# Discovering Discrete Classical (Orthogonal) Polynomials: First Steps

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# Outline

# THE CONSTRUCTION

# They satisfy this Sturm-Liouville SODE

## SODE

$(p_n)$ fulfill the Second Order Differential Equation

$$\phi(x)y''(x) + \psi(x)y'(x) + \lambda y(x) = 0.$$

which, for the COP, is equivalent to

$$\Big(\phi(x)\rho(x)y'(x)\Big)' + \lambda\rho(x)y(x) = 0.$$

All this is possible because there exist a weight function $\rho(x)$ and an interval $(a, b) \subseteq \mathbb{R}$ so that

$$\int_a^b p_n(x)p_m(x)\rho(x)dx = \kappa_n\delta_{n,m}.$$

# From the continuous to the uniform discrete world

The easiest way to discretize the SLE over a uniform lattice. In order to do that we divide the interval (a,b) in subintervals of length $h$, and we approximate

$$y' \sim \frac{1}{2}\left(\frac{y(x+h)-y(h)}{h}+\frac{y(x)-y(x-h)}{h}\right)$$

$$y'' \sim \frac{1}{h}\left(\frac{y(x+h)-y(h)}{h}-\frac{y(x)-y(x-h)}{h}\right).$$

After some corrections we get

$$\phi(x)\Delta\nabla y(x)+\psi(x)\Delta y(x)+\lambda y(x)=0.$$

here

$\Delta f(x)=f(x+1)-f(x)$, $\nabla f(x)=f(x)-f(x-1)$.

In this case we discretize the SLE over a non uniform lattice $\{x(s)\}$ with

$$y'(x) \sim \frac{1}{2}\left(\frac{y(x(s+h))-y(x(s))}{x(x+h)-x(s)}+\frac{y(x(s))-(x(s-h)}{x(s)-x(s-h))}\right),$$

and

$$y''(x) \sim \frac{1}{x(s+h/2)-x(s-h/2)}\left(\frac{y(x(s+h))-y(x(s))}{x(x+h)-x(s)}-\frac{y(x(s))-y(x(s-h))}{x(s)-x(s-h))}\right).$$

We are taking the points $x(s\pm h)$, and $x(s\pm h/2)$!!!. Again, after some corrections, we get

$$\phi(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla}{\nabla x(s)}y(s)+\psi(s)\frac{\Delta}{\Delta x(s)}y(s)+\lambda y(s)=0.$$

# THE BASICS

# Classical Orthogonal Polynomials

- Let $(P_n)$ be a polynomial sequence and $\mathbf{u}$ be a functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1, \ \deg \phi \leq 2.$$

- Three-term recurrence relation:

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

- The weight function $d\mu(z) = \omega(z)\, dz$

$$\langle \mathbf{u}, P \rangle = \int_\Gamma P(z)\, d\mu(z), \qquad \Gamma \subset \mathbb{C}, .$$

1. Continuous classical orthogonal polynomials

   - $\dfrac{d}{dx}(\phi(x)\omega(x)) = \psi(x)\omega(x),$

2. $\Delta$-classical orthogonal polynomials

   - $\nabla(\phi(x)\omega(x)) = \psi(x)\omega(x),$
   - $\Delta f(x) = f(x+1) - f(x),\ \nabla f(x) = f(x) - f(x-1),$

3. $q$-Hahn classical orthogonal polynomials

   - $\dfrac{\Delta[(\phi(s)\omega(s)]}{\Delta x(s-1/2)} = \psi(s)\omega(s),$
   - $x(s) = c_1 q^s + c_2 q^{-s} + c_3.$

# The Favard's theorem

Let $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x).$$

## Favard's theorem

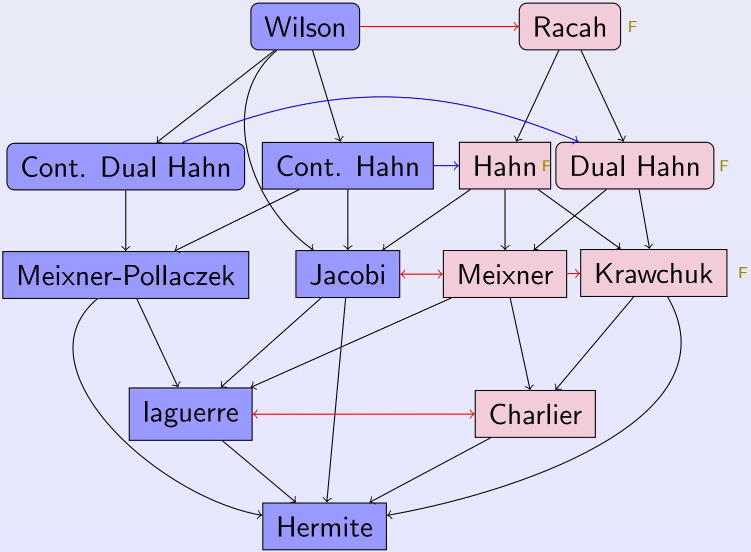
If $\gamma_n \neq 0 \ \forall n \in \mathbb{N}$ then there exists a moments functional $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

with $r_n$ a non-vanishing normalization factor.

# THE RELEVANT FAMILIES

# The Classical Hypergeometric Orthogonal Polynomials



THE SCHEME IS TOO BIG TO PUT IT ON HERE, LET'S GO OUTSIDE TO SEE IT ;)

# SOME RESULTS

Let $(P_n)$ be an OPS with respect to $\omega$. The following statements are equivalent:

1. $P_n$ is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. $(P'_{n+1})$ is a OPS.
3. $(P^{(k)}_{n+k})$ is a OPS for any integer $k$.
4. (First structure relation)

$$\phi(x)P'_n(x) = \widehat{\alpha}_n P_{n+1}(x) + \widehat{\beta}_n P_n(x) + \widehat{\gamma}_n P_{n-1}(x).$$

5. (Second structure relation)

$$P_n(x) = \widetilde{\alpha}_n P'_{n+1}(x) + \widetilde{\beta}_n P'_n(x) + \widetilde{\gamma}_n P'_{n-1}(x).$$

6. (Eigenfunctions of SODE)

$$\phi(x)P''(x) + \psi(x)P'(x) + \lambda P(x) = 0.$$

# Characterization Theorem (cont.)

Let $(P_n)$ be an OPS with respect to $\omega$. The following statements are equivalent:

1. $P_n$ is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. The Rodrigues Formula for $P_n$

$$P_n(x) = \frac{B_n}{\omega(x)} \frac{d^n}{dx^n}(\phi^n(x)\omega(x)), \qquad B_n \neq 0.$$

3. $\phi(x)(P_n P_{n-1})'(x) = g_n P_n^2(x) - (\psi(x) - \phi'(x))P_n(x)P_{n-1}(x) + h_n P_{n-1}^2(x)$

# Hypergeometric and basic hypergeometric representations

The continuous and discrete COP can be written in terms of

$$ {}_rF_s\left( \begin{array}{c} a_1,\ a_2,\ \dots, a_r \\ b_1,\ b_2,\ \dots, b_s \end{array} \middle| z \right) = \sum_{k\geq 0} \frac{(a_1)_k(a_2)_k\dots(a_r)_k}{(b_1)_k(b_2)_k\dots(b_s)_k}\frac{z^k}{k!}. $$

The $q$-discrete COP can be written in terms of

$$ {}_r\varphi_s\left( \begin{array}{c} a_1,\ \dots, a_r \\ b_1,\ \dots, b_s \end{array} \middle| z \right) = \sum_{k\geq 0} \frac{(a_1;q)_k\dots(a_r;q)_k}{(b_1;q)_k\dots(b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}. $$

$$ (a)_k = a(a+1)\cdots(a+k-1) $$

$$ (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) $$

# The Connection Problem

The connection problem is the problem of finding the coefficients $c_{k;n}$ in the expansion of $P_n$ in terms of another sequence of polynomials $R_k$, i.e.

$$P_n(x) = \sum_{k=0}^{n} c_{k;n} R_k(x).$$

We are interested into obtaining such coefficients for Classical orthogonal polynomials in a enough 'general' context.

Again let's go to File 2 :D

# Some References


- (with J.F. Sánchez-Lara) Extensions of discrete classical orthogonal polynomials beyond the orthogonality. *J. Comput. Appl. Math. 225 (2009), no. 2, 440–451*
- (with F. Marcellán) $q$-Classical orthogonal polynomial: A general difference calculus approach. *Acta Appl. Math. 111 (2010), no. 1, 107–128*
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- (with F. Marcellán) The complementary polynomials and the Rodrigues operator of classical orthogonal polynomials. *Proc. Amer. Math. Soc. 140 (2012), no. 10, 3485–3493*

THANK YOU

FOR YOUR ATTENTION !!