Lexicographical manipulations for correctly computing regular tetrahedralizations with incremental topological flipping

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Abstract. Edelsbrunner and Shah have proven that incremental topological flipping works for constructing a regular triangulation for a finite set of weighted points in $d-$dimensional space. This paper describes the lexicographical manipulations employed in a recently completed implementation of their method for correctly computing 3-dimensional regular triangulations. At the start of the execution of this implementation a regular triangulation for the vertices of an artificial cube that contains the points is constructed. Throughout the execution the vertices of this cube are treated in the proper lexicographical manner so that the final triangulation is correct.

Key words. Delaunay triangulation, incremental topological flipping, power diagram, regular triangulation, Voronoi diagram.

1. Introduction

Given integer k, $0 \leq k \leq d$, and a set R of $k+1$ affinely independent points in $d-$dimensional space ($\mathcal{R}^d$), we say that the convex hull of R, denoted by Δ_R, is the $k-$*simplex* for R. Let S be a finite set of points in $\mathcal{R}^d$. By a *triangulation* T for S we mean a finite collection of $k-$simplices for subsets of S, $k = 0, \ldots, d$, that satisfies the following three conditions.

1. If Δ_R is in T then Δ_U is in T for all U, $U \subseteq R$.

2. If Δ_R, Δ_U are in T then $\Delta_R \cap \Delta_U = \Delta_{R \cap U}$.

3. The union of the simplices in T equals the convex hull of S.

Given a triangulation T for S, we say that T is a *Delaunay triangulation* for S if S is the set of $0-$simplices in T, and for each $d-$simplex in T there does not exist a point of S in the interior of the circumsphere of the simplex [2].

A larger class of triangulations that includes the Delaunay triangulations can be defined. Again, let S be a finite set of points in $\mathcal{R}^d$, and for each point p in S let w_p be a real-valued weight assigned to p. Given p in S and a point x in $\mathcal{R}^d$, the *power distance* of x from p, denoted by $\pi_p(x)$, is defined by

$$\pi_p(x) \equiv |xp|^2 - w_p,$$

where $|xp|$ is the Euclidean distance between x and p. Given a set R of $d+1$ affinely independent points in S, a point, denoted by $z(\Delta_R)$, exists in $\mathcal{R}^d$ with the same power distance, denoted by $w(\Delta_R)$, from all $d+1$ points in R. $z(\Delta_R)$ is called the *orthogonal center* of Δ_R. Accordingly, the points in S are said to be in *general position* (in $\mathcal{R}^d$) if every set of $d+1$ points in S is affinely independent, and for every $d+2$ points in S there is no point in $\mathcal{R}^d$ with the same power distance from all $d+2$ points. Given a triangulation T for S, the points in S not necessarily in general position, we then say that T is a *regular triangulation* for S if for each $d-$simplex t in T and each point p in S, $\pi_p(z(t)) \geq w(t)$. We observe that T is unique if the points in S are in general position.

Given points p, q in S, we denote by $H_{p,q}$ the half-space of points x in $\mathcal{R}^d$ for which $\pi_p(x) \leq \pi_q(x)$, and for each p in S, the *power cell* for p, denoted by $P(p)$, is defined by

$$P(p) \equiv \cap_{q \in S \setminus \{p\}} H_{p,q}.$$

The collection of power cells $P(p)$, p in S, is called the *power diagram* of S [1], and if the points in S are in general position in $\mathcal{R}^d$ then it is the dual of the (unique) regular triangulation for S. Indeed the orthogonal center of a $d-$simplex in a regular triangulation for S is a vertex of the power diagram of S. We observe that if the weights of the points in S are all equal then the power diagram of S is identical to the *Voronoi diagram* of S [7], and the regular and Delaunay triangulations for S coincide. In addition, we notice that a point p in S whose power cell is empty cannot be a vertex of any regular triangulation for S. In this case p is said to be *redundant*. However, if p is a vertex of the convex hull of S then its power cell is nonempty so that it must be a vertex of any regular triangulation for S. This makes sense since the union of the simplices in any triangulation for S must equal the convex hull of S.

Let T be a triangulation for a set S of n points in $\mathcal{R}^d$, not necessarily in general position. Given a $d-$simplex t in T we denote by $N(t)$ the set of points in $S \setminus t$ that are vertices of

$d-$simplices in T sharing a $(d-1)-$simplex with t. We then say that t is *locally regular* if for each point p in $N(t)$, $\pi_p(z(t)) \geq w(t)$. By extending results for Delaunay triangulations [5, 6], Edelsbrunner and Shah [3] have proven that if the vertex set of T contains all non-redundant points in S and every $d-$simplex in T is locally regular then T is a regular triangulation for S. They then use this result to generalize to regular triangulations in $\mathcal{R}^d$ a result for computing incrementally Delaunay triangulations in $\mathcal{R}^2$ [4]. Their algorithm is based on an operation referred to as a *flip* that replaces a triangulation for $d+2$ points with the (unique) alternative triangulation for the $d+2$ points [6]. Given a proper subset S' of S and a regular triangulation T' for S', they show how a point p in $S \setminus S'$ can be added to T' through a sequence of flips so that the resulting triangulation for $S' \cup \{p\}$ is regular. They also generalize a two-dimensional technique for efficiently identifying the initial location of the point to be added [4]. Finally, they prove that under the assumption of a random insertion sequence the total expected running time of their algorithm is $O(n \log n + n^{\lceil d/2 \rceil})$.

The algorithm by Edelsbrunner and Shah constructs a regular triangulation for a set S by adding one point at a time into a regular triangulation for the set of previously added points. This implies that before any points in S are added a regular triangulation must be first constructed with vertices at infinity and underlying space equal to $\mathcal{R}^d$. The vertices of this initial triangulation are said to be *artificial*. Throughout the execution of the algorithm artificial points must be treated in the proper lexicographical manner so that the final triangulation does contain a triangulation for S, and this triangulation for S is indeed regular. This is not exactly a trivial undertaking.

In this paper we describe the lexicographical manipulations that are employed in a recently completed implementation of the algorithm by Edelsbrunner and Shah for correctly computing a regular triangulation for an arbitrary set S in $\mathcal{R}^3$. At the start of the execution of the implementation an artificial $3-$dimensional cube that contains S in its interior is constructed, and a regular triangulation for the set of vertices of the cube (weights set to the same number) is computed. The execution then proceeds with the incremental insertion of points in S as suggested by Edelsbrunner and Shah. However, at all times, because of the lexicographical manipulations employed in the presence of artificial points, the artificial points (the eight vertices of the cube) are assumed to be as close to infinity as the manipulations require.

The lexicographical manipulations are divided in two groups. The first group, discussed in Section 3, consists of manipulations for determining the location of a point with respect to a facet of a tetrahedron. The second group, discussed in Section 4, consists of manipulations for determining whether a triangulation for five points is regular or else should be transformed through a flip into the (unique) regular alternative triangulation for the five points. Terminology used throughout the paper is presented in Section 2.

2. Terminology

In this section we introduce terminology that is employed in the sections that follow.

Let S be a finite set of points in $\mathcal{R}^3$, and assign a real valued weight w_p to each point p in S. Real numbers $xmin$, $xmax$, $ymin$, $ymax$, $zmin$, $zmax$ are defined by

$$\begin{aligned}
xmin &\equiv \min\{x : \exists\ y,\ z,\ (x,y,z) \in S\}. \\
xmax &\equiv \max\{x : \exists\ y,\ z,\ (x,y,z) \in S\}. \\
ymin &\equiv \min\{y : \exists\ x,\ z,\ (x,y,z) \in S\}. \\
ymax &\equiv \max\{y : \exists\ x,\ z,\ (x,y,z) \in S\}. \\
zmin &\equiv \min\{z : \exists\ x,\ y,\ (x,y,z) \in S\}. \\
zmax &\equiv \max\{z : \exists\ x,\ y,\ (x,y,z) \in S\}.
\end{aligned}$$

A real number $wmin$ is defined by

$$wmin \equiv \min\{w_p :\ p \in S\}.$$

Real numbers $xctr$, $yctr$,$zctr$ are defined by

$$\begin{aligned}
xctr &\equiv (xmax + xmin)/2. \\
yctr &\equiv (ymax + ymin)/2. \\
zctr &\equiv (zmax + zmin)/2.
\end{aligned}$$

A point $\overline{p}$ in $\mathcal{R}^3$ is defined by

$$\bar{p} \equiv (xctr, yctr, zctr).$$

Vectors e_i, $i = 1, \ldots, 8$, are defined by

$$e_1 \equiv (-1, -1,\ 1).$$

$$e_2 \equiv (-1,\ \ 1,\ \ 1).$$

$$
\begin{aligned}
e_3 &\equiv (\ \ 1,\ \ 1,\ \ 1).\\
e_4 &\equiv (\ \ 1,-1,\ \ 1).\\
e_5 &\equiv (-1,-1,-1).\\
e_6 &\equiv (-1,\ \ 1,-1).\\
e_7 &\equiv (\ \ 1,\ \ 1,-1).\\
e_8 &\equiv (\ \ 1,-1,-1).
\end{aligned}
$$

For each real number μ, $\mu > 0$, the vertices $p_{i\mu}$, $i = 1, \ldots, 8$, of a cube R_μ are defined by

$$p_{i\mu} \equiv \bar{p} + \mu e_i,\ \ i = 1, \ldots, 8.$$

For arbitrarily large μ, $\mu > 0$, R_μ contains S in its interior. Given a real number μ, $\mu > 0$, the points $p_{i\mu}$, $i = 1, \ldots, 8$, are the artificial points, and μ is assumed to be as large as the lexicographical manipulations require. In order to be consistent, given a real number μ, $\mu > 0$, a real number w, $w < wmin$, is selected and assigned as a weight to each of the points $p_{i\mu}$, $i = 1, \ldots, 8$. Since the points $p_{i\mu}$, $i = 1, \ldots, 8$, are the vertices of a cube, it follows easily that any triangulation for these points is regular. In addition, one such triangulation is not hard to compute.

Finally, given a set R of 4 affinely independent weighted points in $\mathcal{R}^3$, denote by $z(\Delta_R)$ the orthogonal center of Δ_R and by $w(\Delta_R)$ the power distance of $z(\Delta_R)$ from any of the points in R.

3. Lexicographical manipulations for point location determination

For arbitrarily large μ, $\mu > 0$, let S' be a proper subset of S, and let T'_μ be a regular triangulation for $S'_\mu \equiv S' \cup \{p_{i\mu},\ i = 1, \ldots, 8\}$ that contains a regular triangulation T' for S'. Let p be a point in $S \setminus S'$, and let t be a tetrahedron in T'_μ. Denote the vertices of t by q_1, q_2, q_3, q_4. Given that p is not a vertex of t, let T_1 and T_2 be the two possible triangulations for $\{q_1, q_2, q_3, q_4, p\}$ [6] and assume t is in T_1. In this section we describe lexicographical manipulations that may be used in the presence of artificial points for identifying T_1 and T_2. For the sake of completeness we also present direct computations that may be used when no artificial points are involved.

For each j, $j = 1, \ldots, 4$, denote by f_j the facet of t that does not contain q_j, and by H_j the plane in $\mathcal{R}^3$ that contains f_j. For each j, $j = 1, \ldots, 4$, denote by H_j^+ the open half-space in $\mathcal{R}^3$ determined by H_j that contains q_j, and by H_j^- the open half-space in $\mathcal{R}^3$ determined by H_j that does not contain q_j. For each j, $j = 1, \ldots, 4$, determining which of H_j, H_j^+, H_j^- contains p can be accomplished through either lexicographical manipulations or direct computations as described below. Indeed it is by ascertaining which of H_j, H_j^+, H_j^- contains p for each j, $j = 1, \ldots, 4$, that one can identify the triangulations T_1 and T_2. Accordingly, the following nine configurations of T_1 and T_2 are possible, each configuration depending on which of H_j, H_j^+, H_j^- contains p for each j, $j = 1, \ldots, 4$.

Configuration 1 (possible '1 to 4' flip): p is in $\cap_{j=1}^4 H_j^+$. Denote by t_1, t_2, t_3, and t_4 the tetrahedra whose vertex sets are $\{q_1, q_2, q_3, p\}$, $\{q_1, q_2, q_4, p\}$, $\{q_1, q_3, q_4, p\}$, and $\{q_2, q_3, q_4, p\}$, respectively. It then follows that T_1 consists exactly of t, and T_2 of t_1, t_2, t_3, and t_4.

Configuration 2 (possible '1 to 3' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \le j_1, j_2, j_3, j_4 \le 4$, p is in $H_{j_1} \cap H_{j_2}^+ \cap H_{j_3}^+ \cap H_{j_4}^+$. Denote by t_1, t_2, and t_3 the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, and $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists exactly of t, and T_2 of t_1, t_2, and t_3.

Configuration 3 (possible '1 to 2' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \le j_1, j_2, j_3, j_4 \le 4$, p is in $H_{j_1} \cap H_{j_2} \cap H_{j_3}^+ \cap H_{j_4}^+$. Denote by t_1 and t_2 the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$ and $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists exactly of t, and T_2 of t_1 and t_2.

Configuration 4 (possible '2 to 3' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \le j_1, j_2, j_3, j_4 \le 4$, p is in $H_{j_1}^- \cap H_{j_2}^+ \cap H_{j_3}^+ \cap H_{j_4}^+$. Denote by t_1, t_2, t_3, and t' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, and $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t and t', and T_2 of t_1, t_2, and t_3.

Configuration 5 (possible '3 to 2' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \le j_1, j_2, j_3, j_4 \le 4$, p is in $H_{j_1}^- \cap H_{j_2}^- \cap H_{j_3}^+ \cap H_{j_4}^+$. Denote by t_1, t_2, t', and t'' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t, t', and t'', and T_2 of t_1 and t_2.

Configuration 6 (possible '2 to 2' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \le j_1, j_2, j_3, j_4 \le 4$, p is in $H_{j_1}^- \cap H_{j_2} \cap H_{j_3}^+ \cap H_{j_4}^+$. Denote by t_1, t_2, and t' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, and $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t and t', and T_2 of t_1 and t_2.

Configuration 7 (possible '4 to 1' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \le j_1, j_2, j_3, j_4 \le 4$, p is in $H_{j_1}^- \cap H_{j_2}^- \cap H_{j_3}^- \cap H_{j_4}^+$. Denote by t_1, t', t'', and t''' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, and $\{q_{j_1}, q_{j_2}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t, t', t'', and t''', and T_2 exactly of t_1.

Configuration 8 (possible '3 to 1' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H_{j_1}^- \cap H_{j_2}^- \cap H_{j_3} \cap H_{j_4}^+$. Denote by t_1, t', and t'' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, and $\{q_{j_1}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t, t', and t'', and T_2 exactly of t_1.

Configuration 9 (possible '2 to 1' flip): For distinct integers j_1, j_2, j_3, j_4, $1 \leq j_1, j_2, j_3, j_4 \leq 4$, p is in $H_{j_1}^- \cap H_{j_2} \cap H_{j_3} \cap H_{j_4}^+$. Denote by t_1 and t' the tetrahedra whose vertex sets are $\{q_{j_1}, q_{j_2}, q_{j_3}, p\}$, and $\{q_{j_2}, q_{j_3}, q_{j_4}, p\}$, respectively. It then follows that T_1 consists of t and t', and T_2 exactly of t_1.

Finally, in what follows, for the purpose of identifying T_1 and T_2 we present lexicographical manipulations and direct computations that may be used for determining which of H_j, H_j^+, H_j^- contains p for a given j, $1 \leq j \leq 4$. We do this by cases, each case depending on the number of artificial vertices of f_j. Here and in the next section we assume without any loss of generality that S' is not empty. It then follows that if the vertices of a facet of a triangle in T'_μ are all artificial then the facet must be contained in its entirety in the boundary of R_μ. We proceed without any loss of generality for the case j equal to 4. We define a vector v by $v \equiv (q_1 - q_3) \times (q_2 - q_3)$, i. e. the cross product of vectors $(q_1 - q_3)$ and $(q_2 - q_3)$, and assume that q_1, q_2, q_3 are ordered in such a way that $v \cdot (q_4 - q_3)$, i. e. the inner product of v and $(q_4 - q_3)$, is positive. Clearly, which of H_4, H_4^+, H_4^- contains p depends on the sign of $v \cdot (p - q_3)$. The solution by cases to the *point location determination problem*, i. e. the problem of determining the sign of $v \cdot (p - q_3)$, follows.

Case 1: None of q_1, q_2, q_3 is artificial. Since none of the vertices of f_4 is artificial the sign of $v \cdot (p - q_3)$ can then be determined through direct computations of v, $p - q_3$, and $v \cdot (p - q_3)$.

Case 2: Exactly one of q_1, q_2, q_3 is artificial. Without any loss of generality we asssume the one point is q_1 so that q_2 and q_3 are in S. Let k be an integer, $1 \leq k \leq 8$, so that q_1 equals $p_{k\mu}$. Accordingly, by definition the vector v must then equal $(\bar{p} + \mu e_k - q_3) \times (q_2 - q_3)$ which in turn reduces to

$$((\bar{p} - q_3) \times (q_2 - q_3)) + \mu(e_k \times (q_2 - q_3)).$$

Define numbers d_0, d_1, as follows:

$$d_0 \equiv ((\bar{p} - q_3) \times (q_2 - q_3)) \cdot (p - q_3).$$

$$d_1 \equiv (e_k \times (q_2 - q_3)) \cdot (p - q_3).$$

The sign of $v \cdot (p - q_3)$ can then be determined as follows:
If d_1 is non-zero then the sign is that of d_1.
Else, if d_1 is zero then it is that of d_0.

Case 3: Exactly two of q_1, q_2, q_3 are artificial. Without any loss of generality we asssume the two points are q_1 and q_2 so that q_3 is in S. Let k and l be integers, $1 \leq k,\ l \leq 8$, so that q_1 equals $p_{k\mu}$ and q_2 equals $p_{l\mu}$. Accordingly, by definition the vector v must then equal $(\bar{p} + \mu e_k - q_3) \times (\bar{p} + \mu e_l - q_3)$ which in turn reduces to

$$\mu((\bar{p} - q_3) \times (e_l - e_k)) + \mu^2(e_k \times e_l).$$

Define numbers d_1, d_2, as follows:

$$\begin{aligned} d_1 &\equiv ((\bar{p} - q_3) \times (e_l - e_k)) \cdot (p - q_3). \\ d_2 &\equiv (e_k \times e_l) \cdot (p - q_3). \end{aligned}$$

The sign of $v \cdot (p - q_3)$ can then be determined as follows:
If d_2 is non-zero then the sign is that of d_2.
Else, if d_2 is zero then it is that of d_1.

Case 4: q_1, q_2, q_3 are all artificial. Since the vertices of f_4 are all artificial it then follows that f_4 must be contained in its entirety in the boundary of R_μ. Since R_μ contains S in its interior and $v \cdot (q_4 - q_3)$ is positive it must then be that $v \cdot (p - q_3)$ is also positive.

4. Lexicographical manipulations for flipping determination

Again, for arbitrarily large μ, $\mu > 0$, let S' be a proper subset of S, and let T'_μ be a regular triangulation for $S'_\mu \equiv S' \cup \{p_{i\mu},\ i = 1, \ldots, 8\}$ that contains a regular triangulation T' for S'. Let p be a point in $S \setminus S'$, and let t be a tetrahedron in T'_μ. Denote the vertices of t by q_1, q_2, q_3, q_4. Given that p is not a vertex of t, let T_1 and T_2 be the two possible triangulations for $\{q_1, q_2, q_3, q_4, p\}$ [6] and assume t is in T_1. In this section we describe lexicographical manipulations that may be used in the presence of artificial points for determining which of T_1 and T_2 is regular. For the sake of completeness we also present direct computations that may be used when no artificial points are involved. We do this by cases, each case depending on the number of artificial vertices of t. First, however, we state and prove three propositions that will be useful during the presentation of these cases.

Proposition 1: Let $\hat{t}$ be a tetrahedron with vertices in $S \cup \{p_{i\mu},\ i = 1, \ldots, 8\}$, μ arbitrarily large. Denote the vertices of $\hat{t}$ by $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$, and $\hat{q}_4$, and assume $\hat{q}_1$ is artificial while $\hat{q}_2$, $\hat{q}_3$, $\hat{q}_4$ are not. In addition, assume $((\hat{q}_2 - \hat{q}_4) \times (\hat{q}_3 - \hat{q}_4)) \cdot (\hat{q}_1 - \hat{q}_4) < 0$. Let k be an integer, $1 \leq k \leq 8$, so that $\hat{q}_1$ equals $p_{k\mu}$. Let $\hat{f}$ be the facet of $\hat{t}$ whose vertices are $\hat{q}_2$, $\hat{q}_3$, and $\hat{q}_4$, and let $\hat{H}$ be the plane in $\mathcal{R}^3$ that contains $\hat{f}$. Denote by $\bar{z}$ the orthogonal center of $\hat{f}$ in the plane $\hat{H}$, and by $\bar{w}$ the power distance of $\bar{z}$ from any of the vertices of $\hat{f}$.
Given a point $\hat{p}$ in S, define a number d by

$$d \equiv ((\hat{q}_2 - \hat{q}_4) \times (\hat{q}_3 - \hat{q}_4)) \cdot (\hat{p} - \hat{q}_4).$$

If d does not equal zero then the sign of $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ is that of d.
Else, if d equals zero then $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ equals $\pi_{\hat{p}}(\bar{z}) - \bar{w}$.

Proof: Define a vector v by $v \equiv (\hat{q}_2 - \hat{q}_4) \times (\hat{q}_3 - \hat{q}_4)$.
Clearly v is perpendicular to $\hat{H}$ and $z(\hat{t})$ equals $\bar{z} + \beta_\mu v$ for some real number β_μ. We then show that given an arbitrary real number $\tilde{\beta}$ then for arbitrarily large μ it must be that $\tilde{\beta} > \beta_\mu$. To this end, denote by $z([\hat{q}_1, \hat{q}_4])$ the orthogonal center of the edge $[\hat{q}_1, \hat{q}_4]$ in the affine hull of $\hat{q}_1$ and $\hat{q}_4$. Since $\hat{q}_4$ is in $\hat{H}$ and by assumption $v \cdot (\hat{q}_1 - \hat{q}_4) < 0$ it follows that for each β, $\beta < \beta_\mu$, it must be that $\bar{z} + \beta v$ is in the interior of $H_{\hat{q}_1,\hat{q}_4}$ (the half-space of points x in $\mathcal{R}^3$ for which $\pi_{\hat{q}_1}(x) \leq \pi_{\hat{q}_4}(x)$). For arbitrarily large μ the Euclidean distance between $\hat{q}_4$ and $z([\hat{q}_1, \hat{q}_4])$ is itself arbitrarily large. In particular, it is larger than the Euclidean distance between $\hat{q}_4$ and $\bar{z} + \tilde{\beta} v$. Thus, $\bar{z} + \tilde{\beta} v$ can not be in $H_{\hat{q}_1,\hat{q}_4}$ and therefore it must be that $\tilde{\beta} > \beta_\mu$.
If d, i. e. $v \cdot (\hat{p} - \hat{q}_4)$, is not zero then for a unique real number $\tilde{\beta}$ it must be that $\pi_{\hat{p}}(\bar{z} + \tilde{\beta} v)$ equals $\pi_{\hat{q}_4}(\bar{z} + \tilde{\beta} v)$. If d is positive it then follows that for each β, $\beta < \tilde{\beta}$, it must be that $\bar{z} + \beta v$ is in the interior of $H_{\hat{q}_4,\hat{p}}$. Since $\beta_\mu < \tilde{\beta}$ for μ arbitrarily large, it then follows that $z(\hat{t})$, i. e. $\bar{z} + \beta_\mu v$, is in the interior of $H_{\hat{q}_4,\hat{p}}$. Therefore $\pi_{\hat{p}}(z(\hat{t})) > \pi_{\hat{q}_4}(z(\hat{t})) = w(\hat{t})$ and $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ is positive. If d is negative it then follows that for each β, $\beta < \tilde{\beta}$, it must be that $\bar{z} + \beta v$ is in the interior of $H_{\hat{p},\hat{q}_4}$. Since $\beta_\mu < \tilde{\beta}$ for μ arbitrarily large, it then follows that $z(\hat{t})$, i. e. $\bar{z} + \beta_\mu v$, is in the interior of $H_{\hat{p},\hat{q}_4}$. Therefore $\pi_{\hat{p}}(z(\hat{t})) < \pi_{\hat{q}_4}(z(\hat{t})) = w(\hat{t})$ and $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ is negative.
If d is zero then $\hat{p}$ is in $\hat{H}$. We then have

$$\begin{aligned}
\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t}) &= \pi_{\hat{p}}(z(\hat{t})) - \pi_{\hat{q}_4}(z(\hat{t})) \\
&= (|z(\hat{t})\hat{p}|^2 - w_{\hat{p}}) - (|z(\hat{t})\hat{q}_4|^2 - w_{\hat{q}_4}) \\
&= (|z(\hat{t})\bar{z}|^2 + |\bar{z}\hat{p}|^2 - w_{\hat{p}}) - (|z(\hat{t})\bar{z}|^2 + |\bar{z}\hat{q}_4|^2 - w_{\hat{q}_4}) \\
&= (|\bar{z}\hat{p}|^2 - w_{\hat{p}}) - (|\bar{z}\hat{q}_4|^2 - w_{\hat{q}_4})
\end{aligned}$$

$$
\begin{aligned}
&= \pi_{\hat{p}}(\bar{z}) - \pi_{\hat{q}_4}(\bar{z}) \\
&= \pi_{\hat{p}}(\bar{z}) - \bar{w}.
\end{aligned}
$$

This completes the proof of the proposition.

Proposition 2: Let $\hat{t}$ be a tetrahedron with vertices in $S \cup \{p_{i\mu},\ i = 1, \ldots, 8\}$, μ arbitrarily large. Denote the vertices of $\hat{t}$ by $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$, and $\hat{q}_4$, and assume $\hat{q}_1$, $\hat{q}_2$ are artificial while $\hat{q}_3$, $\hat{q}_4$ are not. In addition, assume $((\hat{q}_2 - \hat{q}_1) \times (\hat{q}_3 - \hat{q}_4)) \cdot (\hat{q}_1 - \hat{q}_4) < 0$. Let k, l be integers, $1 \leq k, l \leq 8$, so that $\hat{q}_1$ equals $p_{k\mu}$ and $\hat{q}_2$ equals $p_{l\mu}$. Let $\tilde{H}$ be the plane in $\mathcal{R}^3$ that is the *chordale* of $\hat{q}_3$ and $\hat{q}_4$, i. e. the plane of points x in $\mathcal{R}^3$ for which $\pi_{\hat{q}_3}(x) = \pi_{\hat{q}_4}(x)$. Let $\bar{H}$ be the plane in $\mathcal{R}^3$ that is the chordale of $p_{k\mu}$ and $p_{l\mu}$ for all positive values of μ ($p_{k\mu}$ equals $\bar{p} + \mu e_k$ and $p_{l\mu}$ equals $\bar{p} + \mu e_l$), and let $\hat{H}$ be the plane in $\mathcal{R}^3$ that contains $\hat{q}_3$ and $\hat{q}_4$, and is perpendicular to $\tilde{H} \cap \bar{H}$. Denote by $\bar{z}$ the point that is the intersection of $\tilde{H}$, $\bar{H}$, and $\hat{H}$, and by $\bar{w}$ the power distance of $\bar{z}$ from either $\hat{q}_3$ or $\hat{q}_4$

Given a point $\hat{p}$ in S, define a number d by

$$d \equiv ((e_l - e_k) \times (\hat{q}_3 - \hat{q}_4)) \cdot (\hat{p} - \hat{q}_4).$$

If d does not equal zero then the sign of $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ is that of d.
Else, if d equals zero then $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ equals $\pi_{\hat{p}}(\bar{z}) - \bar{w}$.

Proof: Define a vector v by $v \equiv (e_l - e_k) \times (\hat{q}_3 - \hat{q}_4)$.
The rest of the proof follows as in the proof of Proposition 1 above.

Proposition 3: Let $\hat{t}$ be a tetrahedron with vertices in $S \cup \{p_{i\mu},\ i = 1, \ldots, 8\}$, μ arbitrarily large. Denote the vertices of $\hat{t}$ by $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$, and $\hat{q}_4$, and assume $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$ are artificial while $\hat{q}_4$ is not. In addition, assume $((\hat{q}_2 - \hat{q}_1) \times (\hat{q}_3 - \hat{q}_1)) \cdot (\hat{q}_1 - \hat{q}_4) < 0$. Let k, l, m be integers, $1 \leq k, l, m \leq 8$, so that $\hat{q}_1$ equals $p_{k\mu}$, $\hat{q}_2$ equals $p_{l\mu}$, and $\hat{q}_3$ equals $p_{m\mu}$. Let $\tilde{H}$ and $\bar{H}$ be the planes in $\mathcal{R}^3$ that are the chordales, respectively, of $p_{k\mu}$ and $p_{l\mu}$, and $p_{k\mu}$ and $p_{m\mu}$, for all positive values of μ. Let $\hat{H}$ be the plane in $\mathcal{R}^3$ that contains $\hat{q}_4$ and is perpendicular to $\tilde{H} \cap \bar{H}$. Denote by $\bar{z}$ the point that is the intersection of $\tilde{H}$, $\bar{H}$, and $\hat{H}$, and by $\bar{w}$ the power distance of $\bar{z}$ from $\hat{q}_4$.

Given a point $\hat{p}$ in S, define a number d by

$$d \equiv ((e_l - e_k) \times (e_m - e_k)) \cdot (\hat{p} - \hat{q}_4).$$

If d does not equal zero then the sign of $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ is that of d.
Else, if d equals zero then $\pi_{\hat{p}}(z(\hat{t})) - w(\hat{t})$ equals $\pi_{\hat{p}}(\bar{z}) - \bar{w}$.

Proof: Define a vector v by $v \equiv (e_l - e_k) \times (e_m - e_k)$.
The rest of the proof follows as in the proof of Proposition 1 above.

We now present the solution by cases to the *flipping determination problem*, i. e. the problem of determining which of T_1 and T_2 is regular.

Case 1: None of q_1, q_2, q_3, q_4 is artificial. Compute $z(t)$ and $w(t)$. If $\pi_p(z(t)) \geq w(t)$ then T_1 is regular. Else, T_2 is regular.

Case 2: Exactly one of q_1 q_2, q_3, q_4 is artificial. Without any loss of generality assume q_1 is artificial, and let k be an integer, $1 \leq k \leq 8$, so that q_1 equals $p_{k\mu}$.
Assume $((q_2 - q_4) \times (q_3 - q_4)) \cdot (q_1 - q_4) < 0$.
Compute $d \equiv ((q_2 - q_4) \times (q_3 - q_4)) \cdot (p - q_4)$, and apply Proposition 1 as follows:
If $d > 0$ then $\pi_p(z(t)) > w(t)$ so that T_1 is regular.
Else, if $d < 0$ then $\pi_p(z(t)) < w(t)$ so that T_2 is regular.
Finally, if d is zero then let f be the facet of t whose vertices are q_2, q_3, and q_4, and let H be the plane in $\mathcal{R}^3$ that contains f. Compute $\bar{z}$, the orthogonal center of f in the plane H, and $\bar{w}$, the power distance of $\bar{z}$ from any of the vertices of f.
If $\pi_p(\bar{z}) \geq \bar{w}$ then $\pi_p(z(t)) \geq w(t)$ so that T_1 is regular.
Else, if $\pi_p(\bar{z}) < \bar{w}$ then $\pi_p(z(t)) < w(t)$ so that T_2 is regular.

Case 3: Exactly two of q_1 q_2, q_3, q_4 are artificial. Without any loss of generality assume q_1 and q_2 are artificial, and let k, l be integers, $1 \leq k, l \leq 8$, so that q_1 equals $p_{k\mu}$ and q_2 equals $p_{l\mu}$.
Assume $((q_2 - q_1) \times (q_3 - q_4)) \cdot (q_1 - q_4) < 0$.
Compute $d \equiv ((e_l - e_k) \times (q_3 - q_4)) \cdot (p - q_4)$, and apply Proposition 2 as follows:
If $d > 0$ then $\pi_p(z(t)) > w(t)$ so that T_1 is regular.
Else, if $d < 0$ then $\pi_p(z(t)) < w(t)$ so that T_2 is regular.
Finally, if d is zero then let $\tilde{H}$ be the plane in $\mathcal{R}^3$ that is the chordale of q_3 and q_4, let $\bar{H}$ be the plane in $\mathcal{R}^3$ that is the chordale of $p_{k\mu}$ and $p_{l\mu}$ for all positive values of μ, and let H be the plane in $\mathcal{R}^3$ that contains q_3 and q_4, and is perpendicular to $\tilde{H} \cap \bar{H}$. Compute $\bar{z}$, the point that is the intersection of $\tilde{H}$, $\bar{H}$, and H, and $\bar{w}$, the power distance of $\bar{z}$ from either q_3 or q_4
If $\pi_p(\bar{z}) \geq \bar{w}$ then $\pi_p(z(t)) \geq w(t)$ so that T_1 is regular.
Else, if $\pi_p(\bar{z}) < \bar{w}$ then $\pi_p(z(t)) < w(t)$ so that T_2 is regular.

Case 4: Exactly three of q_1, q_2, q_3, q_4 are artificial. Without any loss of generality assume q_1, q_2 and q_3 are artificial, and let k, l, m be integers, $1 \leq k, l, m \leq 8$, so that q_1 equals $p_{k\mu}$, q_2 equals $p_{l\mu}$, and q_3 equals $p_{m\mu}$.
Assume $((q_2 - q_1) \times (q_3 - q_1)) \cdot (q_1 - q_4) < 0$.
Compute $d \equiv ((e_l - e_k) \times (e_m - e_k)) \cdot (p - q_4)$, and apply Proposition 3 as follows:
If $d > 0$ then $\pi_p(z(t)) > w(t)$ so that T_1 is regular.
Else, if $d < 0$ then $\pi_p(z(t)) < w(t)$ so that T_2 is regular.
Finally, if d is zero then let $\tilde{H}$ and $\bar{H}$ be the planes in $\mathcal{R}^3$ that are the chordales, respectively, of $p_{k\mu}$ and $p_{l\mu}$, and $p_{k\mu}$ and $p_{m\mu}$, for all positive values of μ. Let H be the plane in $\mathcal{R}^3$ that contains q_4 and is perpendicular to $\tilde{H} \cap \bar{H}$. Compute $\bar{z}$, the point that is the intersection of $\tilde{H}$, $\bar{H}$, and H, and $\bar{w}$, the power distance of $\bar{z}$ from q_4.
If $\pi_p(\bar{z}) \geq \bar{w}$ then $\pi_p(z(t)) \geq w(t)$ so that T_1 is regular.
Else, if $\pi_p(\bar{z}) < \bar{w}$ then $\pi_p(z(t)) < w(t)$ so that T_2 is regular.

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