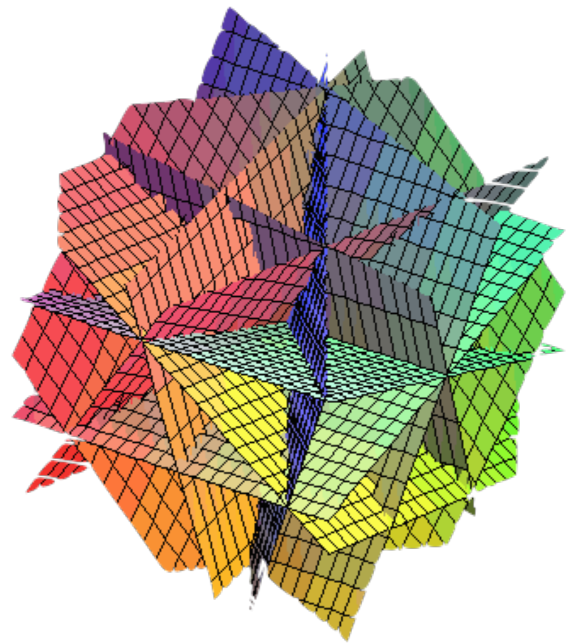
(Enumeration Results for) Signed Graphs

Matthias Beck

San Francisco State University

math.sfsu.edu/beck



[John Stembridge]

Why Graphs?

A [directed] graph $G = (V, E)$ consists of

- a node set V
- an edge set $E \subseteq \binom{V}{2}$ $[V^2]$

So... why?

- Modeling (directional) relations
- Fascinating theorems & conjectures
- ... including of computational nature

Signed Graph Concepts

A signed graph $\Sigma = (G, \sigma)$ consists of:

- a graph $G = (V, E)$ which may have multiple edges, loops (which together form E_*), half edges, and loose edges

- a signature $\sigma : E_* \to \{\pm\}$

Balance

Earliest appearance of signed graphs: social psychology (Heider 1946, Cartwright–Harary 1956) "The enemy of my enemy is my friend"

A simple cycle is balanced if its product of signs is $+$. A signed graph is balanced if it contains no half edges and all of its simple cycles are balanced.

Remark An all-negative signed graph is balanced if and only if it is bipartite.

Theorem (Harary 1953, anticipated by König 1936) A signed graph is balanced if and only if V can be bipartitioned such that each edge between the parts is negative and each edge within a part is positive.

The frustration index of the signed graph Σ is the smallest number of edges whose negation makes Σ balanced.
(Finding the frustration index is NP-hard: for an all-negative signed graph it is equivalent to the maximum cut problem.)

Switching

Switching Σ at $v \in V$ means switching the sign of each edge incident with v. Note that switching does not alter balance.

Exercise A signed graph is balanced if and only if it has no half edges and can be switched to an all-positive signed graph. ($\longrightarrow$ Harary's Theorem)

Other Applications

- Knot theory (positive/negative crossings)
- Biology (perturbed large-scale biological networks
- Chemistry (Möbius systems)
- Physics (spin glasses—mixed Ising model)*
- Computer science (correlation clustering)

* Finding the ground state energy of an Ising model means finding the frustration index of a signed graph.

Incidence Matrices of Graphs

Two versions of incidence matrix (a_{ve}) of a graph $G = (V, E)$:

- $a_{ve} = 1$ if v and e are incident, 0 otherwise
- orient G and define $a_{ve} = \pm 1$ according to whether v points into or away from e and 0 if v and e are not incident

A matrix is totally unimodular if all its minors are 0 or ± 1. Examples:

- unoriented incidence matrix of a bipartite graph
- oriented incidence matrix of any graph

Incidence Matrices of Signed Graphs

Orienting a signed graph gives rise to a bidirected graph (first introduced by Edmonds–Johnson 1970)

$$\begin{array}{lcl} \sigma_e = + & & e \text{ becomes directed} \\ & \longrightarrow & \\ \sigma_e = - & & e \text{ becomes extra- or introverted} \end{array}$$

Define $a_{ve} = \pm 1$ according to whether v points into or away from e, and 0 if v and e are not incident.

Theorem (Heller–Tompkins, Gale–Hoffman 1956) The incidence matrix of a bidirected graph is totally unimodular if and only if the corresponding signed graph is balanced.

Theorem (Appa–Kotnyek 2006, following Lee 1989) The inverse of any maximal minor of the incidence matrix of a bidirected graph is half integral.

Magic Labelings of Graphs

An edge labeling $E \to \{1, 2, \ldots, k\}$ is magic if each sum of all labels incident to a node is the same.

Theorem (Stanley 1973) The number $m_G(k)$ of all magic k-labelings is a quasipolynomial in k with period ≤ 2. It is a polynomial if G is bipartite.

Corollary (conjectured by Anand–Dumir–Gupta 1966) The number of semimagic squares with row/column sum k is a polynomial in k.

The geometry behind this corollary concerns the Birkhoff–von Neumann polytope

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\ge 0} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \le k \le n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \le j \le n \end{array} \right\}$$

Ehrhart (Quasi-)Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$.
Equivalently, $\mathcal{P} = \left\{ \boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \right\}$ for some unimodular matrix $\boldsymbol{A}$.

For $k \in \mathbb{Z}_{>0}$ let $\operatorname{ehr}_{\mathcal{P}}(k) := \# \left(k\mathcal{P} \cap \mathbb{Z}^d \right)$.

Theorem (Ehrhart 1962) If $\mathcal{P}$ is a lattice polytope, $\operatorname{ehr}_{\mathcal{P}}(k)$ is a polynomial. If $\mathcal{P}$ is a rational polytope, $\operatorname{ehr}_{\mathcal{P}}(k)$ is a quasipolynomial whose period divides the denominator of $\mathcal{P}$.

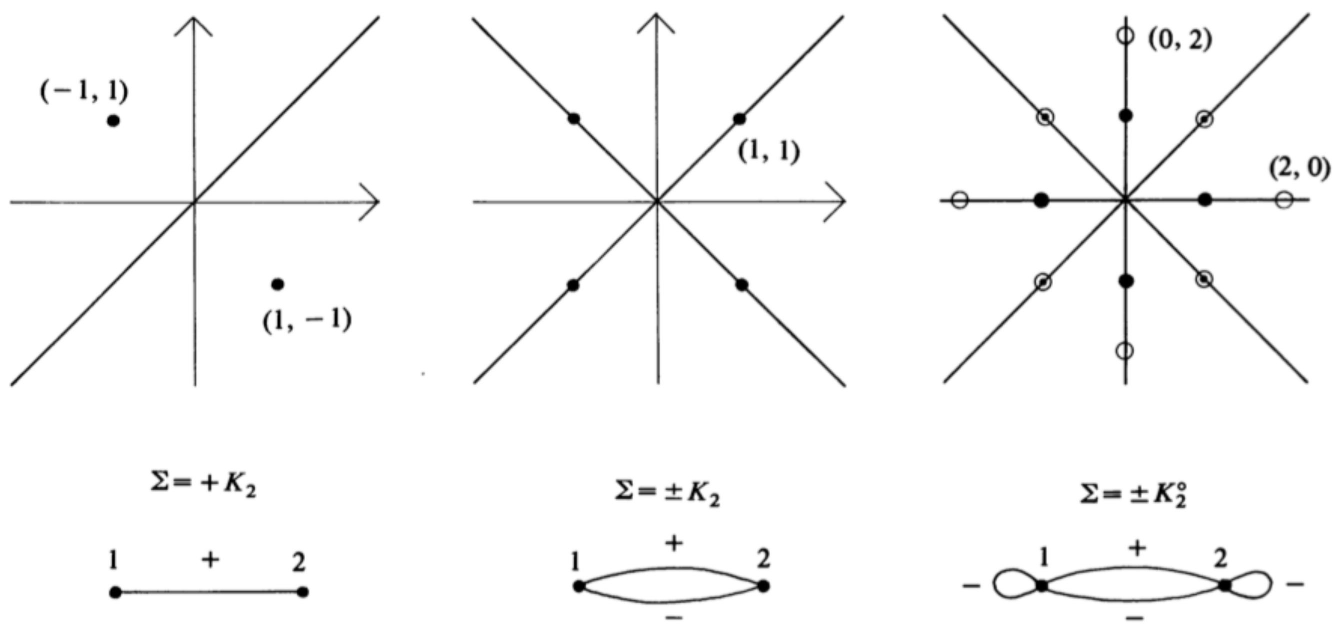
Magic Labelings Revisited

Theorem (Stanley 1973) The number $m_G(k)$ of all magic k-labelings is a quasipolynomial in k with period ≤ 2. It is a polynomial if G is bipartite.

(Signed) Graphic Arrangements

$\mathcal{H}_G := \{x_j = x_k : jk \in E\}$ is a hyperplane arrangement in $\mathbb{R}^V$,
a subarrangement of the (real) braid arrangement $\{x_j = x_k : j \neq k\}$

$\mathcal{H}_\Sigma := \{x_j = \sigma_e x_k : e = jk \in E\}$ is a subarrangement of the type-B/C arrangement $\{x_j = \pm x_k,\ x_j = 0 : j \neq k\}$



[Thomas Zaslavsky, *Amer. Math. Monthly* 1981]

Signed Graphic Arrangements

A bidirected graph is acyclic if every simple cycle has a source or sink.

Observation (Greene–Zaslavsky 1970s) The regions of

$$\mathcal{H}_\Sigma = \{x_j = \sigma_e \, x_k : \, e = jk \in E\}$$

are in one-to-one correspondence with the acyclic orientations of Σ.

Chromatic Polynomials of Signed Graphs

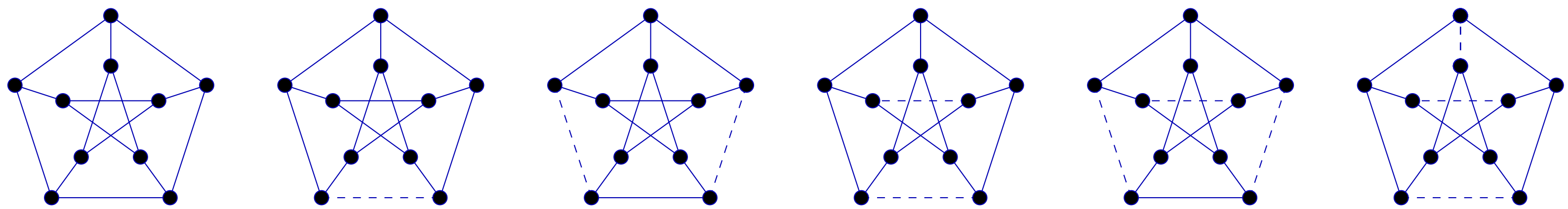
Proper k-coloring of Σ — mapping $x : V \to \{-k, -k+1, \dots, k\}$ such that for any edge $e = ij$ we have $x_i \neq \sigma_e \, x_j$

$$\chi_\Sigma(2k+1) := \# \, (\text{proper } k\text{-colorings of } \Sigma)$$
$$\chi^*_\Sigma(2k) := \# \, (\text{proper zero-free } k\text{-colorings of } \Sigma)$$

Theorem (Zaslavsky 1982) $\chi_\Sigma(2k+1)$ and $\chi^*_\Sigma(2k)$ are polynomials. Moreover, $(-1)^{|V|}\chi_\Sigma(-(2k+1))$ equals the number of pairs (α, x) consisting of an acyclic orientation α of Σ and a compatible k-coloring x. In particular, $(-1)^{|V|}\chi_\Sigma(-1)$ equals the number of acyclic orientations of Σ.

Signed Petersen Graphs

Theorem (Zaslavsky 2012) There are precisely six signed Petersen graphs that are not switching isomorphic:



Theorem (MB–Meza–Nevarez–Shine–Young 2015, conjectured and partially proved by Zaslavsky 2012) The six signed Petersen graphs can be told apart by any positive integer evaluation of their (zero-free) chromatic polynomials.

DIY Proof Sage code at math.sfsu.edu/beck/papers/signedpetersen.sage

Open Problems

- Is there a combinatorial interpretation of $\chi^*_\Sigma(-1)$?
- MB–Zaslavsky (2006) introduced a $\mathbb{Z}_{2k+1}$-flow polynomial for signed graphs. Is there any flow polynomial for evaluations at even integers?
- Computations: Birkhoff–von Neumann polytope, flow polytopes, flow polynomials
- Four-Color Theorem? Without computers?
- Five-Flow Conjecture? Antimagic-Graph Conjecture?