31^3	0.3082	0.3243	0.3264	0.3153	0.3191	0.3156	0.3177	0.3188
32^3	0.3148	0.3139	0.3079	0.3030	0.3151	0.3259	0.3064	0.3226
33^3	0.2988	0.3300	0.3000	0.3096	0.2982	0.3036	0.3064	0.3065
36^3	0.3060	0.3040	0.3003	0.3102	0.2941	0.3037	0.2981	0.3072
39^3	0.2856	0.2871	0.2904	0.2953	0.2939	0.2908	0.2920	0.2957
42^3	0.2863	0.2813	0.2822	0.2891	0.2842	0.2866	0.2901	0.2774
48^3	0.2740	0.2664	0.2676	0.2759	0.2686	0.2654	0.2667	0.2648

Table 1: Computing time per site.

n	*set 1*	*set 2*	*set 3*	*set 4*	*set 5*	*set 6*	*set 7*	*set 8*
8^3	6.4043	6.3301	6.3496	6.3398	6.3789	6.4141	6.4141	6.3555
9^3	6.4595	6.3937	6.4033	6.4719	6.4225	6.4170	6.4115	6.4472
10^3	6.4710	6.4430	6.4610	6.4830	6.4610	6.4810	6.4440	6.5030
11^3	6.4936	6.5177	6.5297	6.5229	6.5289	6.5289	6.5177	6.5177
12^3	6.5758	6.5336	6.5706	6.5336	6.5156	6.5538	6.5538	6.5752
13^3	6.5772	6.5576	6.5544	6.5603	6.5872	6.5284	6.5690	6.5899
14^3	6.6148	6.5911	6.5652	6.6020	6.6148	6.6017	6.5860	6.6323
15^3	6.6216	6.6071	6.5609	6.6406	6.6308	6.6139	6.5961	6.6394
16^3	6.6394	6.5989	6.5972	6.6492	6.6316	6.6265	6.6445	6.6316
17^3	6.6585	6.6237	6.6076	6.6304	6.6381	6.6507	6.6640	6.6332
18^3	6.6408	6.6502	6.6487	6.6476	6.6626	6.6619	6.6493	6.6408
19^3	6.6553	6.6615	6.6545	6.6673	6.6602	6.6620	6.6606	6.6434
20^3	6.6583	6.6700	6.6524	6.6546	6.6621	6.6640	6.6610	6.6574
21^3	6.6622	6.6656	6.6628	6.6754	6.6758	6.6553	6.6634	6.6589
22^3	6.6767	6.6697	6.6595	6.6657	6.6802	6.6828	6.6665	6.6686
23^3	6.6869	6.6815	6.6615	6.6667	6.6763	6.6822	6.6694	6.6734
24^3	6.6811	6.6736	6.6748	6.6763	6.6778	6.6768	6.6782	6.6823
25^3	6.6797	6.6807	6.6799	6.6768	6.6733	6.6825	6.6845	6.6798
26^3	6.6758	6.6912	6.6880	6.6804	6.6912	6.6925	6.6907	6.6838
27^3	6.6815	6.6925	6.6803	6.6843	6.6888	6.6964	6.6901	6.6827
28^3	6.6824	6.6959	6.6893	6.6879	6.6969	6.7047	6.6959	6.6795
29^3	6.6953	6.6996	6.6899	6.6907	6.6975	6.7084	6.6962	6.6835
30^3	6.6991	6.7053	6.6976	6.6893	6.7071	6.7116	6.6984	6.6986
31^3	6.7004	6.7010	6.6961	6.7050	6.7007	6.7160	6.7073	6.6960
32^3	6.6978	6.7031	6.7079	6.7088	6.7041	6.7028	6.7031	6.7047
33^3	6.6955	6.7014	6.7113	6.7142	6.7017	6.7125	6.7148	6.7062
36^3	6.7185	6.7140	6.7090	6.7092	6.7166	6.7114	6.7128	6.7194
39^3	6.7193	6.7171	6.7188	6.7162	6.7144	6.7172	6.7164	6.7165
42^3	6.7160	6.7264	6.7205	6.7218	6.7226	6.7232	6.7219	6.7191
48^3	6.7262	6.7301	6.7221	6.7289	6.7303	6.7274	6.7244	6.7290

Table 2: Number of 0−dimensional faces per site.