Approximation and Convergence of the First Intrinsic Volume

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IST Austria

I An Example

II Intrinsic Volume

III Persistent Homology

IV Convergence

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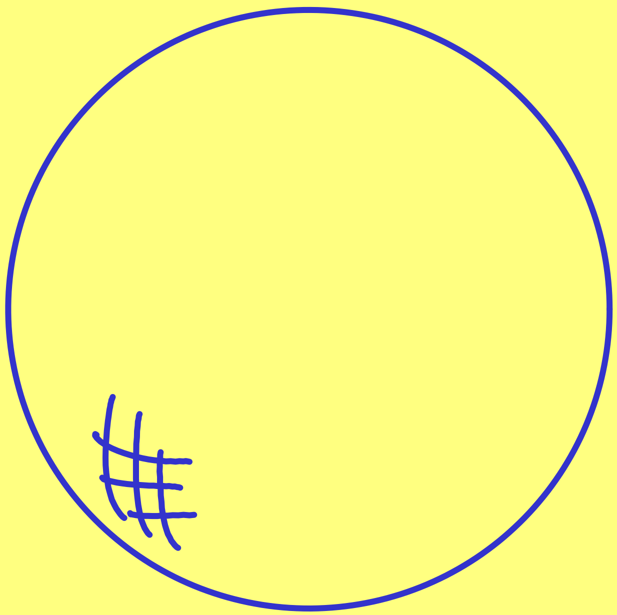
IV Convergence

I.1 Volume of Unit Ball



$\mathbb{B}^3 : \ \|x\| \leq 1.$

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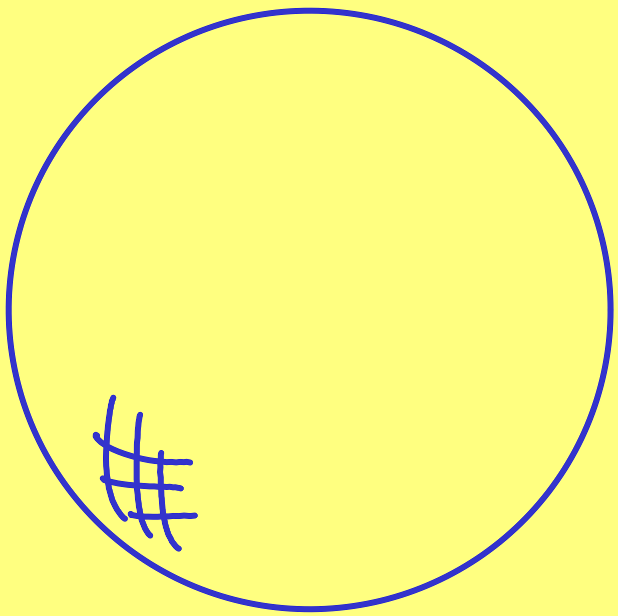

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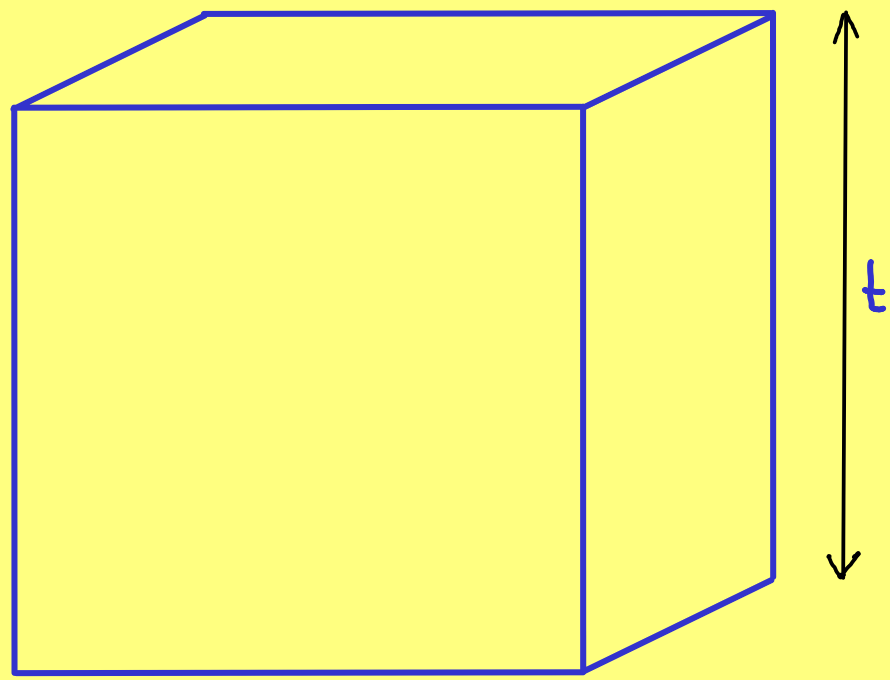

$\mathbb{B}^3_t$ is resolution-t approximation

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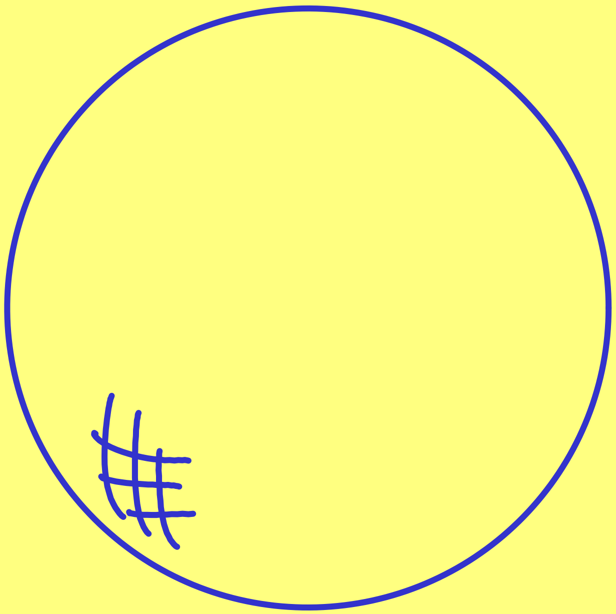

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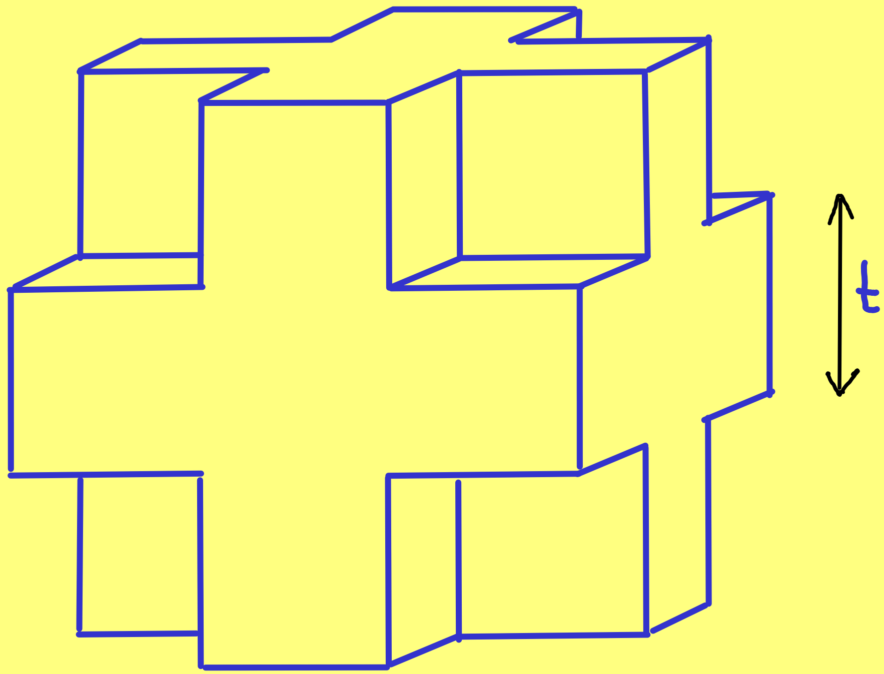

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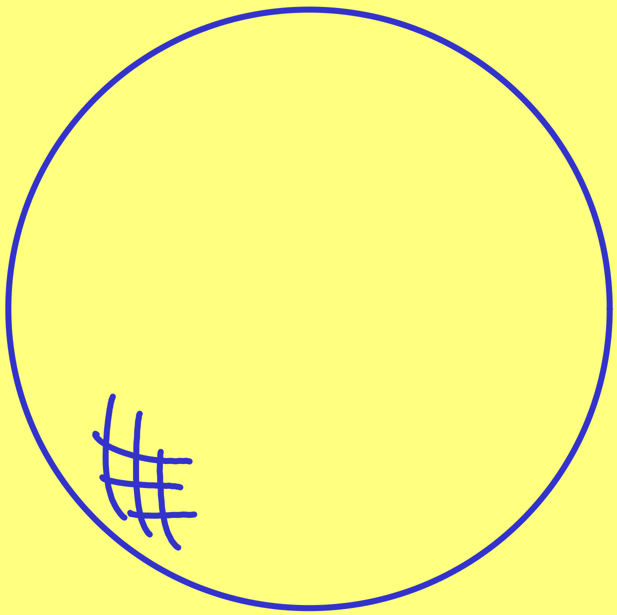
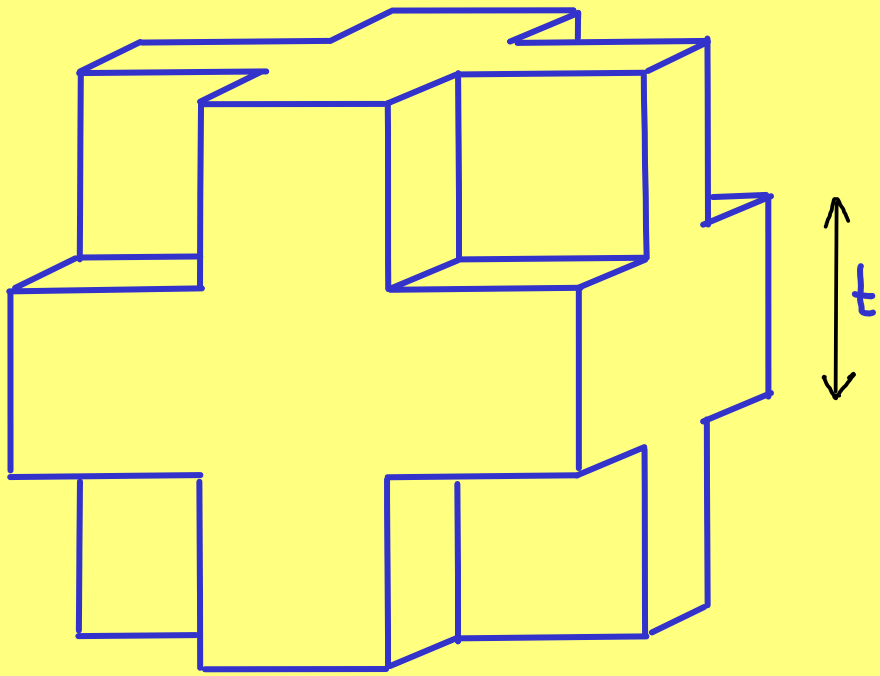


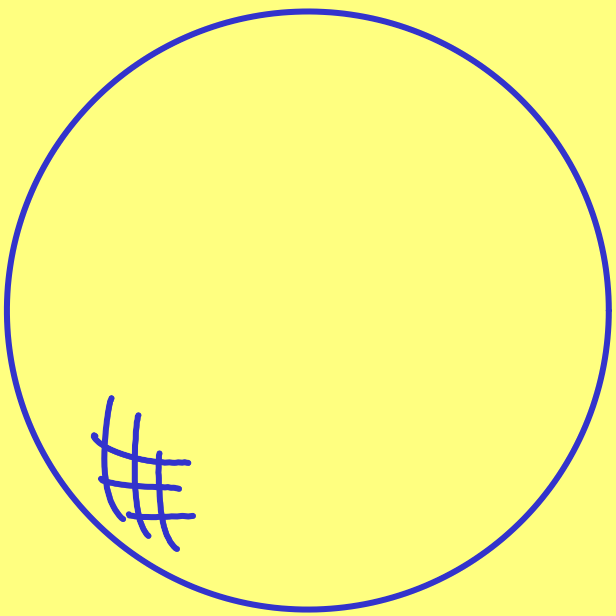


$\mathbb{B}^3 : \|x\| \leq 1.$

$\mathbb{B}^3_t$ is resolution-t approximation

$$\lim_{t \to 0} \mathrm{Vol}(\mathbb{B}^3_t) = \lim_{t \to 0} t^3 \#(\mathbb{B}^3 \cap t\mathbb{Z}^3) = \mathrm{Vol}(\mathbb{B}^3) = \frac{4}{3}\pi$$

I.2 Area of Unit Ball



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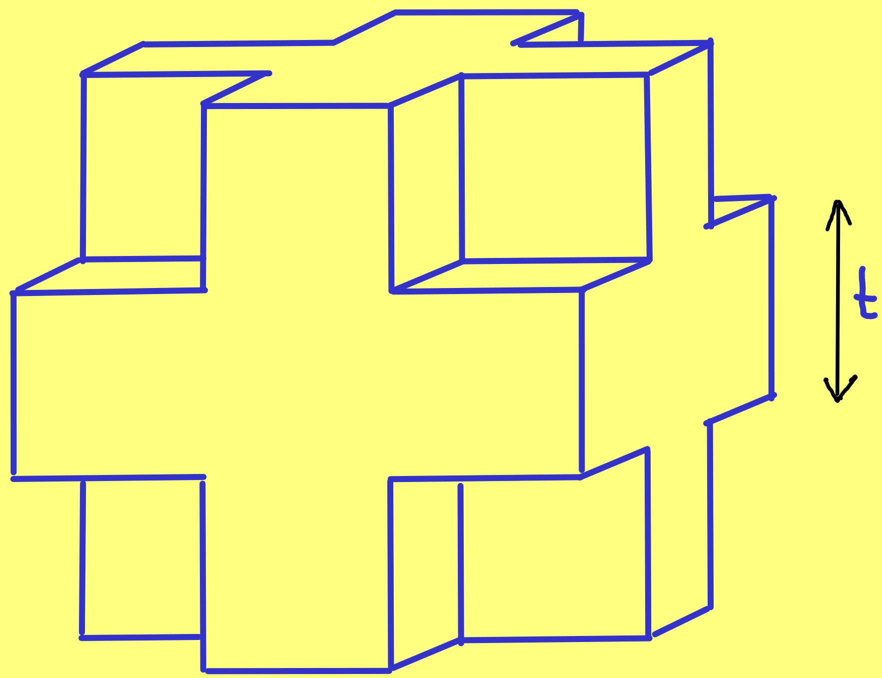

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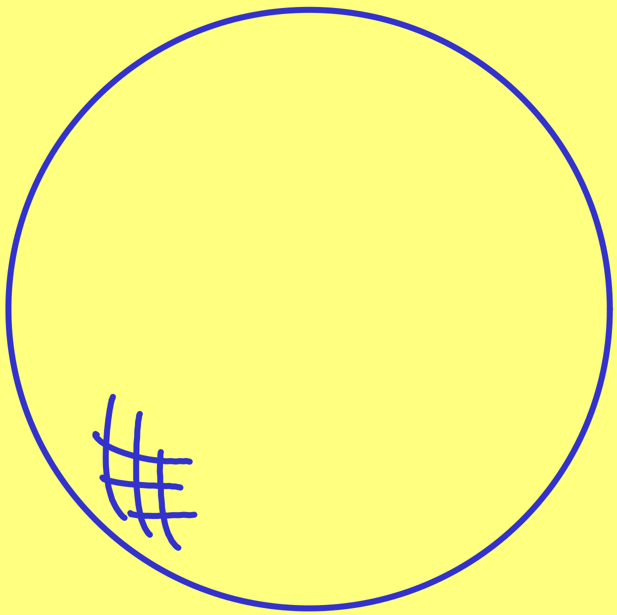

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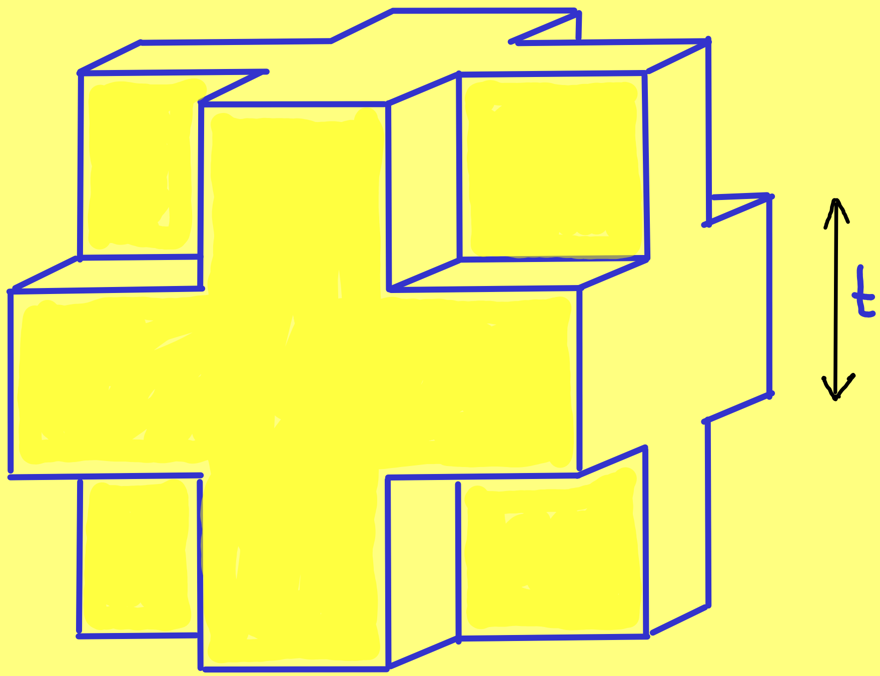

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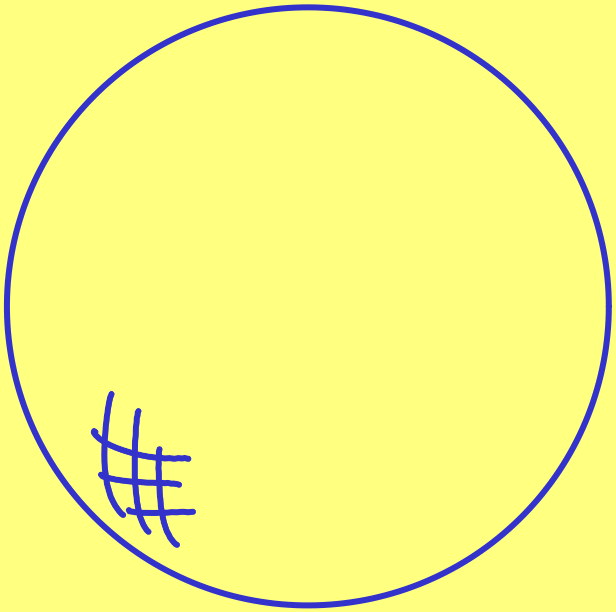
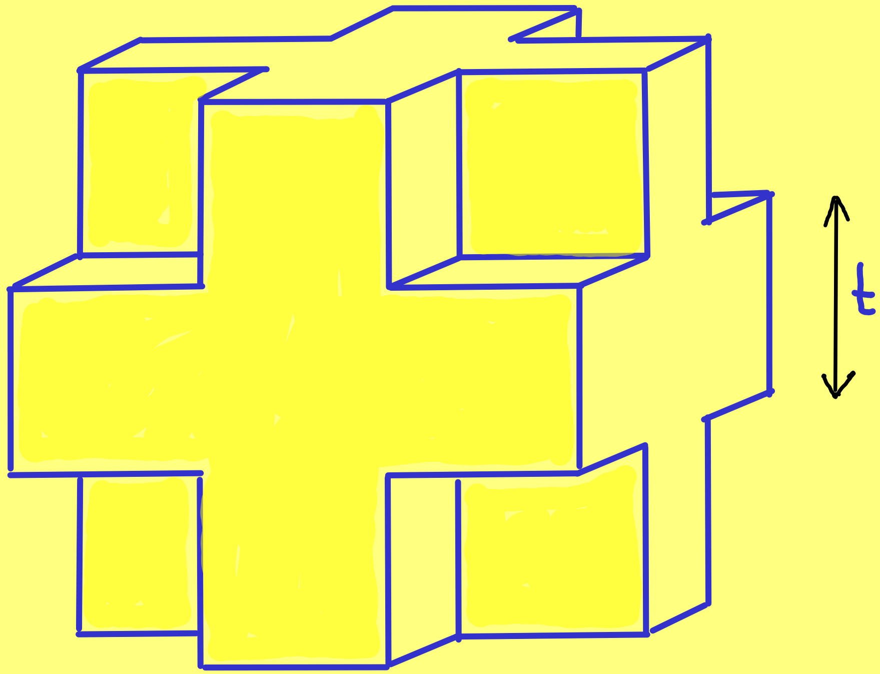


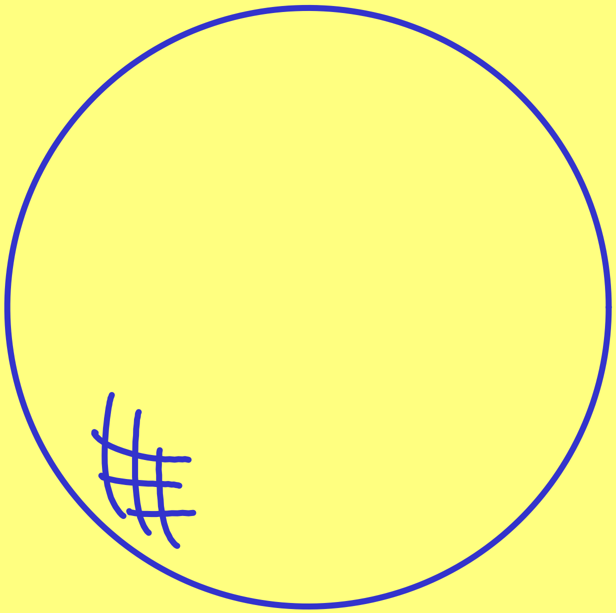


$\mathbb{B}^3 : \|x\| \leq 1.$

$\mathbb{B}^3_t$ is resolution-t approximation

$$\lim_{t\to 0} \mathrm{Area}(\mathbb{B}^3_t) = \lim_{t\to 0} 6t^2 \#(\mathbb{B}^2 \cap t\mathbb{Z}^2) = 6\,\mathrm{Area}(\mathbb{B}^2) = 6\pi$$

I.3 Mean Curvature of Unit Ball



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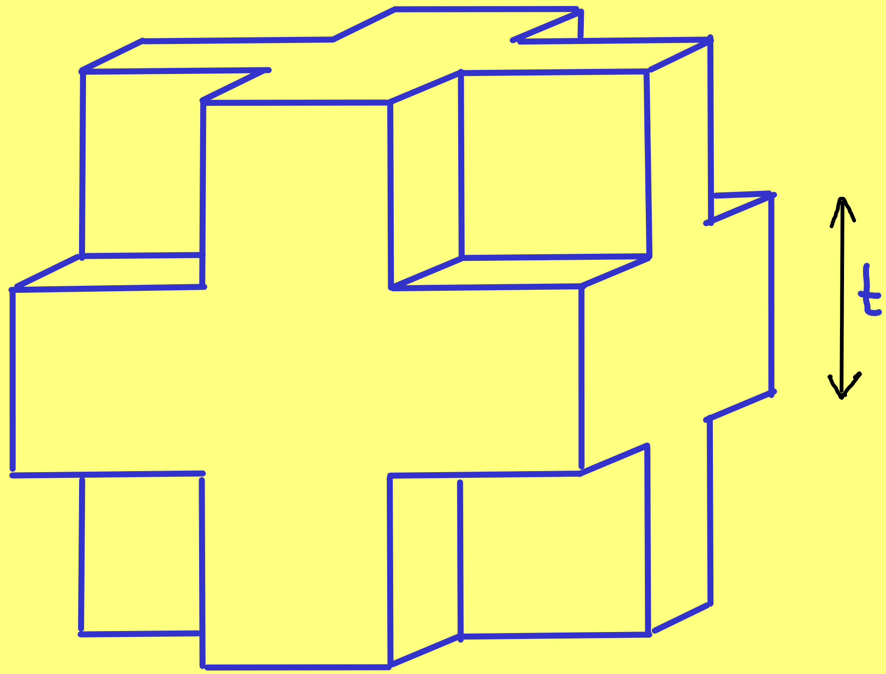

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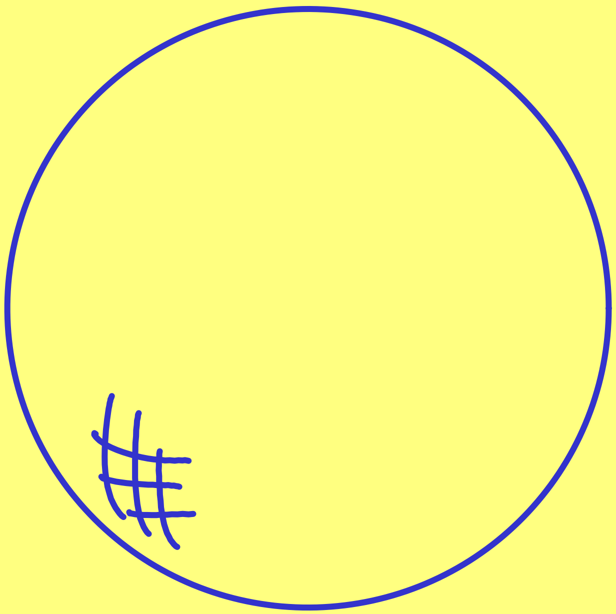

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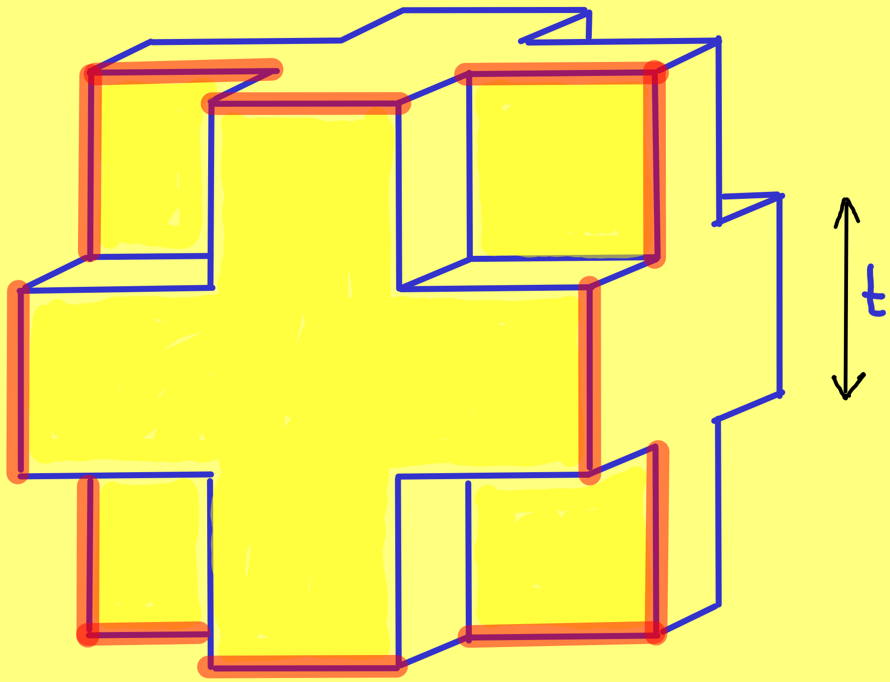

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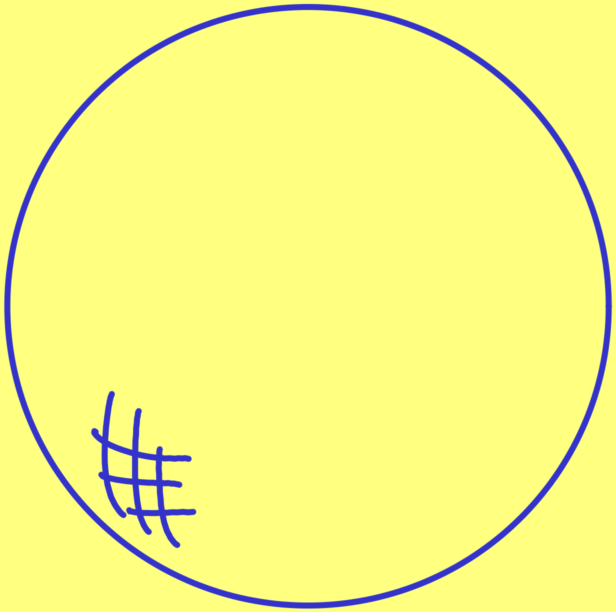

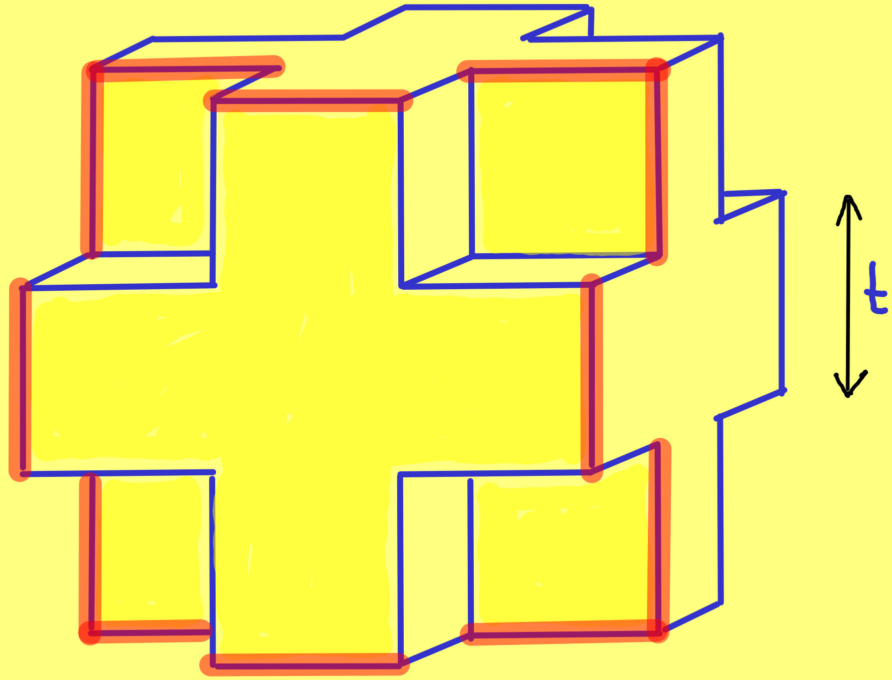

$\mathbb{B}^3 : \|x\| \leq 1.$

$\mathbb{B}^3_t$ is resolution-t approximation

$$\lim_{t\to 0} \mathrm{Mean}(\mathbb{B}^3_t) = \lim_{t\to 0} 3\pi t\, \#(\mathbb{B}^1 \cap t\mathbb{Z}^1) = 3\pi\, \mathrm{Length}(\mathbb{B}^1) = 6\pi$$

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II.1 Steiner Polynomial (1796-1863)

$\mathbb{M} \subseteq \mathbb{R}^n$.

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b_k: volume of $\mathbb{B}^k$

V_{n-k}: $(n-k)$-th intrinsic volume

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$$\mathrm{Vol}(\mathbb{M} + r\mathbb{B}^n) = \sum_{k=0}^{n} \underbrace{b_k}_{\substack{\text{volume} \\ \text{of } \mathbb{B}^k}} \underbrace{V_{n-k}(\mathbb{M})}_{\substack{(n-k)\text{-th} \\ \text{intrinsic volume}}} r^k$$

in $\mathbb{R}^3$: $V_3 =$ volume, $V_1 = \frac{1}{\pi}$ mean curvature

$V_2 = \frac{1}{2}$ area, $V_0 = \frac{1}{4\pi}$ Gaussian curvature

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Hadwiger (1951) : characterization theorem.

Weyl (1939) : generalization to tubes.

Federer (1969) : to positive reach.

II.2 GRASSMANIAN (1809 - 1877)

$\mathcal{L}_k^n \subseteq \mathcal{E}_k^n$ spaces of k-planes in $\mathbb{R}^n$

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Haar measure normalized s.t.

$$\nu(\mathcal{L}^n_k) = 1,$$

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$\mathcal{L}_k^n \subseteq \mathcal{E}_k^n$ spaces of k-planes in $\mathbb{R}^n$

linear affine

Haar measure normalized s.t.

$$\nu(\mathcal{L}_k^n) = 1,$$

$$\mu(E \in \mathcal{E}_k^n \mid E \cap \mathbb{B}^n \neq \emptyset) = b_{n-k}.$$

II.3 CROFTON FORMULA (1826-1915)

$$V_{n-k}(M) = c_{k,n} \cdot \int_{E \in \mathcal{E}^n_k} \chi(M \cap E)\, dE$$

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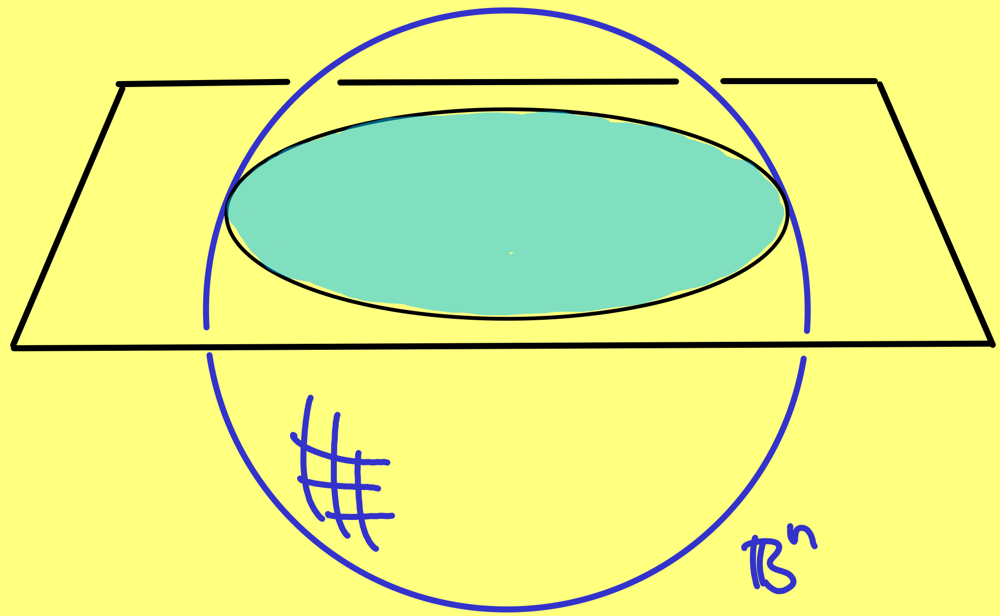
$$V_{n-k}(\mathbb{B}^n) = \binom{n}{k} \frac{b_n}{b_k}.$$

II.3 Crofton Formula (1826-1915)

$$V_{n-k}(\mathbb{M}) = c_{k,n} \cdot \int_{E \in \mathcal{E}^n_k} \chi(\mathbb{M} \cap E)\, dE$$

$$c_{k,n} = \binom{n}{k} \frac{b_n}{b_k \cdot b_{n-k}}$$

$$V_{n-k}(\mathbb{B}^n) = \binom{n}{k} \frac{b_n}{b_k}.$$



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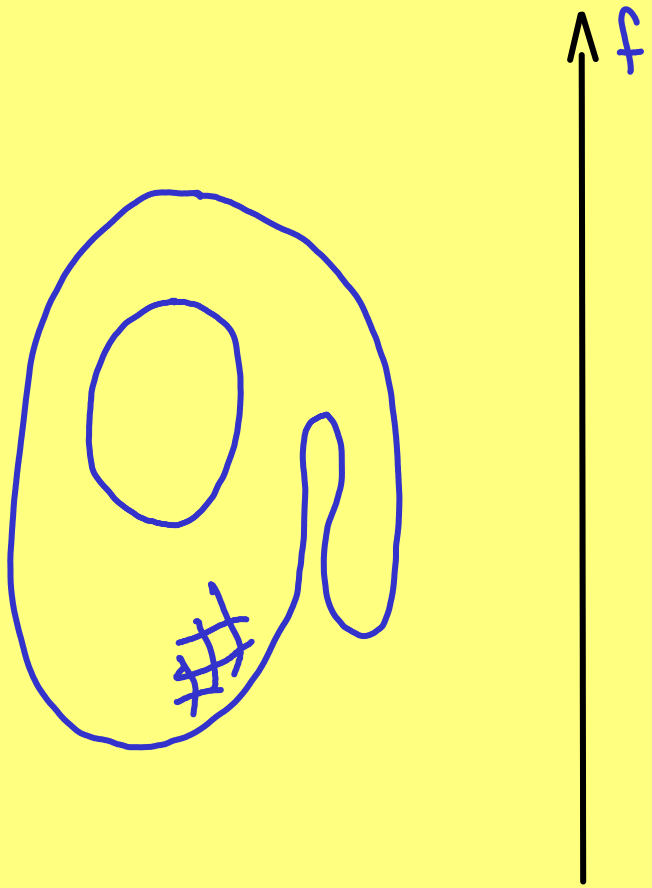

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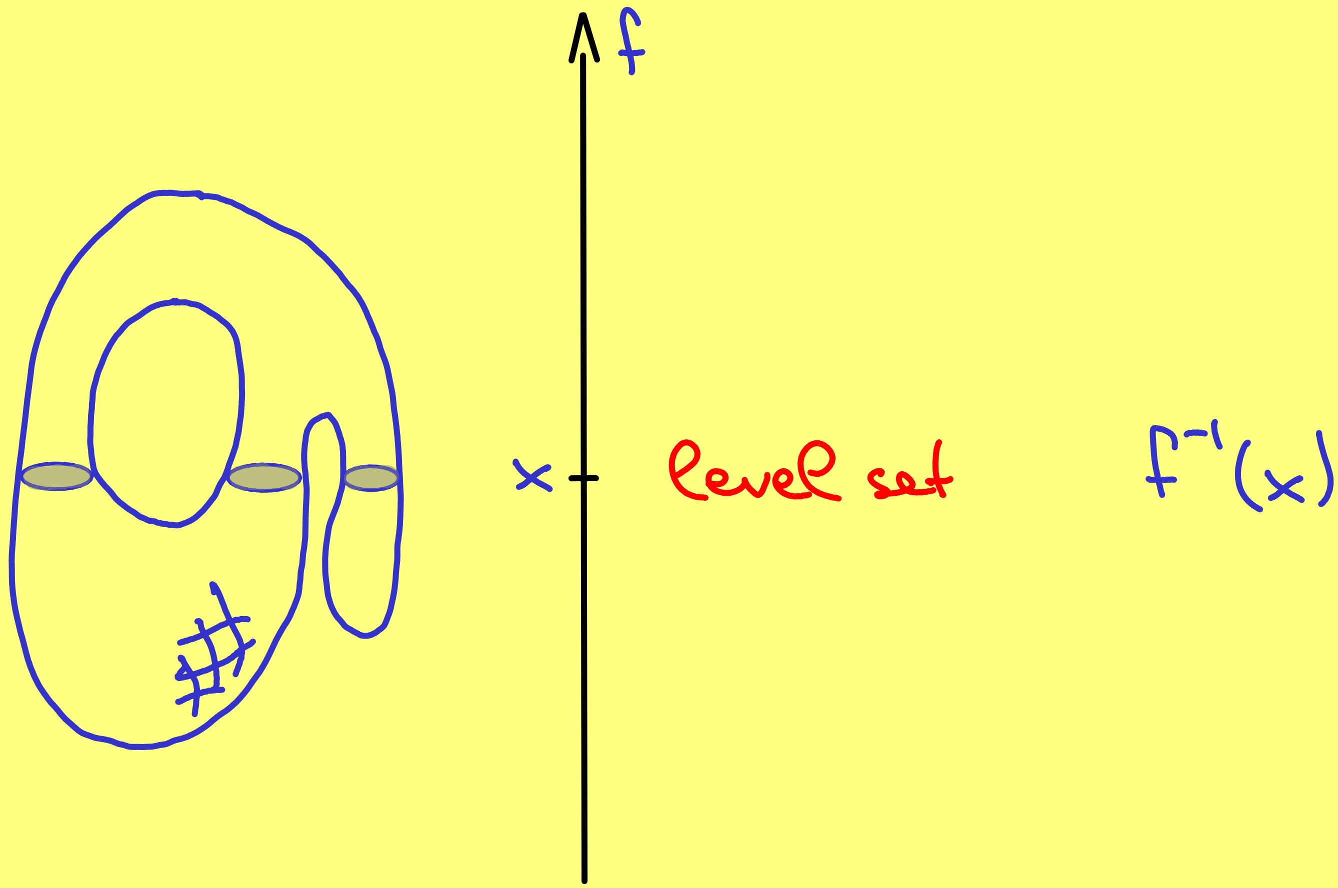

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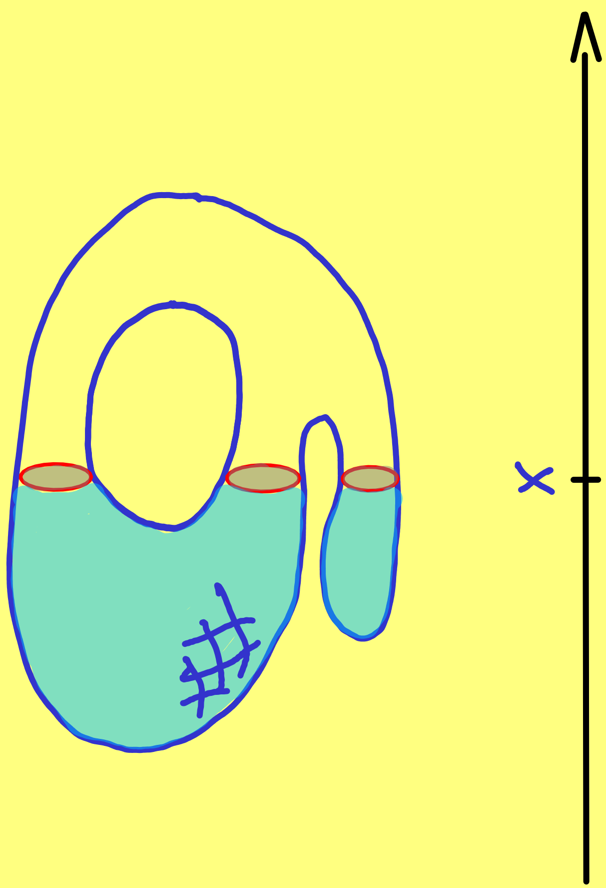
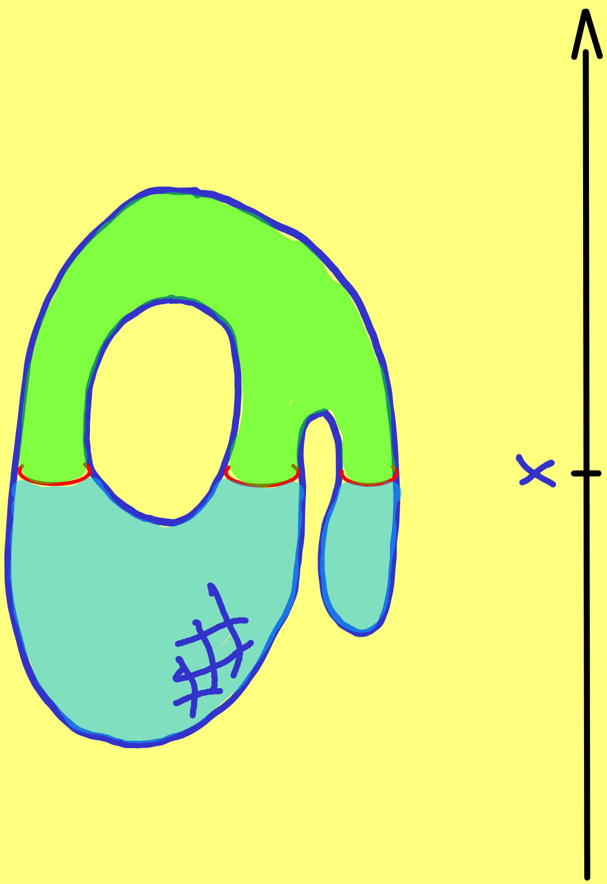


level set $f^{-1}(x)$

sublevel set $f^{-1}(-\infty, x]$

III.1 Height Function

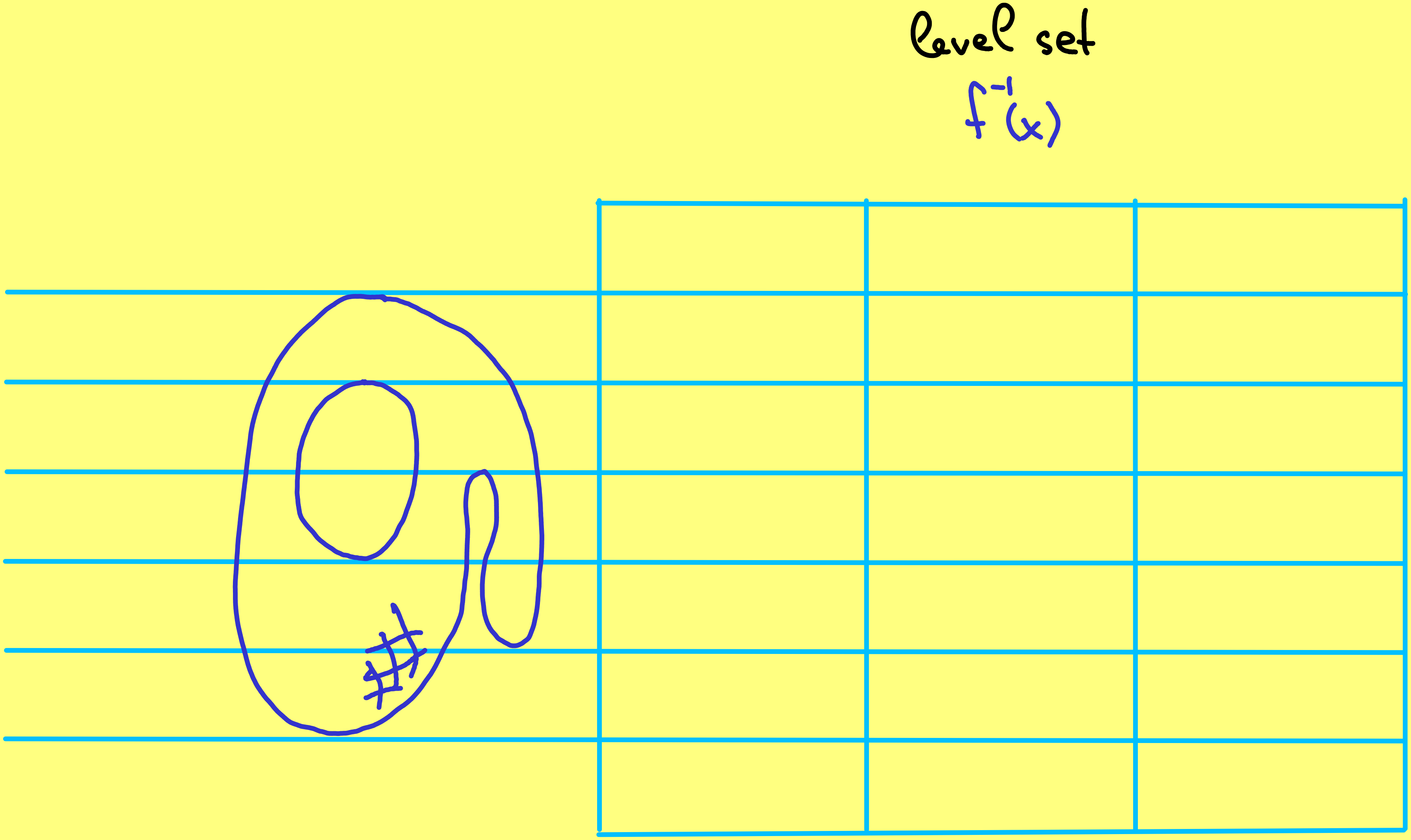


superlevel set $f^{-1}[x,\infty)$

level set $f^{-1}(x)$

sublevel set $f^{-1}(-\infty,x]$

III.2 LEVEL SETS



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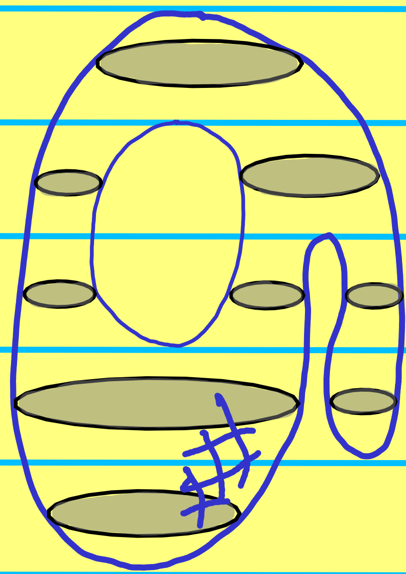
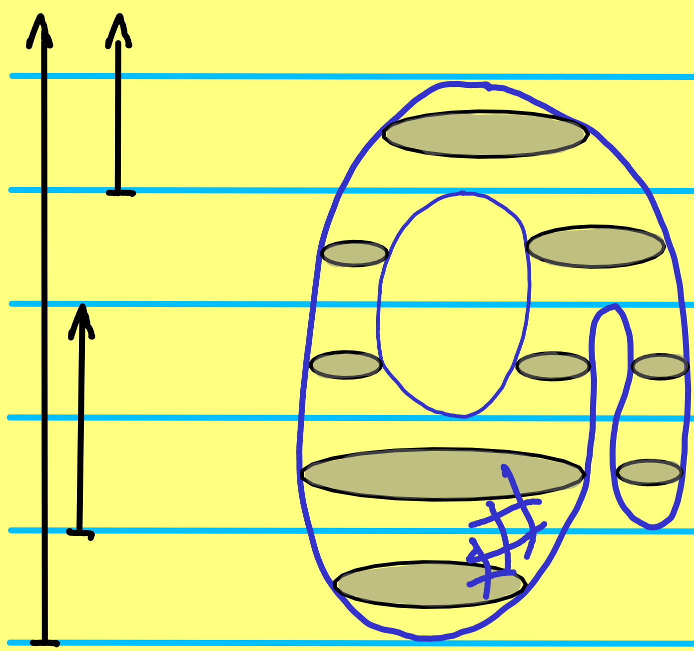


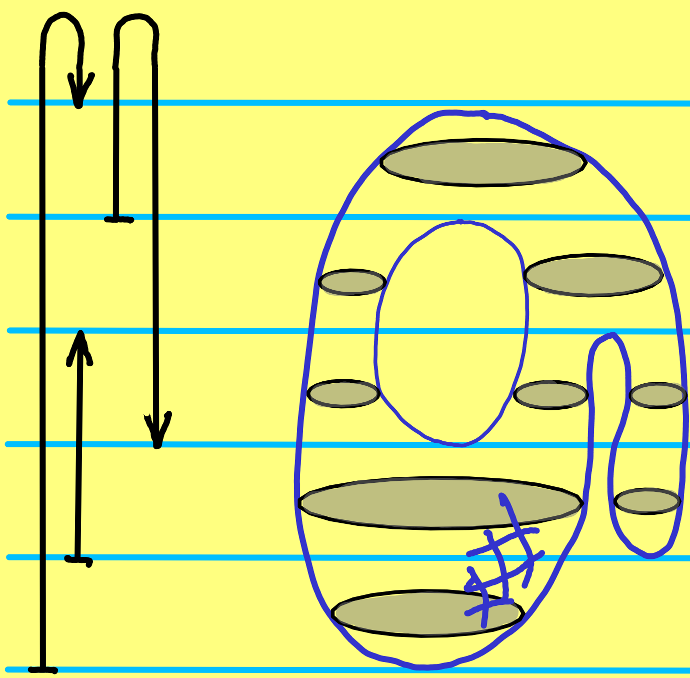
	level set $f^{-1}(x)$	
	0 0 0	
	1 0 0	
	2 0 0	
	3 0 0	
	2 0 0	
	1 0 0	
	0 0 0	

III.2 Level Sets



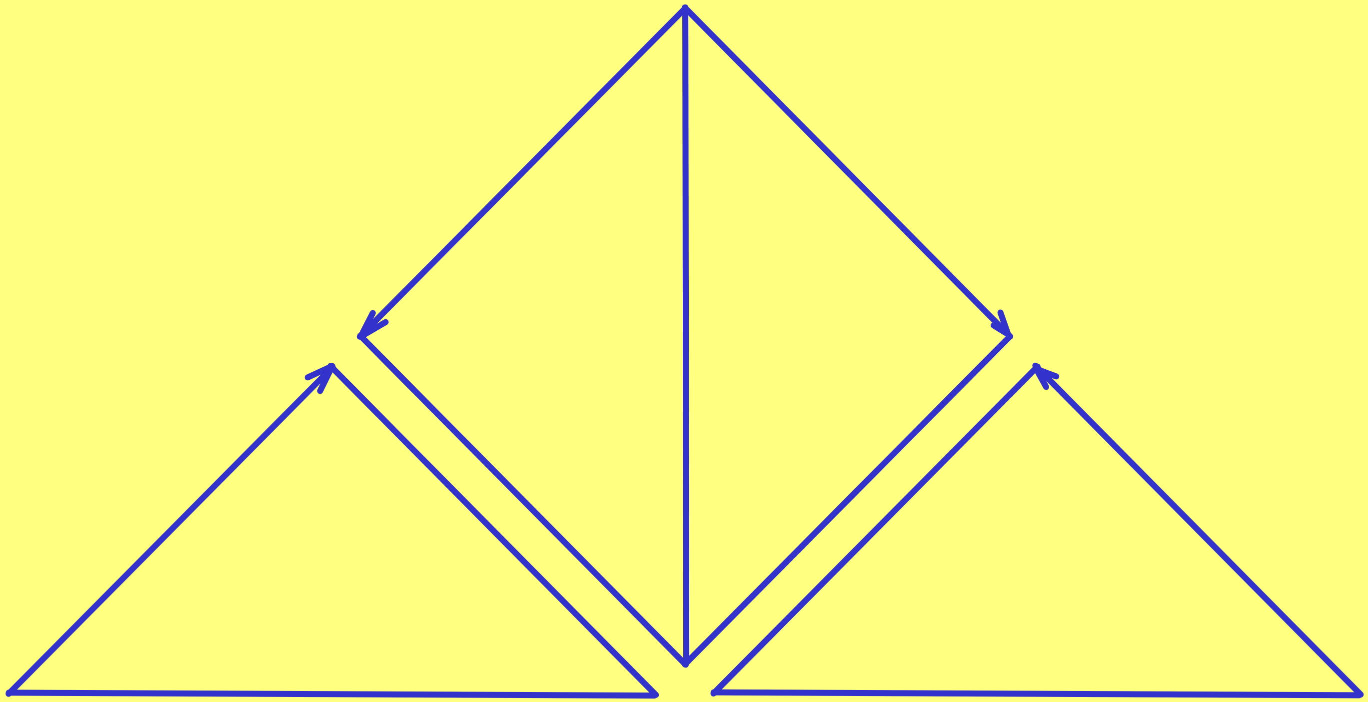
sublevel set $f^{-1}(-\infty, x]$	level set $f^{-1}(x)$	
1 1 0	0 0 0	
1 1 0	1 0 0	
1 0 0	2 0 0	
2 0 0	3 0 0	
2 0 0	2 0 0	
1 0 0	1 0 0	
0 0 0	0 0 0	

III.2 Level Sets



sublevel set $f^{-1}(-\infty, x]$	level set $f^{-1}(x)$	superlevel set $f^{-1}[x, \infty)$
1 1 0	0 0 0	1 1 0
1 1 0	1 0 0	0 1 0
1 0 0	2 0 0	0 1 0
2 0 0	3 0 0	0 1 0
2 0 0	2 0 0	0 0 0
1 0 0	1 0 0	0 0 0
0 0 0	0 0 0	0 0 0

III.3 Diagram



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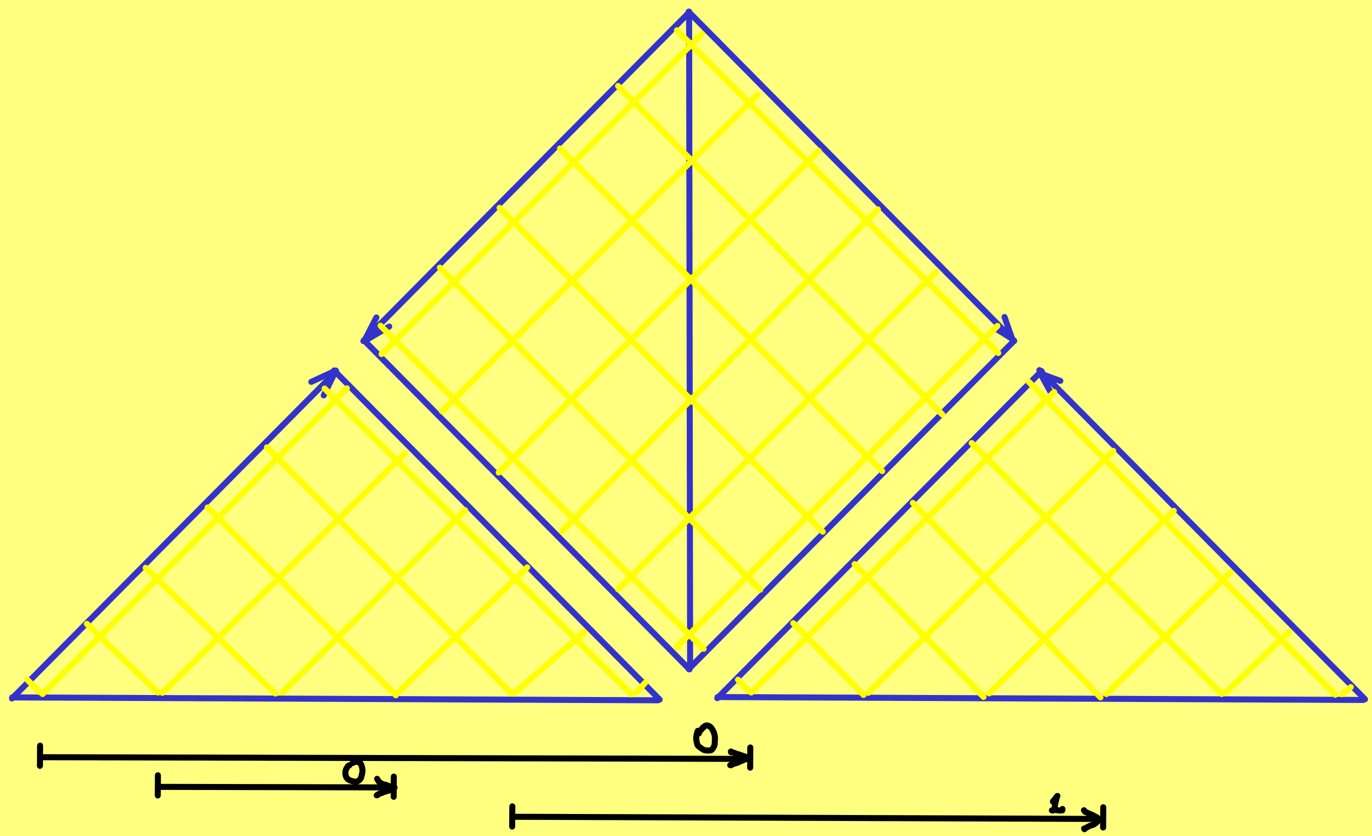

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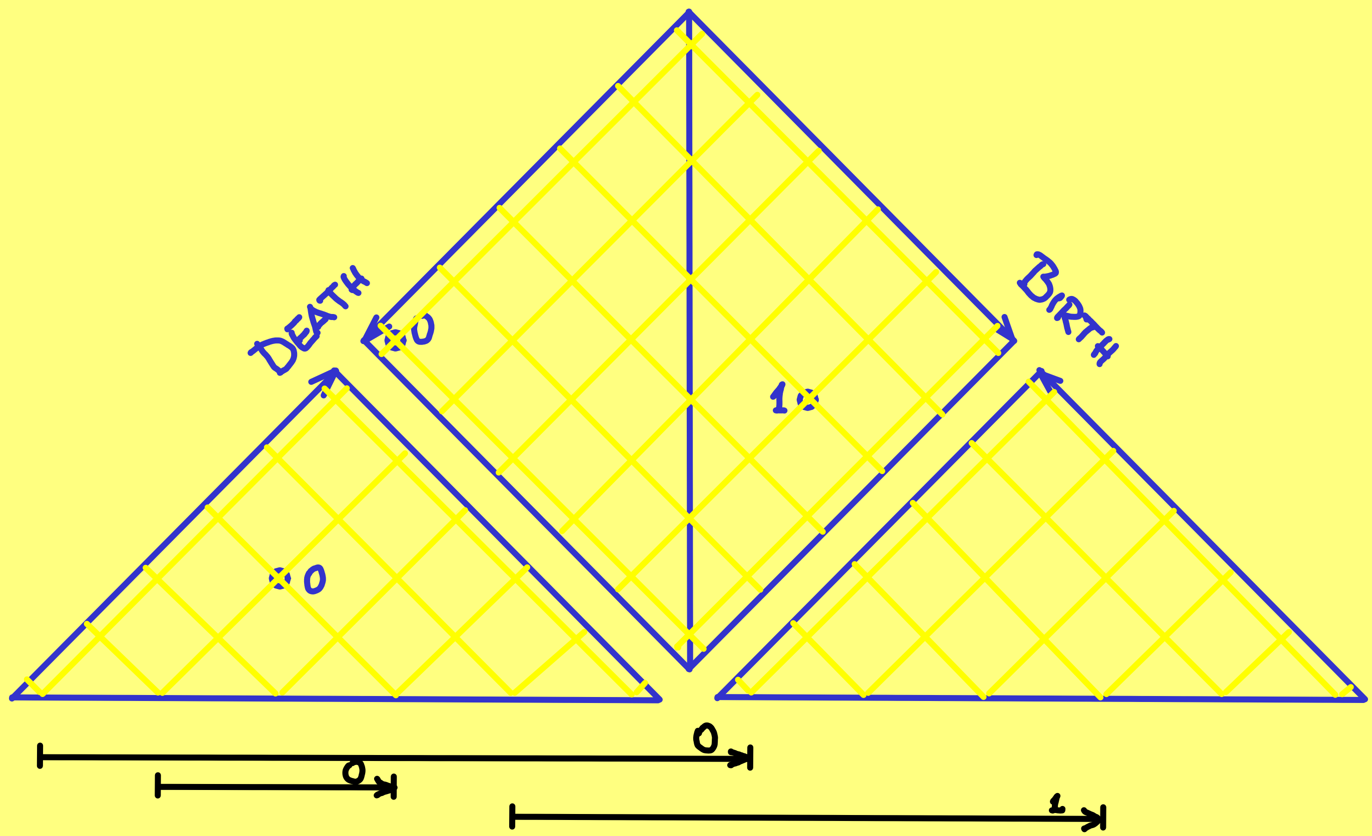

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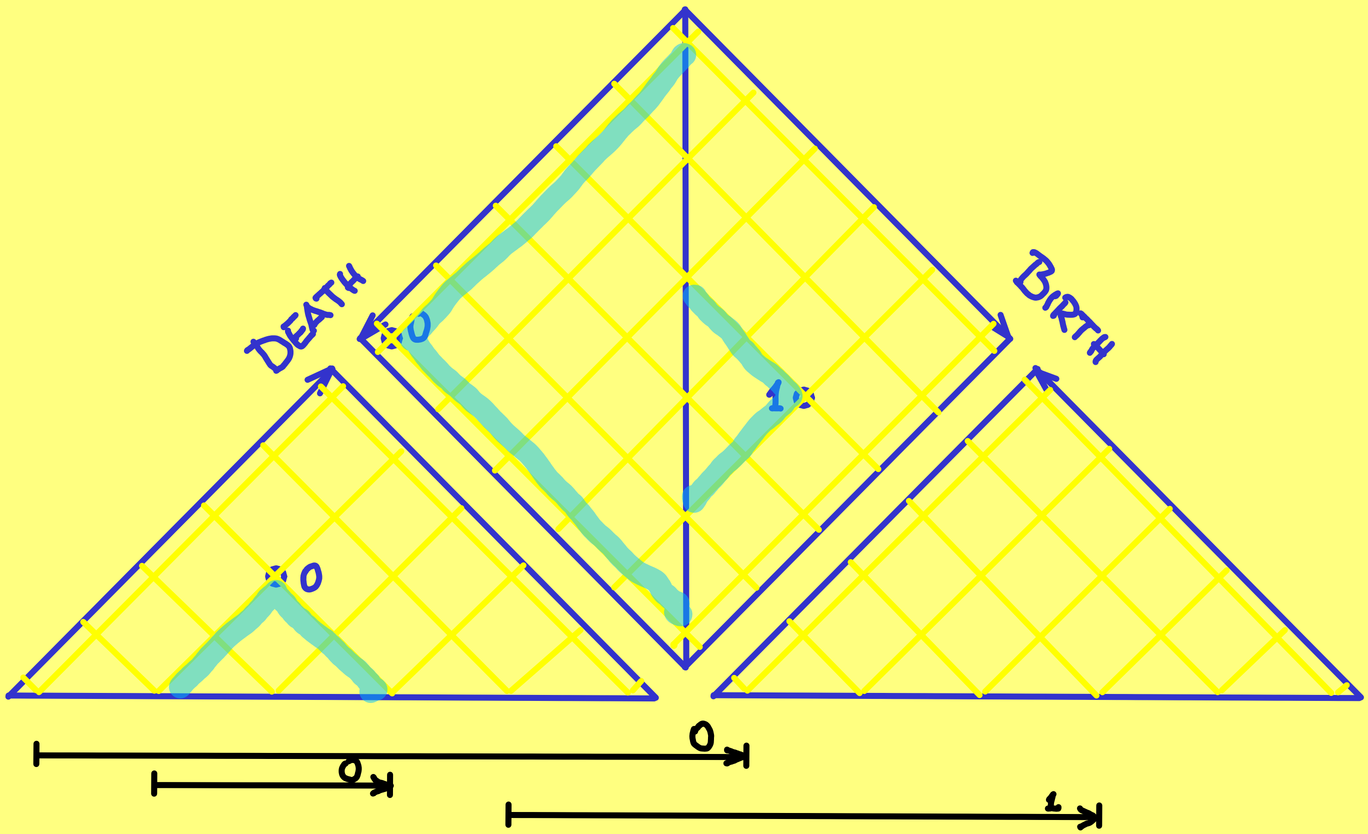

$$A = (x_b, x_d), \quad \text{pers}(A) = |x_d - x_b|.$$

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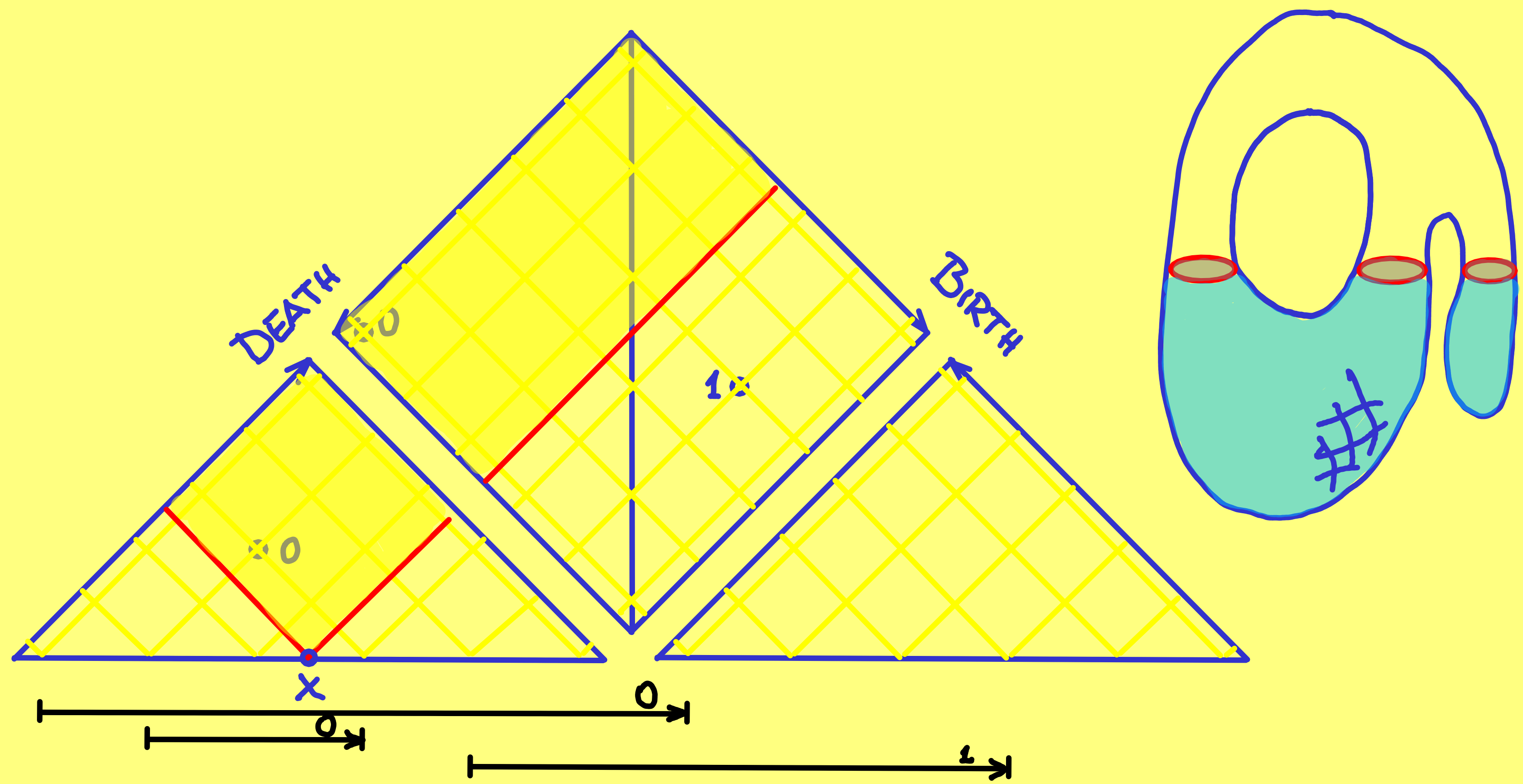

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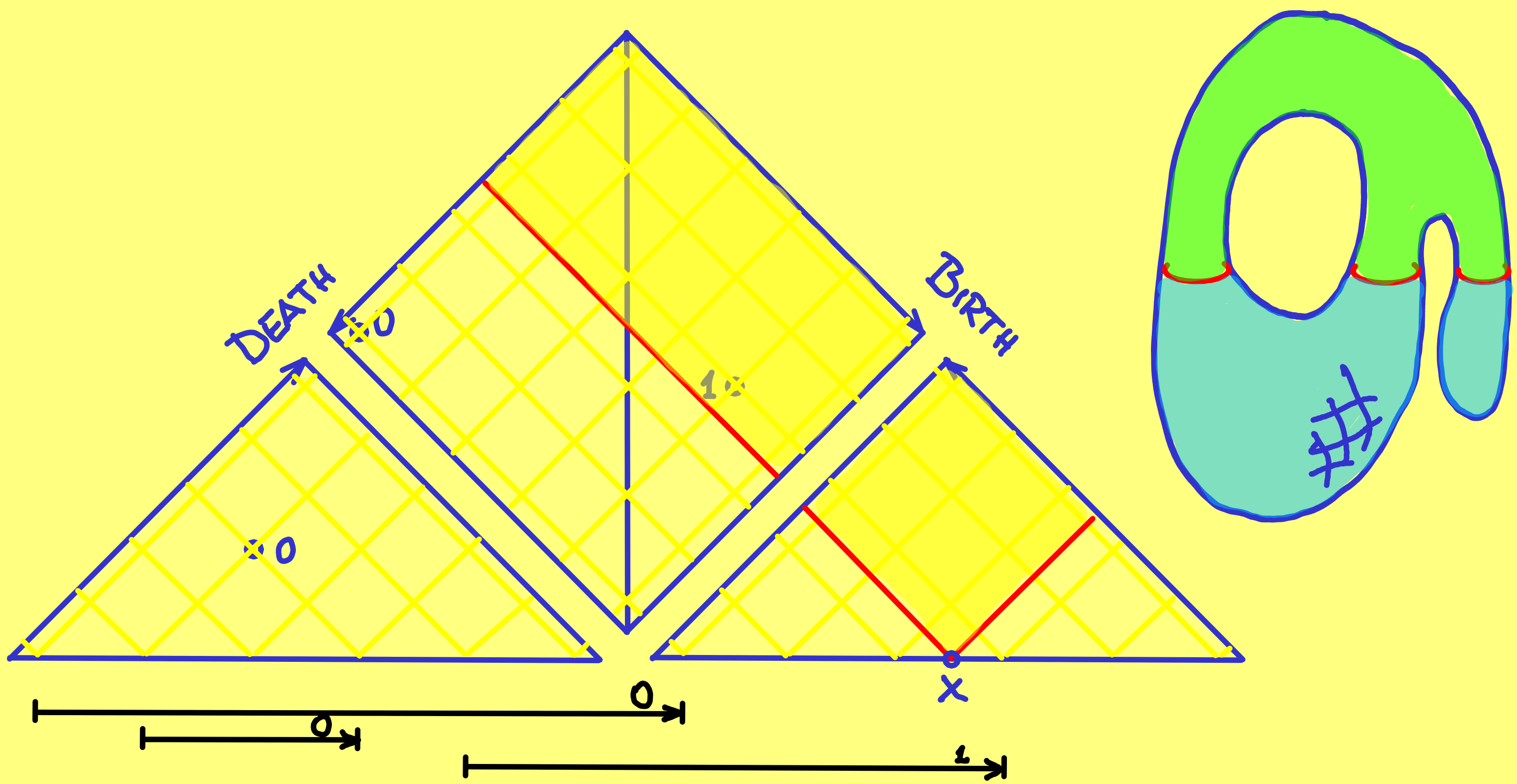

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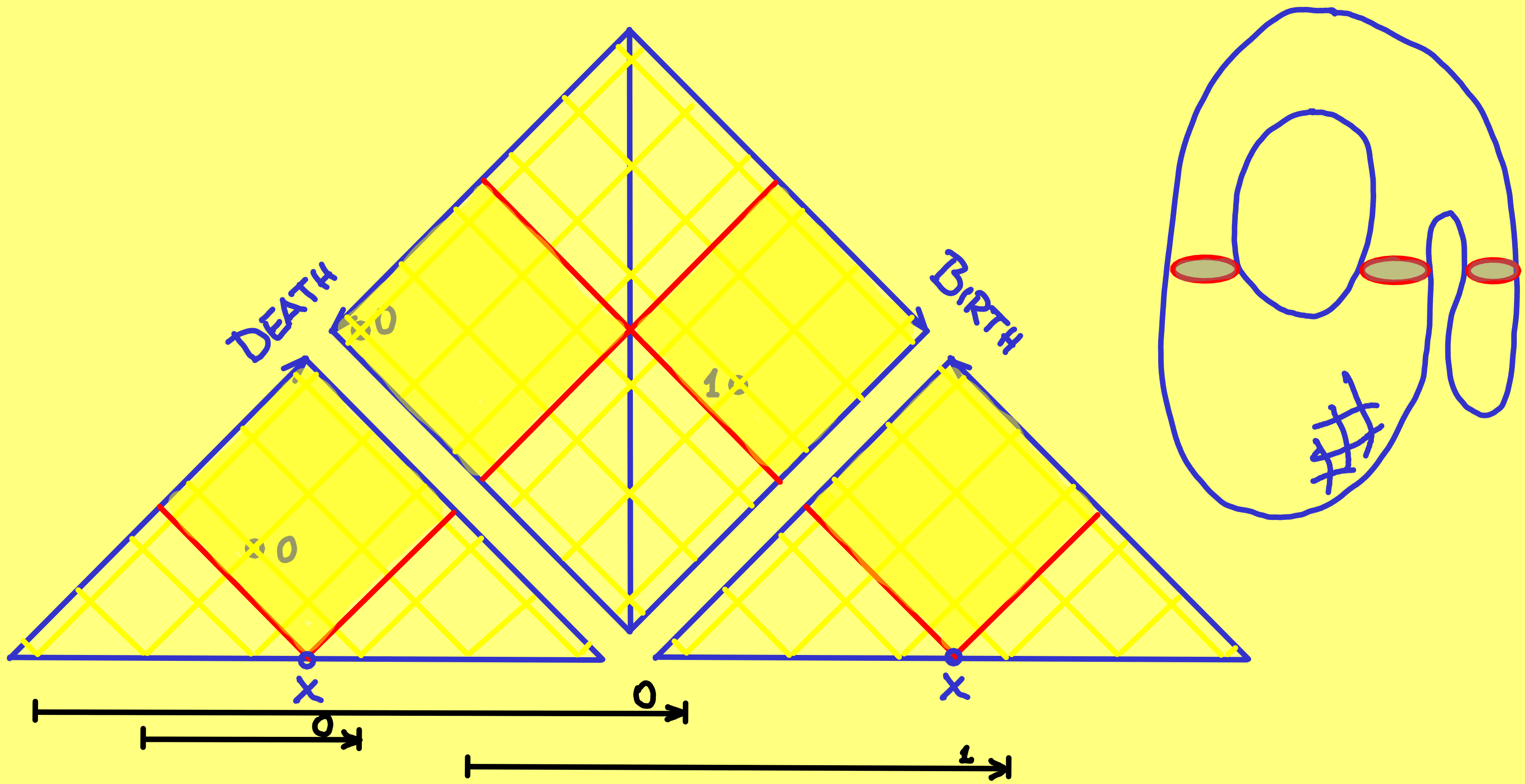

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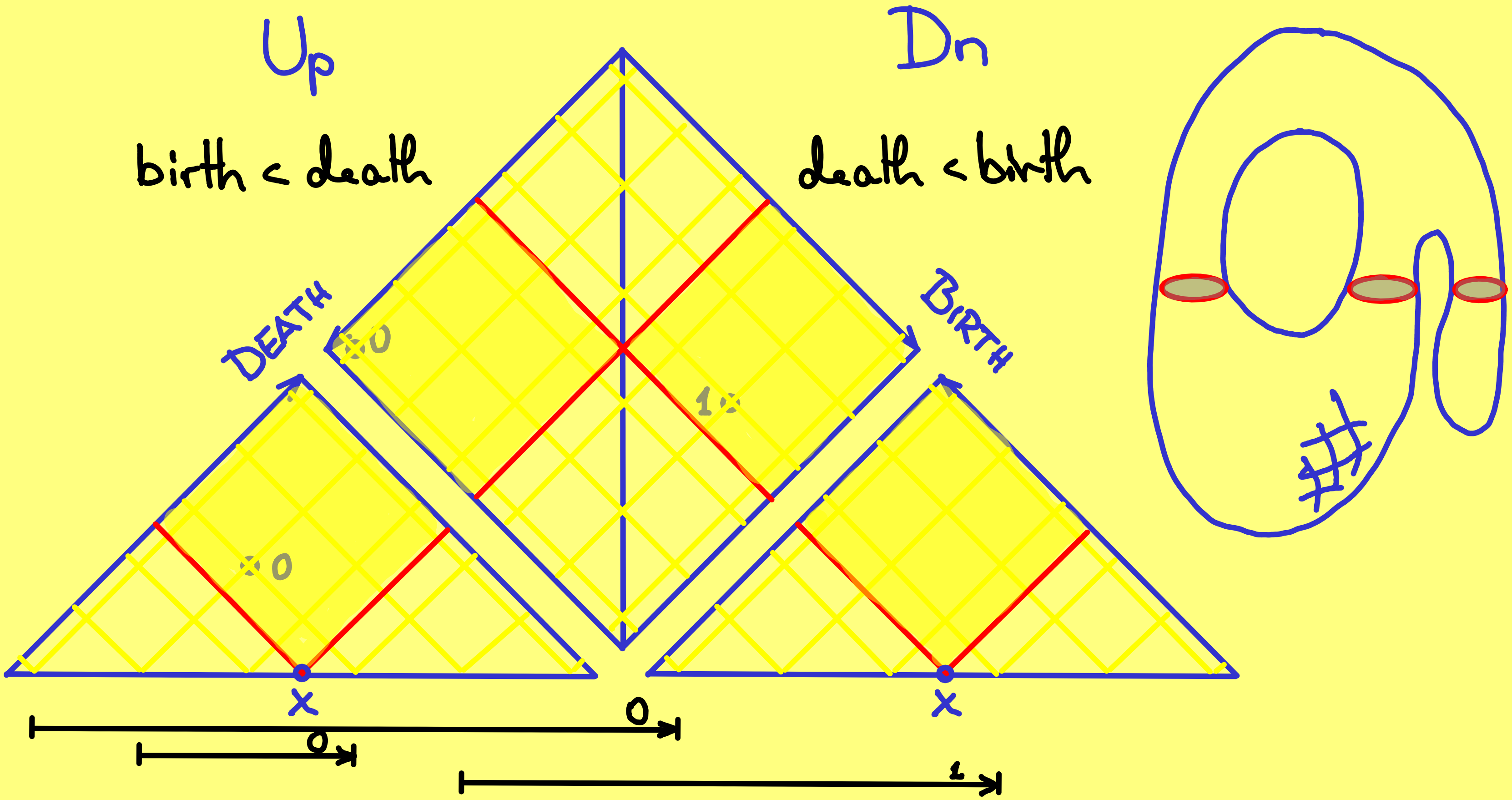
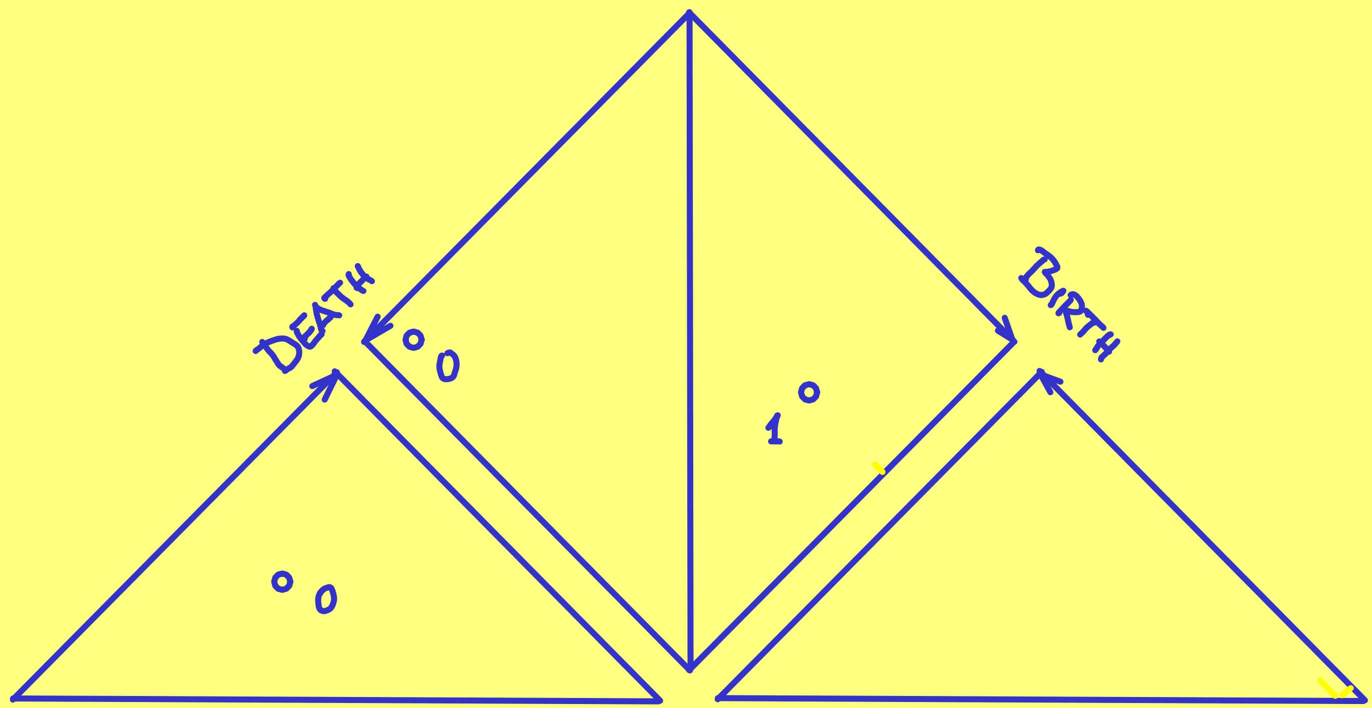


THM. $\beta_k(f^{-1}(x)) = \#Up_k(f,x) + \#Dn_{k+1}(f,x).$

III.3 DIAGRAM STABILITY



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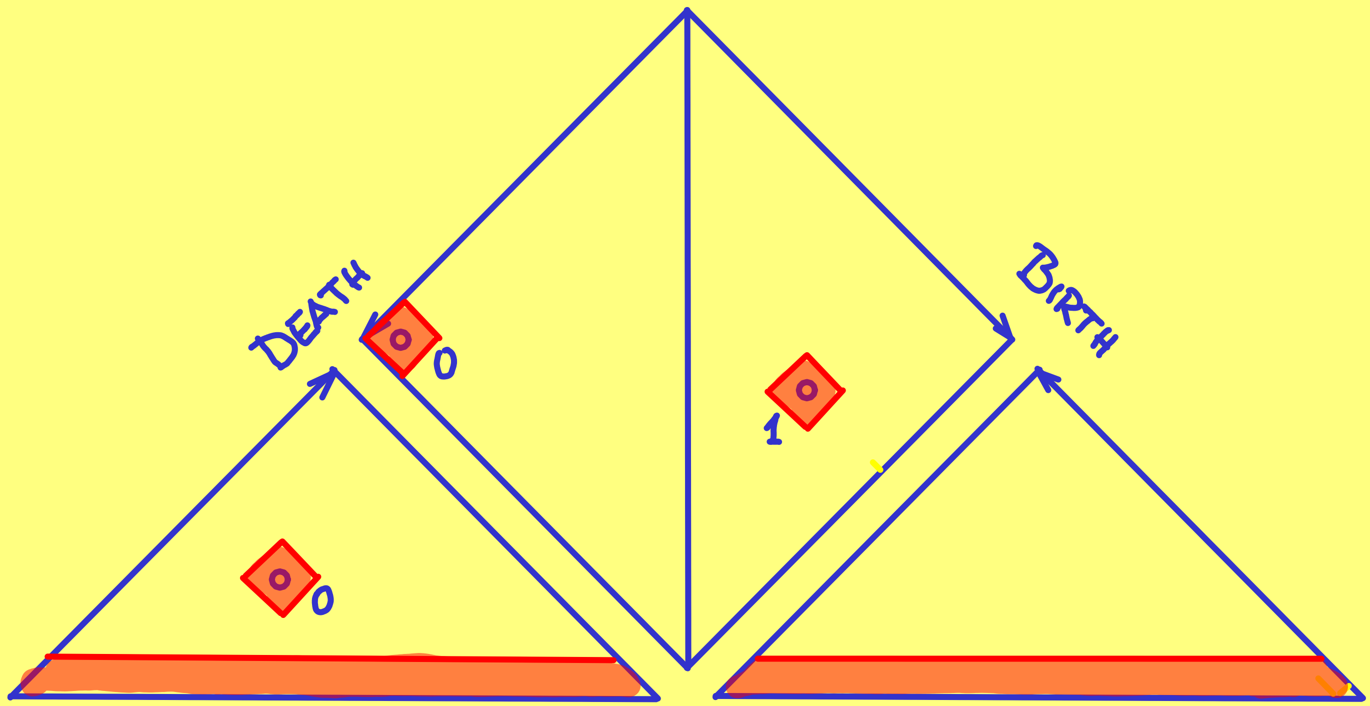

Thm. $W_\infty(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq \|f-g\|_\infty$.

III.4 # MOMENT

$$\chi(f^{-1}(x)) = \sum_{k=0}^{n} (-1)^k \left[\# Up_k(f,x) - \# Dn_k(f,x)\right].$$

III.4

Moment

$$\chi(f^{-1}(x)) = \sum_{k=0}^{n} (-1)^k \left[\#\mathrm{Up}_k(f,x) - \#\mathrm{Dn}_k(f,x) \right].$$

The χ-moment of $\mathrm{Dgm}(f)$ is

$$X(f \quad) = \int_{x=-\infty}^{\infty} \chi(f^{-1}(x))\, dx = \sum_{k=0}^{n} (-1)^k \sum_{A \in \mathrm{Dgm}_k} (x_d - x_b)$$

III.4 MODIFIED MOMENT

$$\chi(f^{-1}(x)) = \sum_{k=0}^{n} (-1)^k \left[\#\mathrm{Up}_k(f,x) - \#\mathrm{Dn}_k(f,x)\right].$$

The modified χ-moment of $\mathrm{Dgm}(f)$ and $\varepsilon > 0$ is

$$X(f,\varepsilon) = \cancel{\int_{x=-\infty}^{\infty} \chi(f^{-1}(x))\,dx} = \sum_{k=0}^{n} (-1)^k \sum_{\substack{A \in \mathrm{Dgm}_k \\ |x_d - x_b| > \varepsilon}} (x_d - x_b)$$

III.5 # Intrinsic Volume

The 1st intrinsic volume

of $\mathbb{M} \subseteq \mathbb{R}^n$ is

$$V_1(\mathbb{M} \quad) = c_{n-1,n} \cdot \int_{E \in \mathcal{E}^n_{n-1}} \chi(\mathbb{M} \cap E) \, dE$$

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$$= c_{n-1,n} \cdot \int_{L \in \mathcal{L}^n_{n-1}} \int_{x=-\infty}^{\infty} \chi(f_L^{-1}(x))\, dx\, dL$$

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$$= c_{n-1,n} \cdot \int_{L \in \mathcal{L}^n_{n-1}} \int_{x=-\infty}^{\infty} \chi(f_L^{-1}(x))\, dx\, dL$$

$$= c_{n-1,n} \cdot \int_{L \in \mathcal{L}^n_{n-1}} X(f_L \quad)\, dL.$$

III.5 Modified Intrinsic Volume

Def. The modified 1st intrinsic volume of $\mathbb{M} \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ is

$$V_1(\mathbb{M}, \varepsilon) = \cancel{c_{n-1,n} \cdot \int_{E \in \mathcal{E}^n_{n-1}} \chi(\mathbb{M} \cap E)\, dE}$$

$$= \cancel{c_{n-1,n} \cdot \int_{L \in \mathcal{L}^n_{n-1}} \int_{x=-\infty}^{\infty} \chi(f_L^{-1}(x))\, dx\, dL}$$

$$= c_{n-1,n} \cdot \int_{L \in \mathcal{L}^n_{n-1}} X(f_L, \varepsilon)\, dL.$$

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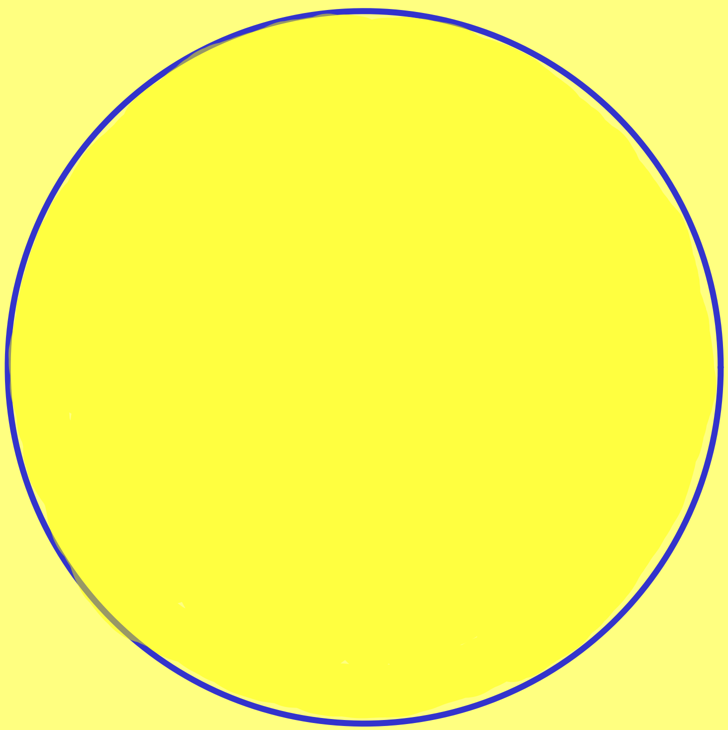
IV.1 Unit Ball

Thm. $\left| V_1(\mathbb{B}^n) - V_1(\mathbb{B}^n_{t, t\sqrt{n}}) \right| \leq c_{n-1,n} \cdot t\sqrt{n}.$

IV.1 Unit Ball

Thm. $\left| V_1(\mathbb{B}^n) - V_1(\mathbb{B}^n_t, t\sqrt{n}) \right| \leq c_{n-1,n} \cdot t\sqrt{n}.$



IV.1 Unit Ball

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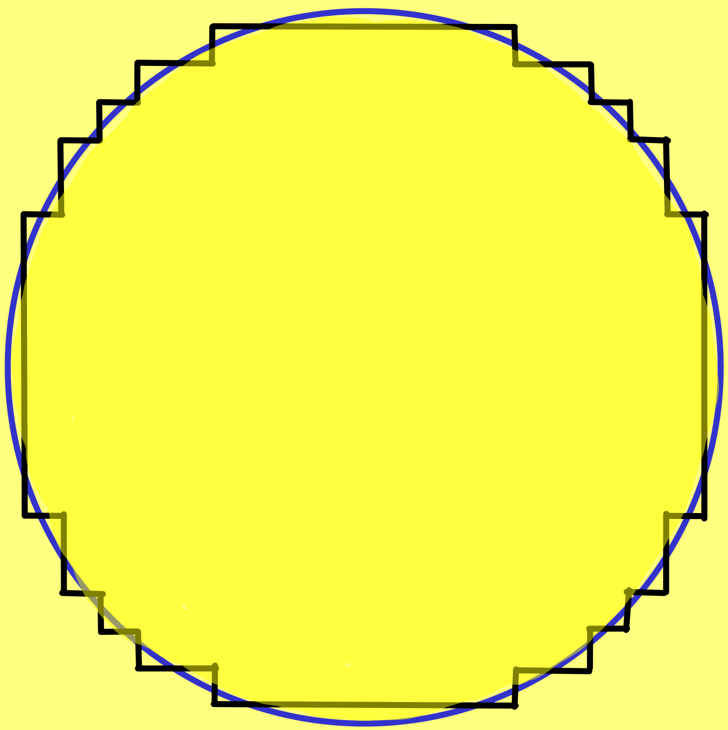
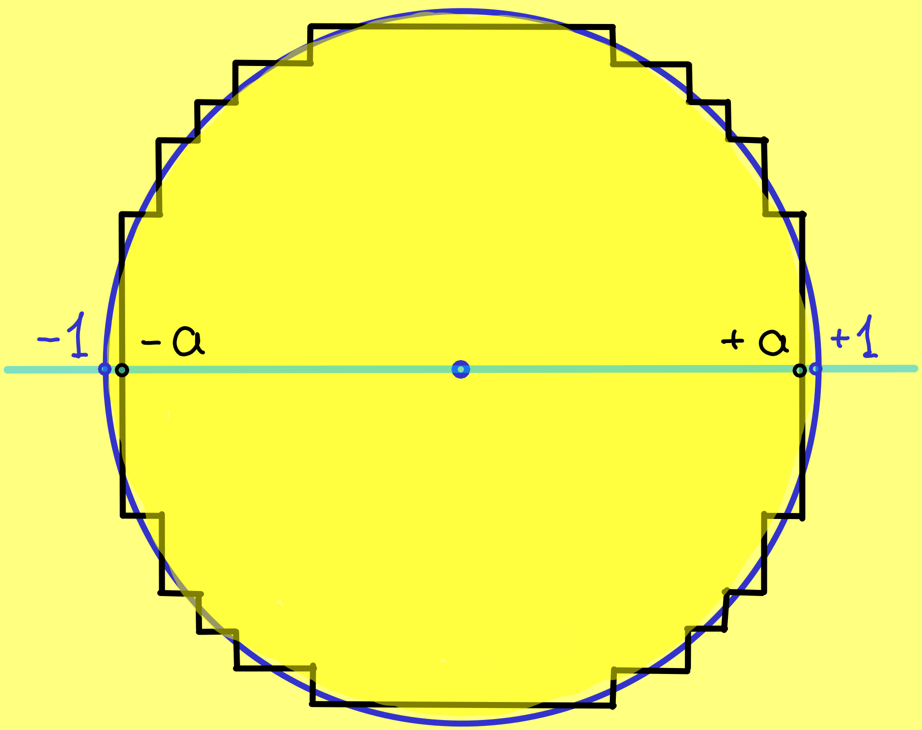


IV.1 Unit Ball

Thm. $\left| V_1(\mathbb{B}^n) - V_1(\mathbb{B}^n_t, t\sqrt{n}) \right| \leq c_{n-1,n} \cdot t\sqrt{n}.$



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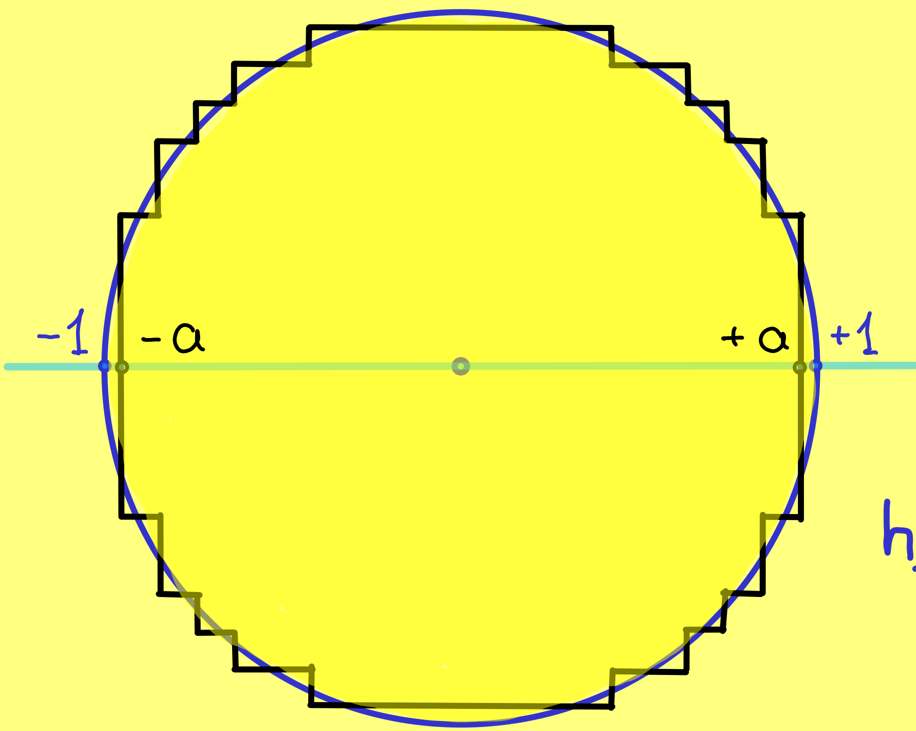

$h_t : \mathbb{B}^n \to \mathbb{B}^n_t$ with distortion

$\sup_{x \in \mathbb{B}^n} \| x - h_t(x) \| \leq \frac{1}{2} t\sqrt{n}.$

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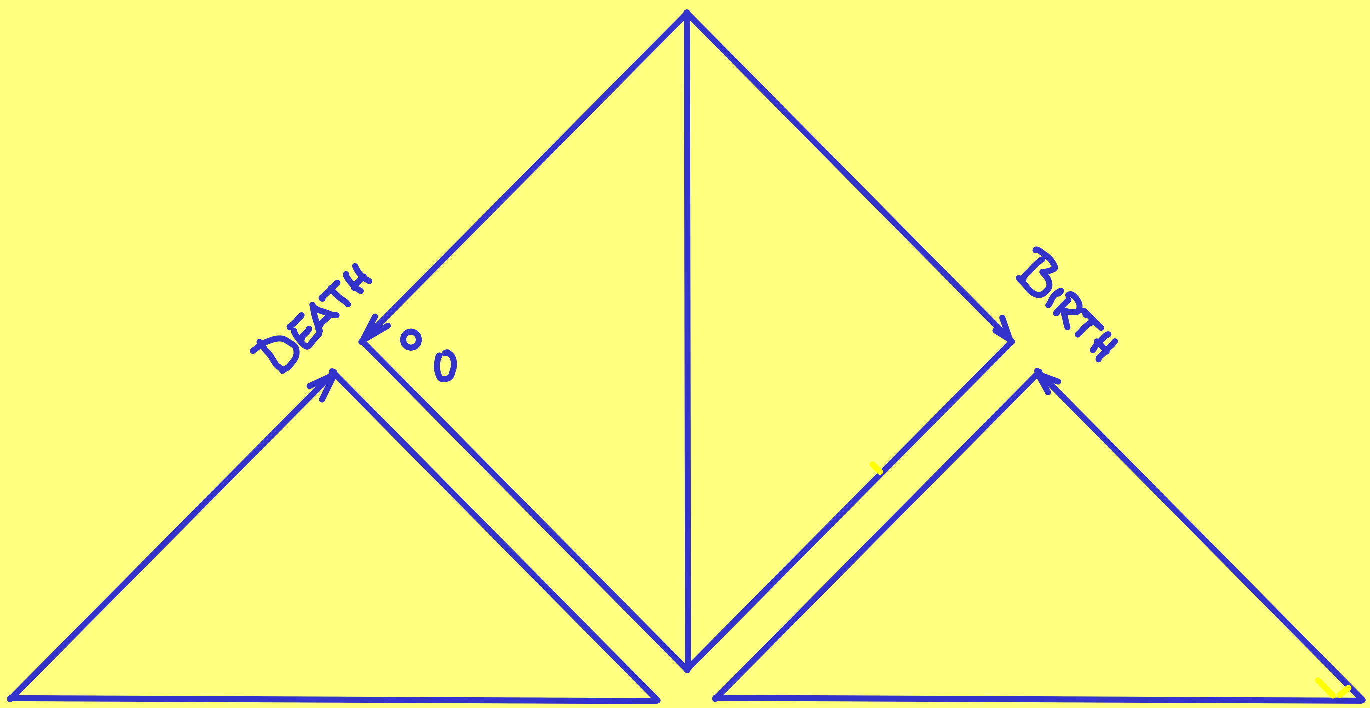

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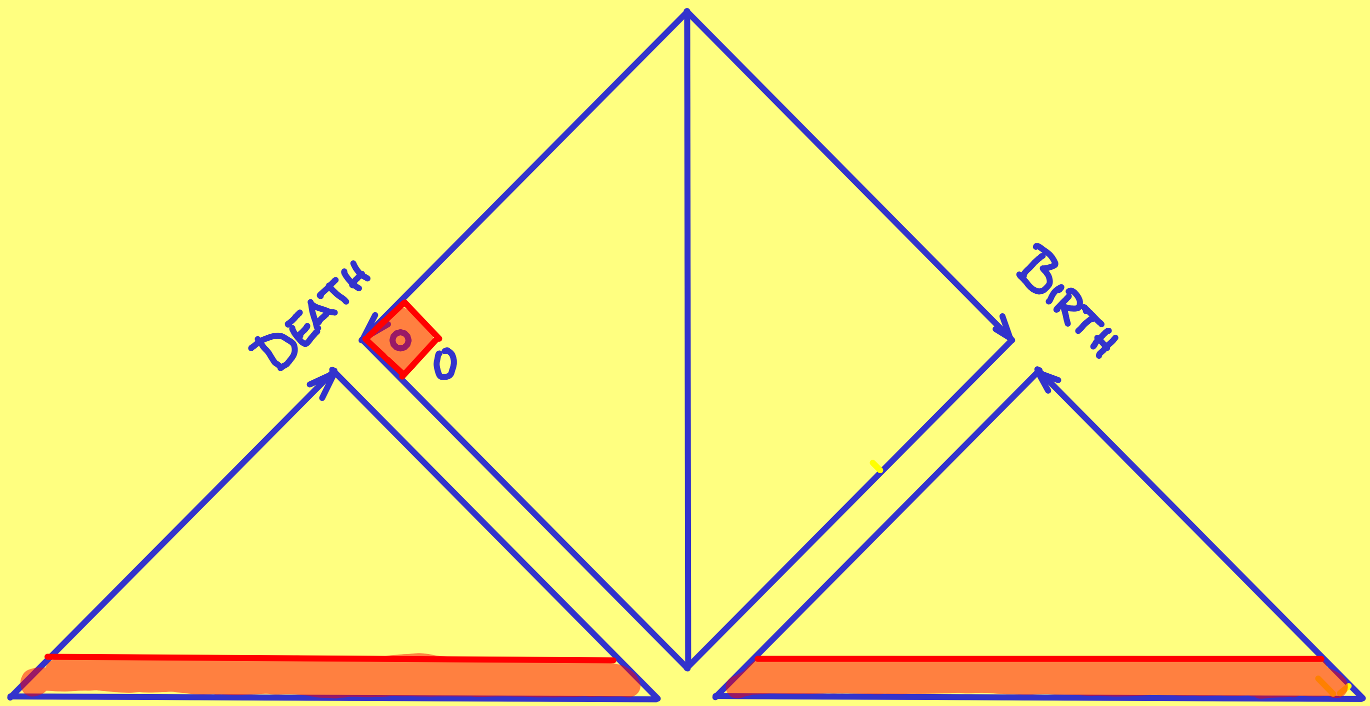

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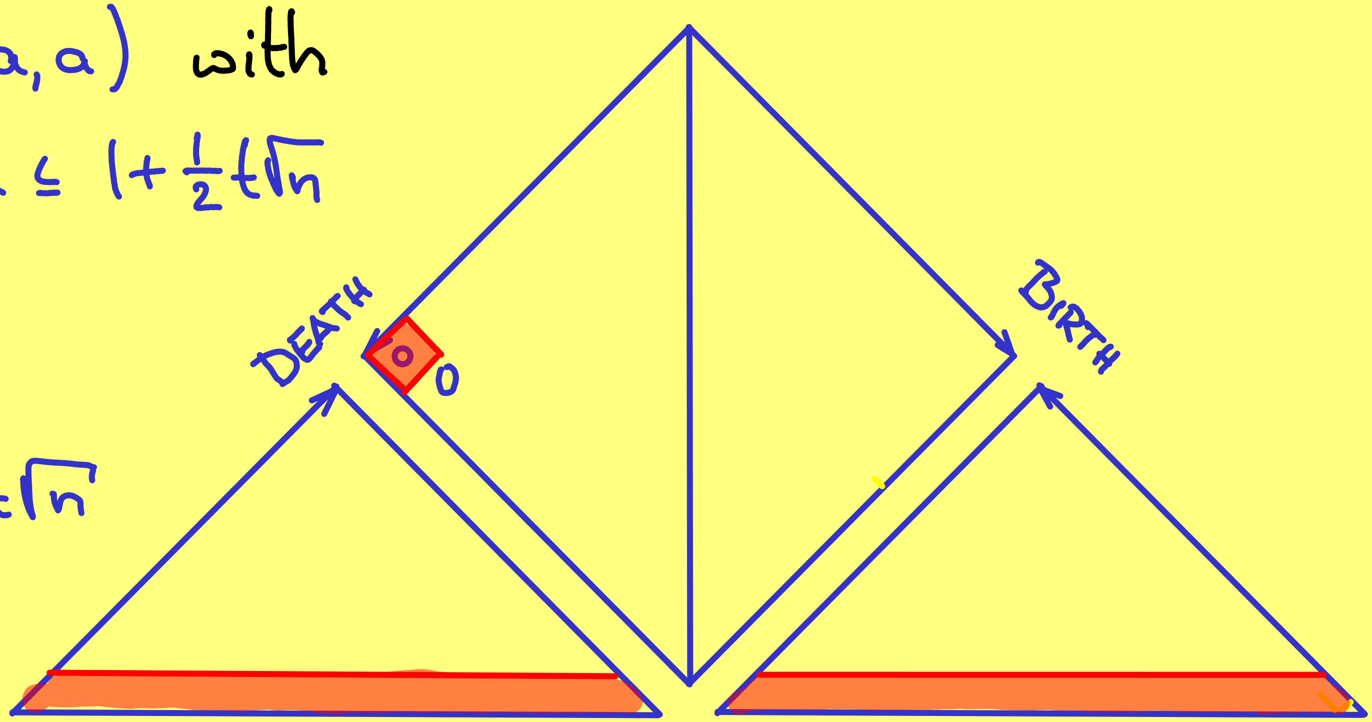
$(-a, a)$ with

$1 - \frac{1}{2} t\sqrt{n} \leq a \leq 1 + \frac{1}{2} t\sqrt{n}$

$2a - 2 \leq t\sqrt{n}$



IV.2 Solid Bodies

Def. A solid body is a compact set $\mathbb{M} \subseteq \mathbb{R}^n$ s.t.

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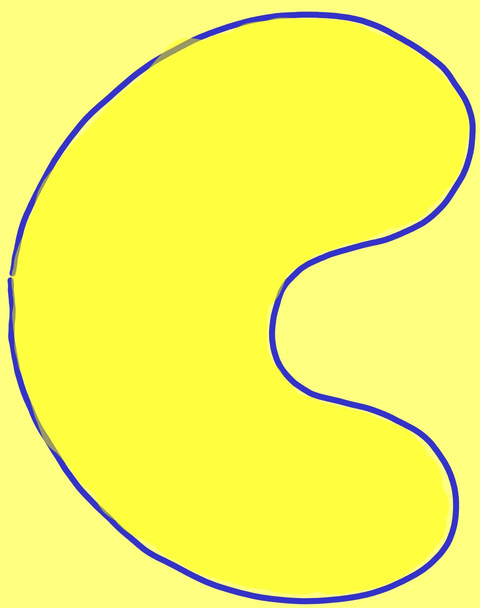

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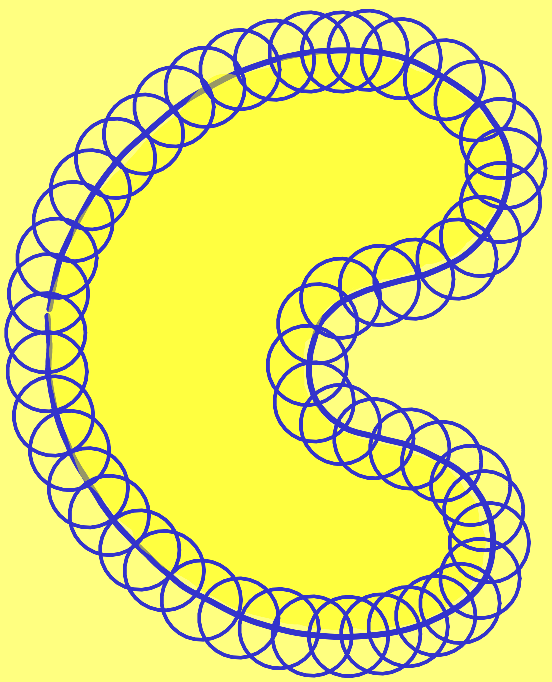

$\mathrm{reach}(\partial\mathbb{M}) \cdot \kappa_{max}(\partial\mathbb{M}) \leq 1.$

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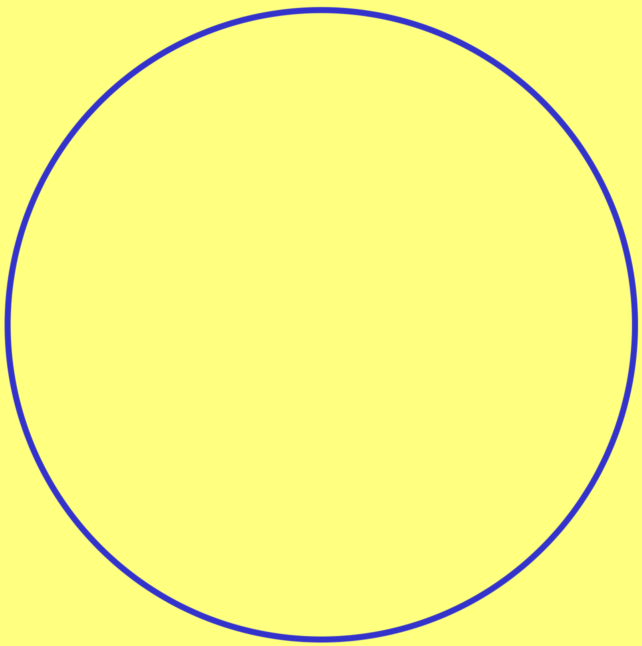
Thm. Solid body $\mathbb{M} \subseteq \mathbb{R}^n$, $1/\kappa_{max} \geq 3tn$, reach $> 2t\sqrt{n}$, C finite.

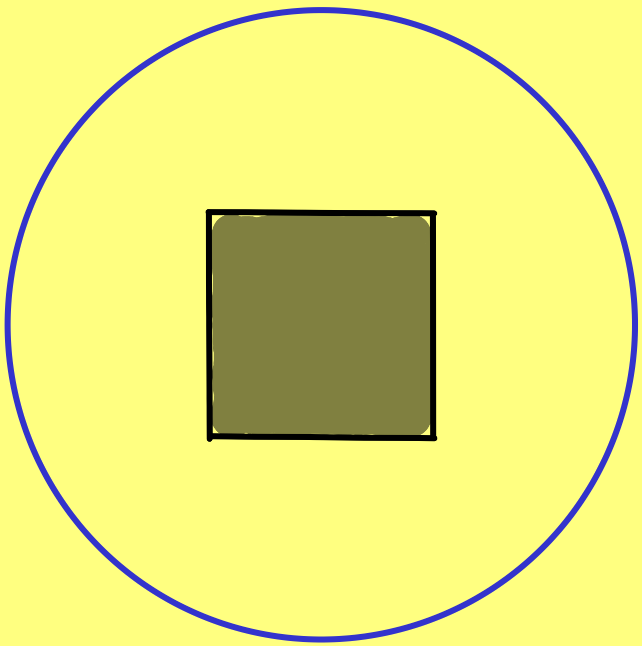
Then $|V_1(\mathbb{M}) - V_1(\mathbb{M}_t, t\sqrt{n})| \leq c_{n-1,n} \cdot t\sqrt{n} \cdot C$.

IV.3 Distorted Normal Bundle



IV.3 Distorted Normal Bundle

$\square^n = [-1,1]^n/(2\sqrt{n})$



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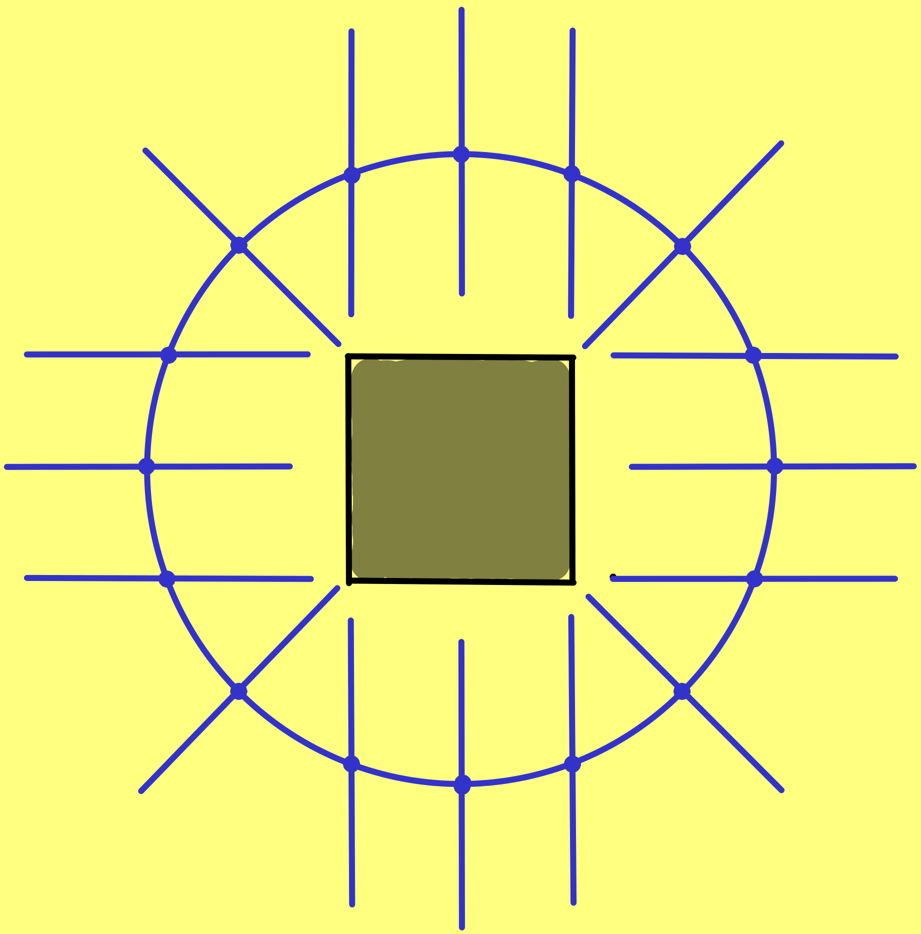

$\square^n = [-1,1]^n/(2\sqrt{n})$

$D_n : S^{n-1} \to S^{n-1}$ dfnd by

$D_n(u) = (u - NN(u))/\|u - NN(u)\|$

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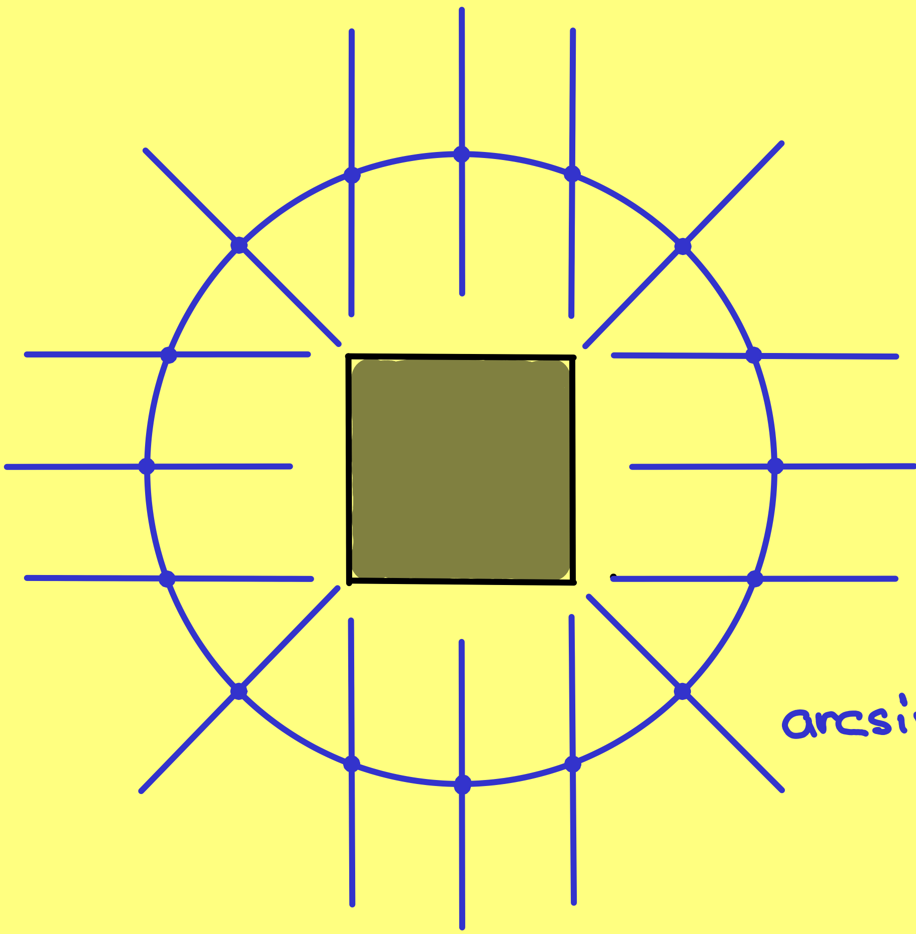

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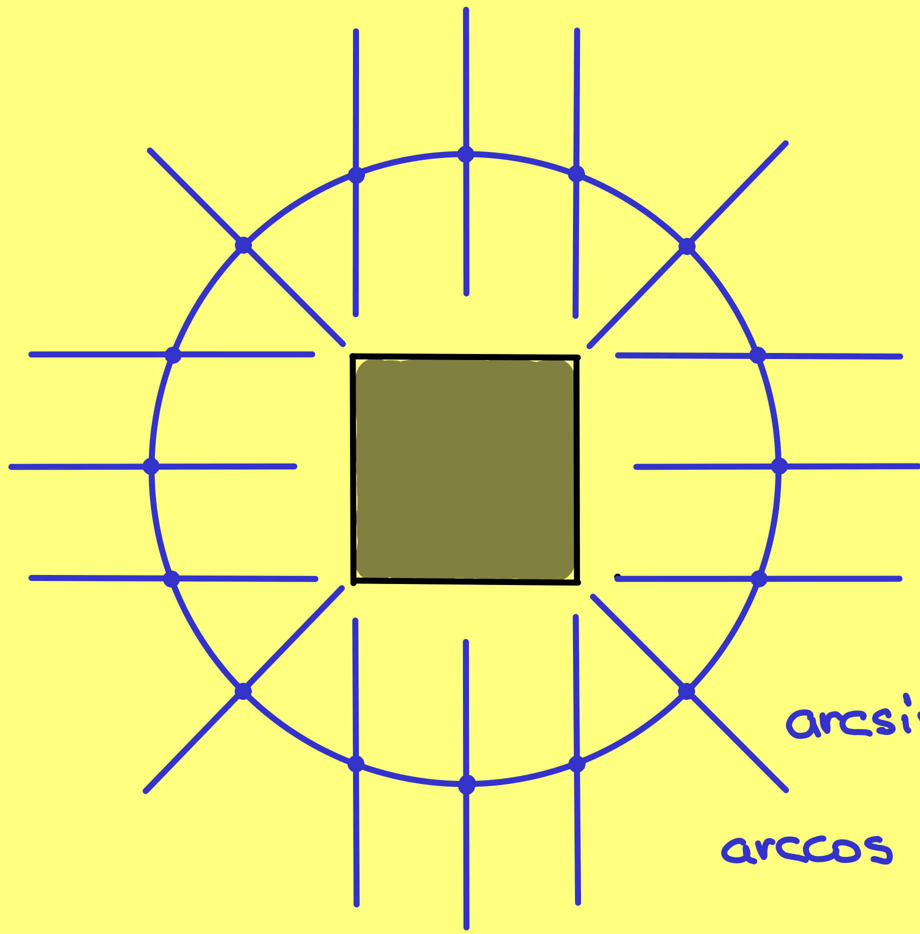

$\square^n = [-1,1]^n / (2\sqrt{n})$

$D_n : S^{n-1} \to S^{n-1}$ dfnd by

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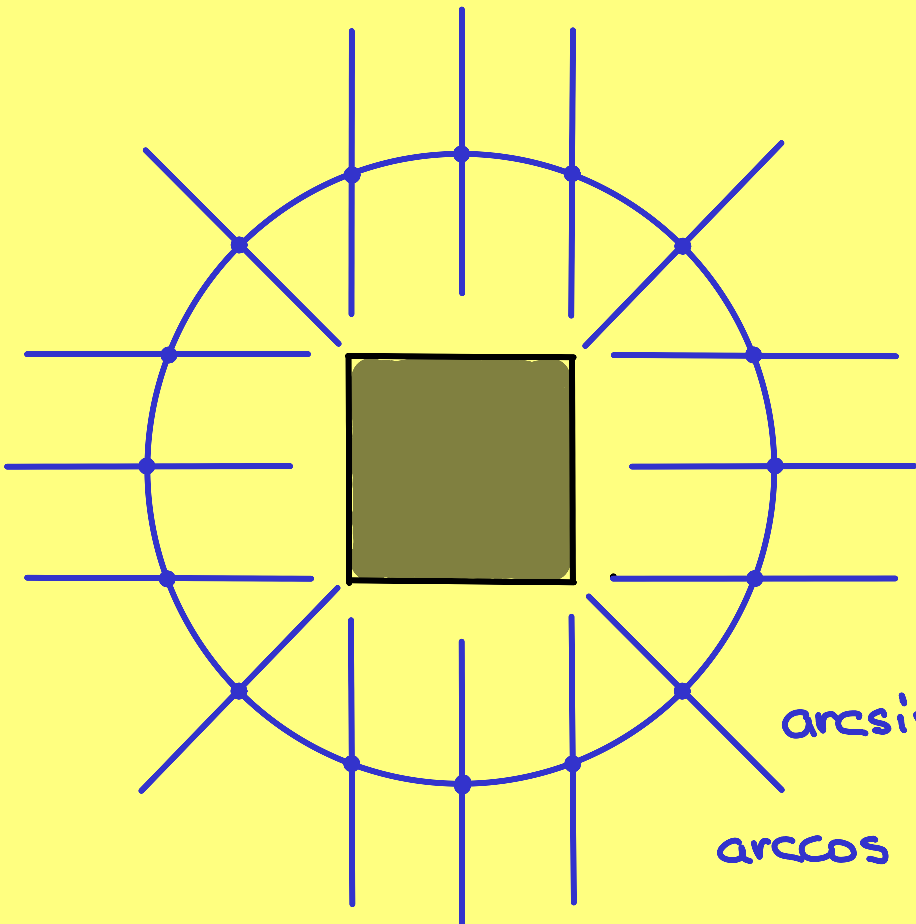
$\arccos \langle u, D_n(u) \rangle \leq \sqrt{\frac{3n+1}{4n}}$

IV.3 DISTORTED NORMAL BUNDLE



$\square^n = [-1,1]^n/(2\sqrt{n})$

$D_n : S^{n-1} \to S^{n-1}$ dfnd by

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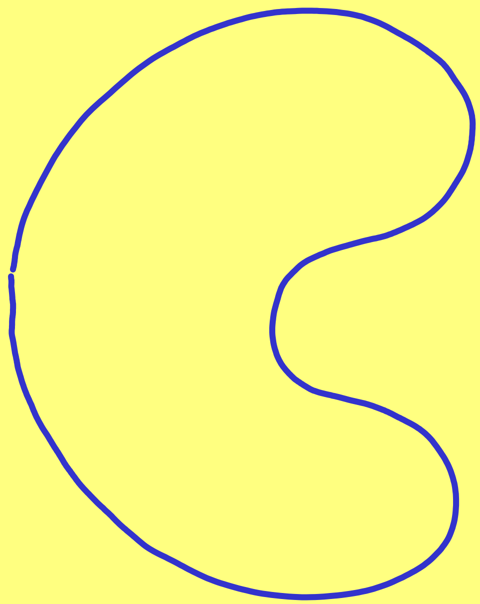
$\arccos \langle D_n(u), D_n(v) \rangle \leq 2 \arccos \langle u, v \rangle$

IV.4 Symmetric Difference



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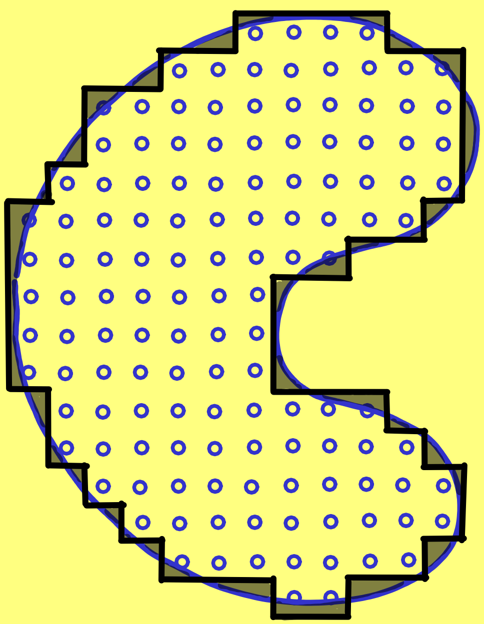

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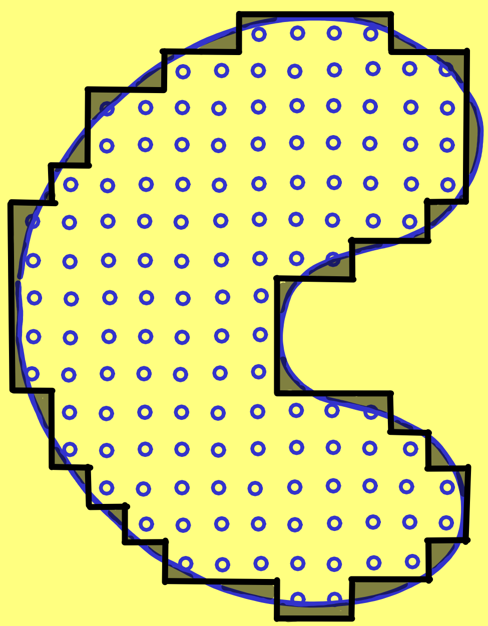

Coord.-aligned normal bundle

maps $x \in \partial M \mapsto D_n(N(x))$.

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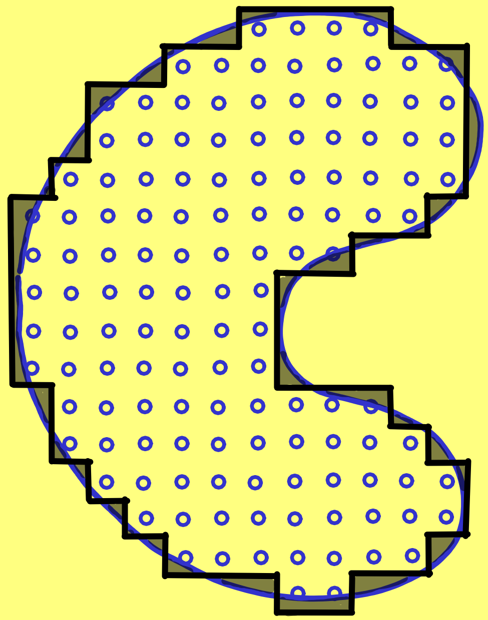

Coord.-aligned normal bundle

maps $x \in \partial M \mapsto D_n(N(x))$.

$$L_x = \{x + \lambda D_n(N(x)) \mid -t\sqrt{n} \le \lambda \le t\sqrt{n}\}$$

$$F_x = L_x \cap \left[(M \setminus M_t) \cup (M_t \setminus M) \right].$$

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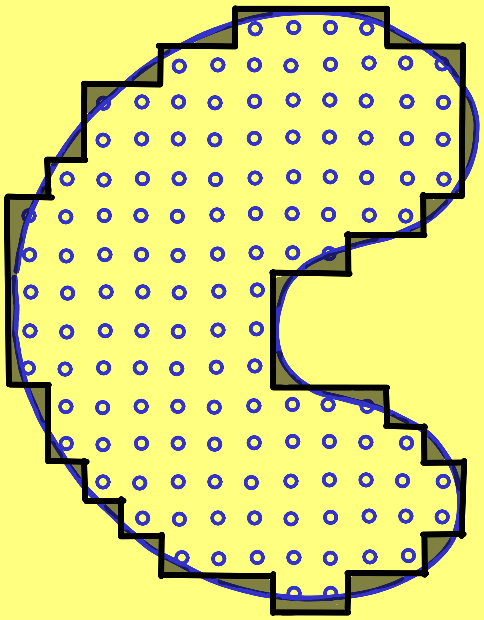

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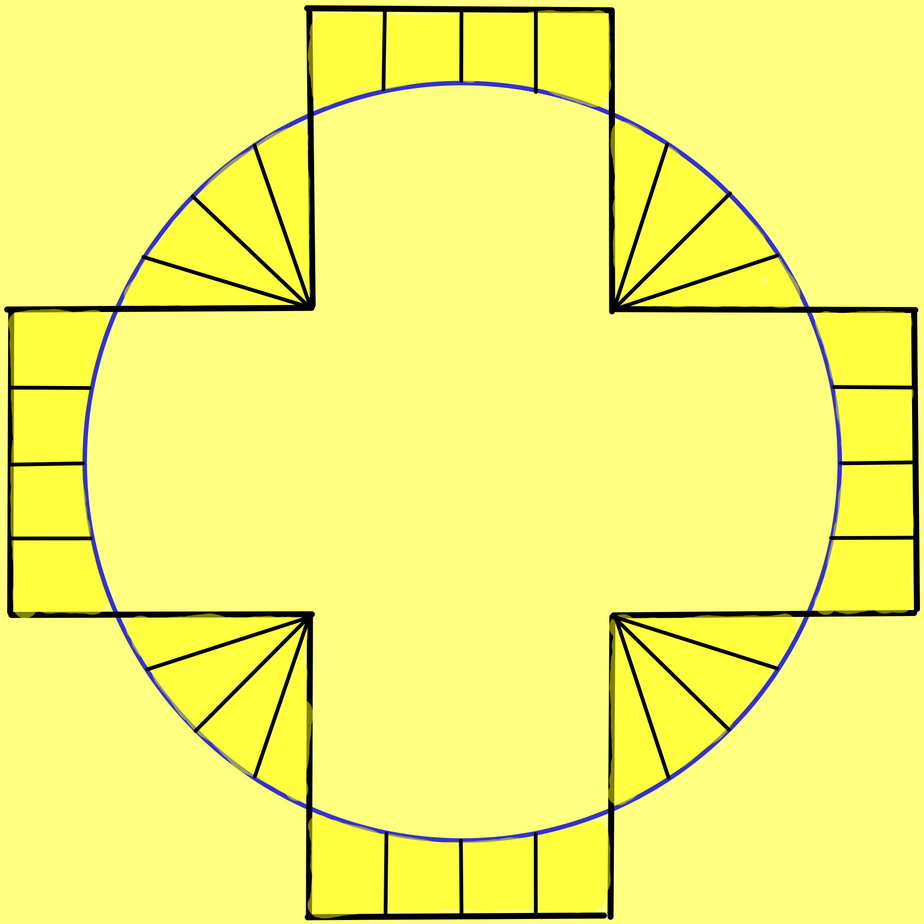
$$F_x = L_x \cap [(M \setminus M_t) \cup (M_t \setminus M)].$$

If $\partial M \subseteq \mathbb{R}^n$ s.t. $1/k_{max} \geq 3tn$, reach $> 2t\sqrt{n}$,

then $\{F_x \mid x \in \partial M\}$ is a fibration of symm. diff.

IV.5 Homotopy Equivalence



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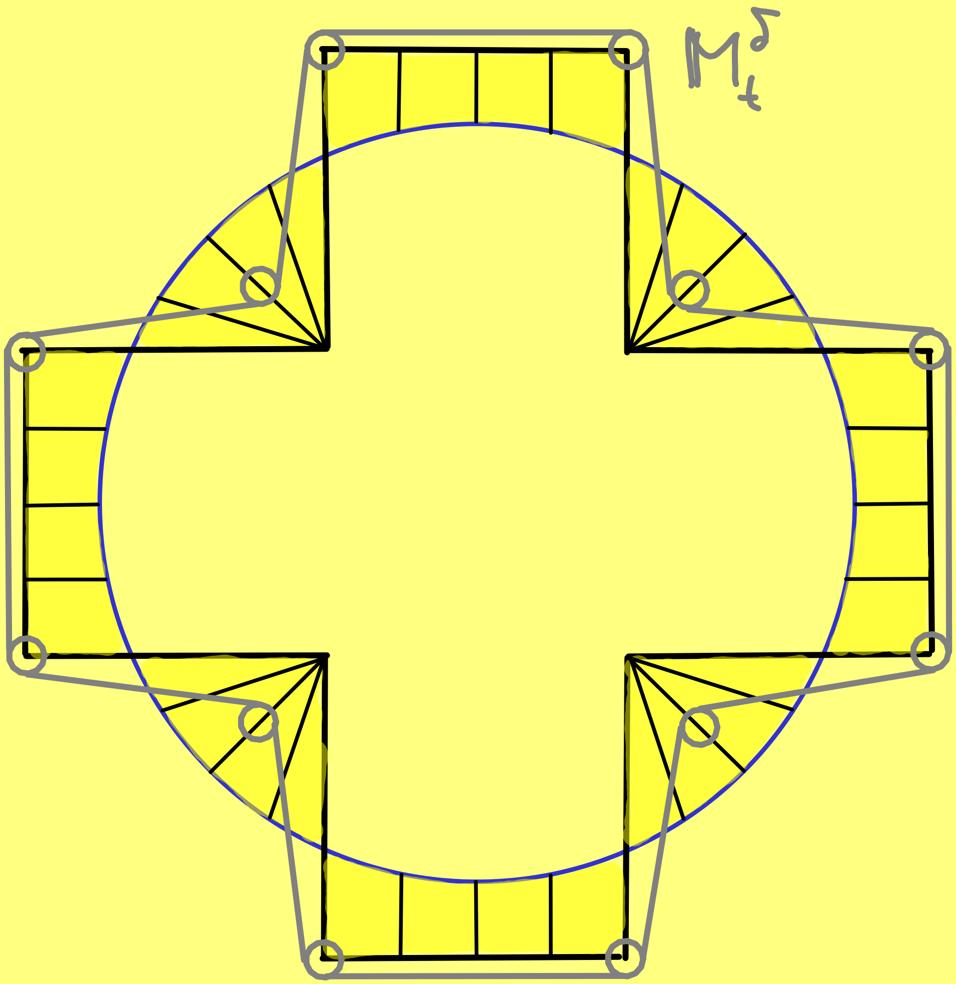

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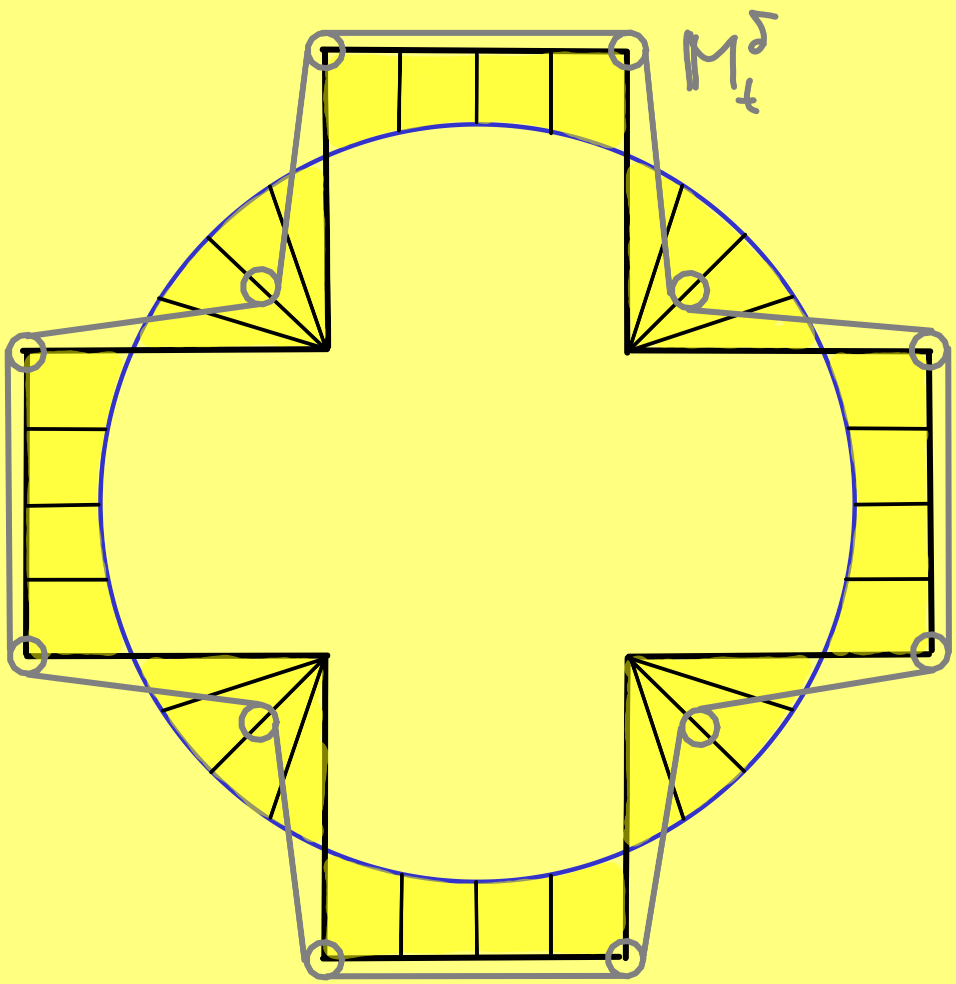

maps

$f : M \to M_t^\delta$

$g : M_t^\delta \to M_t$

$h : M_t^\delta \to M$

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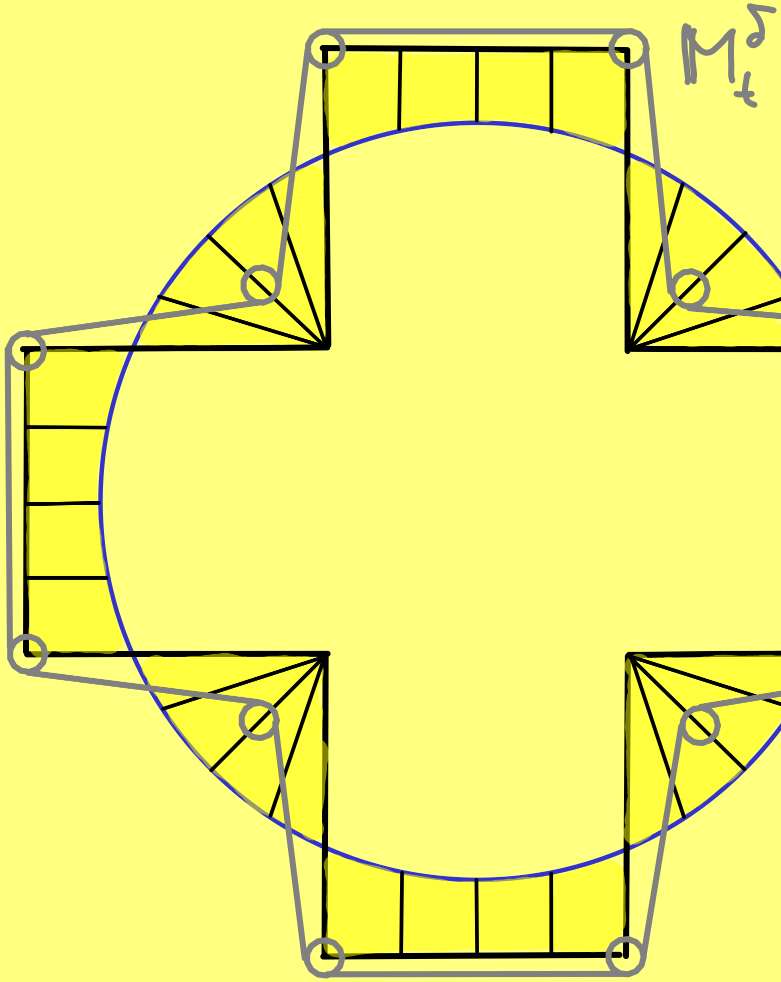

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THM. If $\partial M \subseteq \mathbb{R}^n$ s.t.

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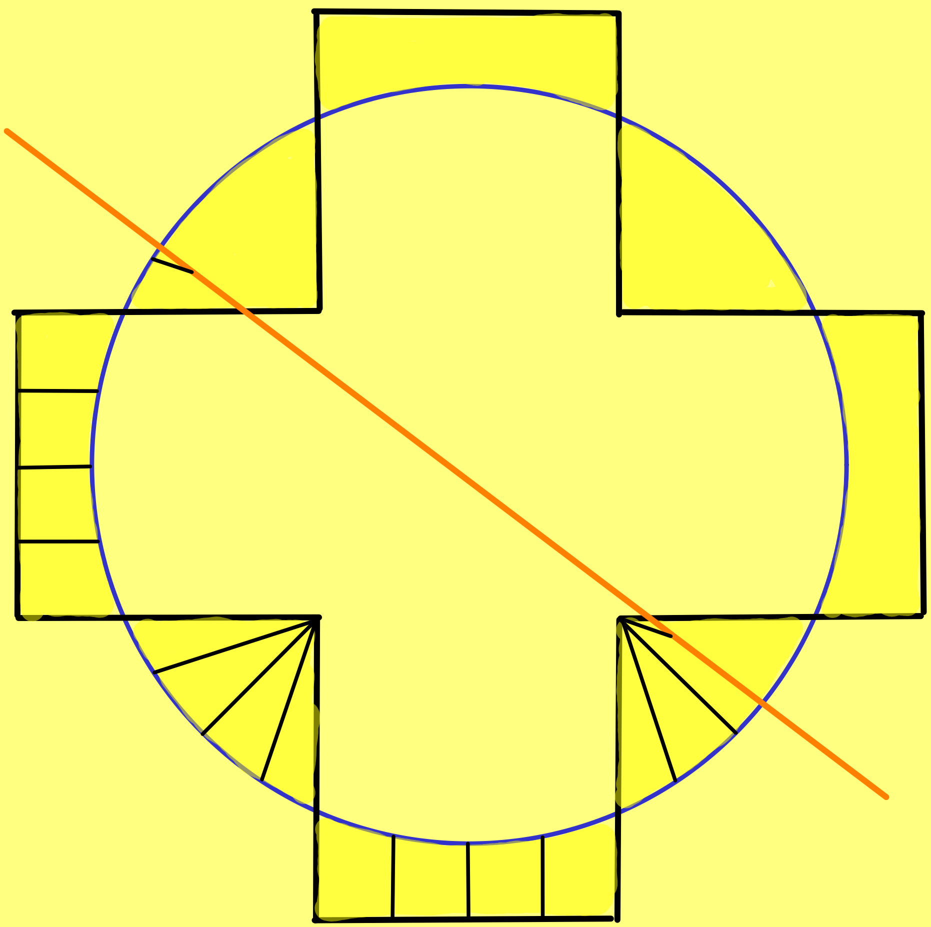
then $M \simeq M_t$.

IV.6 Sub- and Superlevel Sets



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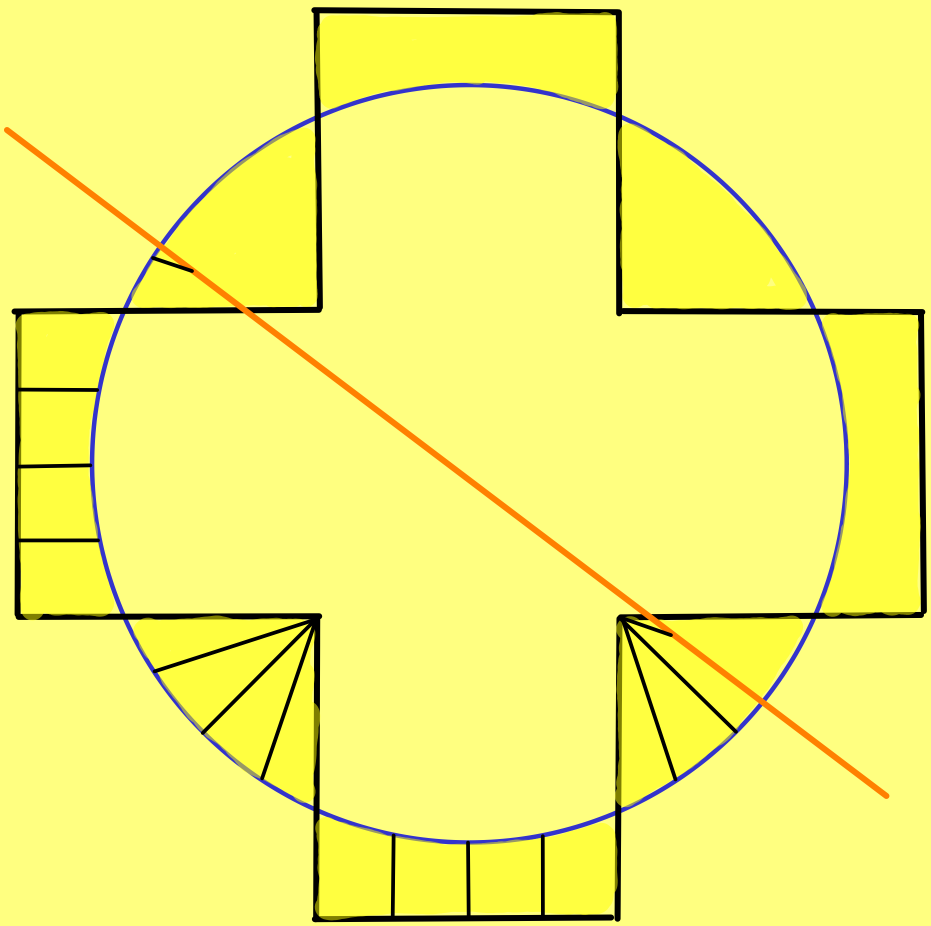

height functions

$f_u : \mathbb{M} \to \mathbb{R}$

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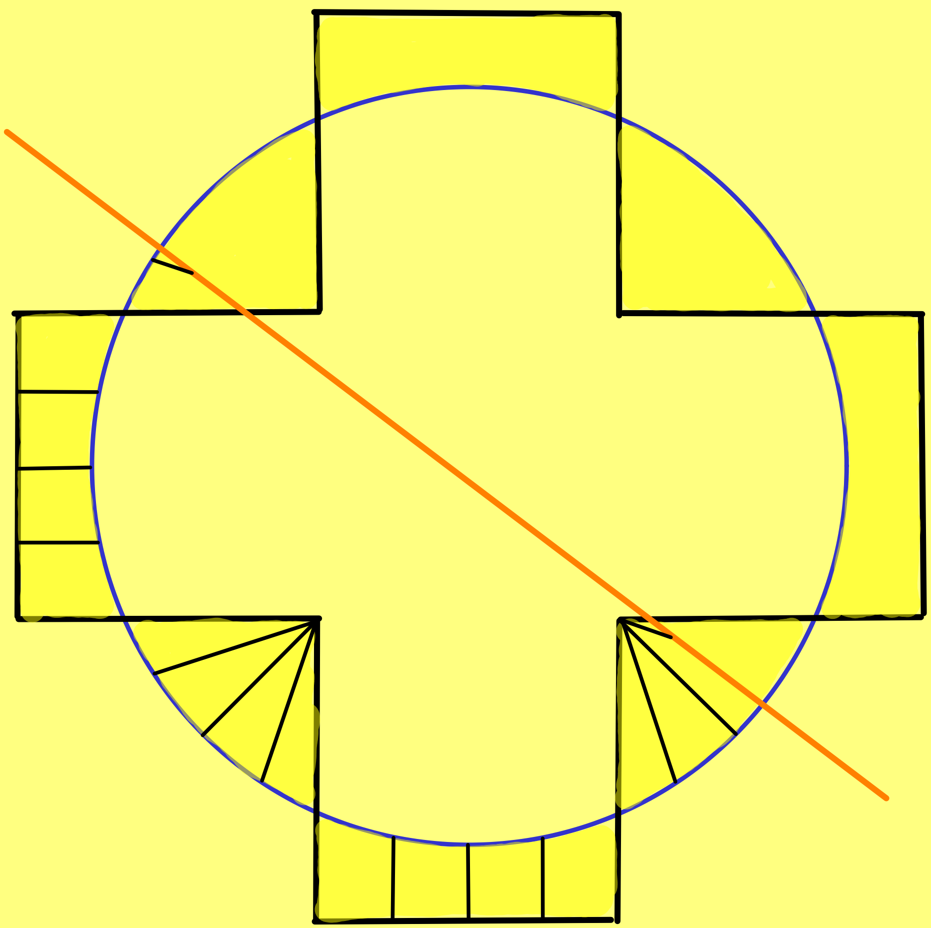

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$F_r = H(f_u(-\infty, r])$

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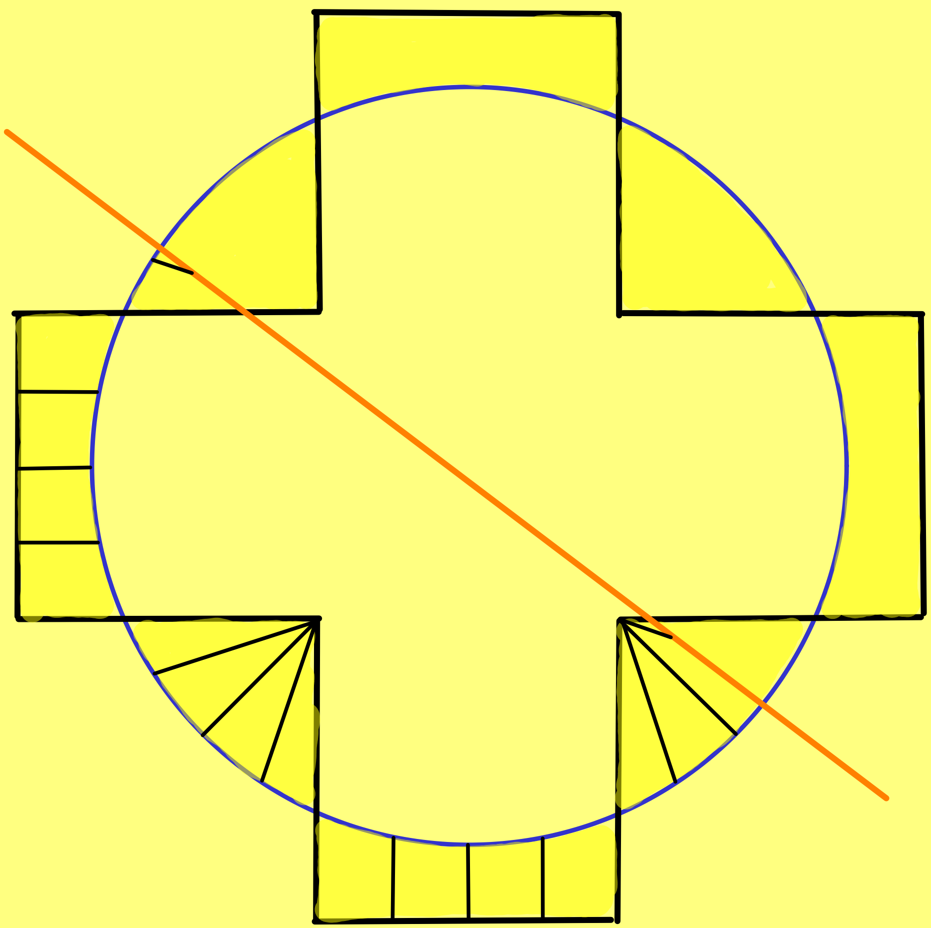

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$F_r = H(f_u(-\infty, r])$
$G_r = H(g_u(-\infty, r])$

$F^r = H(\mathbb{M}, f_u[r, \infty))$
$G^r = H(\mathbb{M}_t, g_u[r, \infty))$

IV.7 Towers

$$\mathcal{F}: \quad 0 \to \cdots \to F_r \to F_{r+p} \to \cdots \to F^r \to F^{r+p} \to \cdots \to 0$$

$$\mathcal{G}: \quad 0 \to \cdots \to G_r \to G_{r+p} \to \cdots \to G^r \to G^{r+p} \to \cdots \to 0$$

IV.7 Towers

$$
\begin{tikzcd}
\mathcal{F}: & 0 \arrow[r] & \cdots \arrow[r] & F_r \arrow[r] \arrow[dr] & F_{r+\rho} \arrow[r] & \cdots \arrow[r] & F^r \arrow[r] \arrow[dr] & F^{r+\rho} \arrow[r] & \cdots \arrow[r] & 0 \\
\mathcal{G}: & 0 \arrow[r] & \cdots \arrow[r] & G_r \arrow[r] \arrow[ur] & G_{r+\rho} \arrow[r] & \cdots \arrow[r] & G^r \arrow[r] \arrow[ur] & G^{r+\rho} \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}
$$

IV.7 Towers

$$
\begin{tikzcd}
\mathcal{F}: & 0 \arrow[r] & \cdots \arrow[r] & F_r \arrow[r] \arrow[dr] & F_{r+\rho} \arrow[r] & \cdots \arrow[r] & F^r \arrow[r] \arrow[dr] & F^{r+\rho} \arrow[r] & \cdots \arrow[r] & 0 \\
\mathcal{G}: & 0 \arrow[r] & \cdots \arrow[r] & G_r \arrow[r] \arrow[ur] & G_{r+\rho} \arrow[r] & \cdots \arrow[r] & G^r \arrow[r] \arrow[ur] & G^{r+\rho} \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}
$$

$\mathcal{F}, \mathcal{G}$ are ρ-interleaving $\Rightarrow W_\infty(\mathrm{Dgm}(\mathcal{F}), \mathrm{Dgm}(\mathcal{G})) \leq \rho$

IV.7 TOWERS

$$\begin{tikzcd}
\mathcal{F}: & 0 \arrow[r] & \cdots \arrow[r] & F_r \arrow[r] \arrow[dr] & F_{r+\varepsilon} \arrow[r] & \cdots \arrow[r] & F^r \arrow[r] \arrow[dr] & F^{r+\varepsilon} \arrow[r] & \cdots \arrow[r] & 0 \\
\mathcal{G}: & 0 \arrow[r] & \cdots \arrow[r] & G_r \arrow[r] \arrow[ur] & G_{r+\varepsilon} \arrow[r] & \cdots \arrow[r] & G^r \arrow[r] \arrow[ur] & G^{r+\varepsilon} \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

$\mathcal{F}, \mathcal{G}$ are ε-interleaving $\Rightarrow W_\infty(\mathrm{Dgm}(\mathcal{F}), \mathrm{Dgm}(\mathcal{G})) \le \varepsilon$

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Then $|V_1(\mathbb{M}) - V_1(\mathbb{M}_t, t\sqrt{n})| \leq c_{n-1,n} \cdot t\sqrt{n} \cdot C$.

THANK YOU