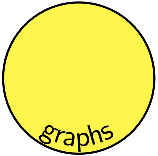
Problems of Enumeration and Realizability on Matroids, Simplicial Complexes, and Graphs

Yvonne Kemper
August 6, 2014

In Honor of a Diagram



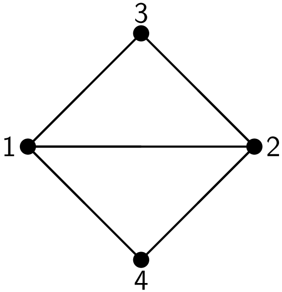
Wait, what were those things again?

Definition

A **graph** $G = (V, E)$ is a set of **vertices** $V = \{v_1, \ldots, v_n\}$ and a set of **edges** $E = \{v_i v_j : v_i, v_j \in V\}$.

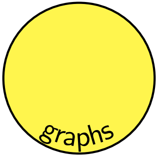
Example

Here is a graph!



$$
\begin{aligned}
G &= (V, E) \\
&= (\{1, 2, 3, 4\}, \\
&\quad \{12, 13, 14, 23, 24\})
\end{aligned}
$$

In Honor of a Diagram



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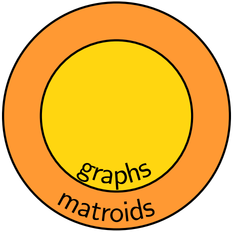

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Definition

A **matroid** $M = (E(M), \mathcal{I}(M))$ consists of a **ground set** $E(M)$ and a family of subsets $\mathcal{I}(M) \subseteq 2^{E(M)}$ called **independent sets** such that

(1) $\emptyset \in \mathcal{I}$;

(2) if $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$; and

(3) if $I, J \in \mathcal{I}$, and $|J| < |I|$, then there exists some $e \in I \setminus J$ such that $J \cup \{e\} \in \mathcal{I}$.

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Example

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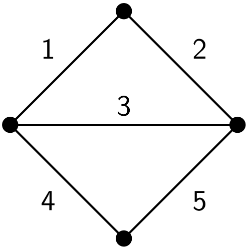

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&\quad 25, 34, 35, 45, 124, 125, 134, \\
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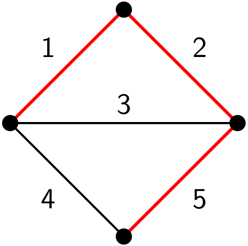

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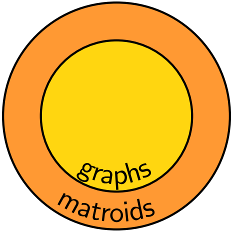

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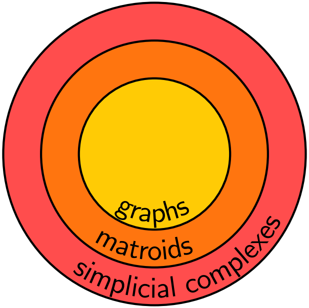

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Definition

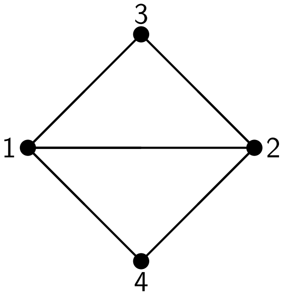
An **(abstract) simplicial complex** Δ on a **vertex set** V is a set of subsets of V. These subsets are called the **faces** of Δ, and we require that

(1) for all $v \in V$, $\{v\} \in \Delta$, and

(2) for all $F \in \Delta$, if $G \subseteq F$, then $G \in \Delta$.

Example

Here is a ~~graph~~ ~~matroid~~ simplicial complex!



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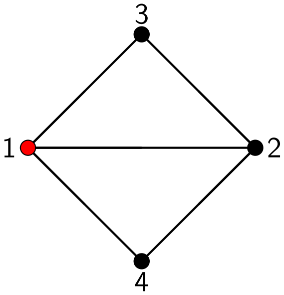

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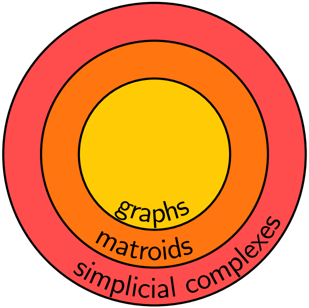

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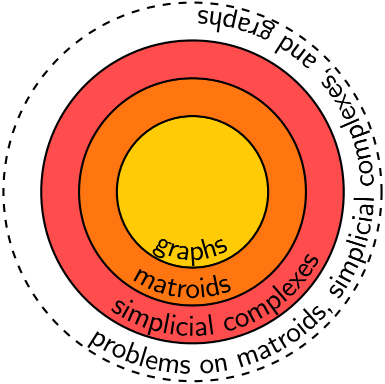

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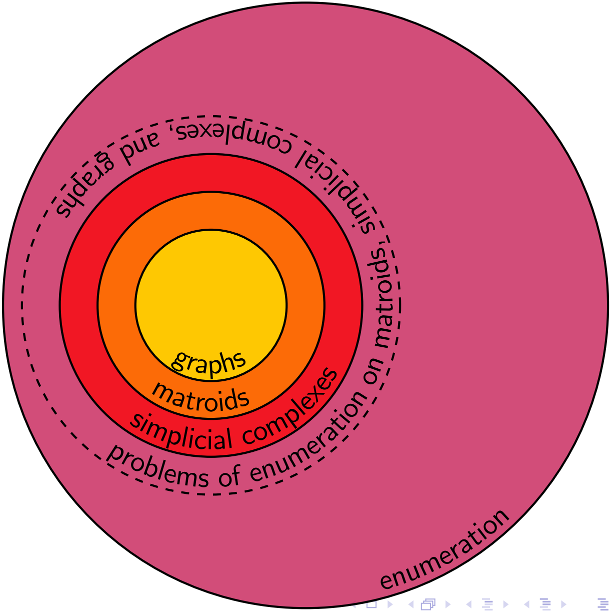

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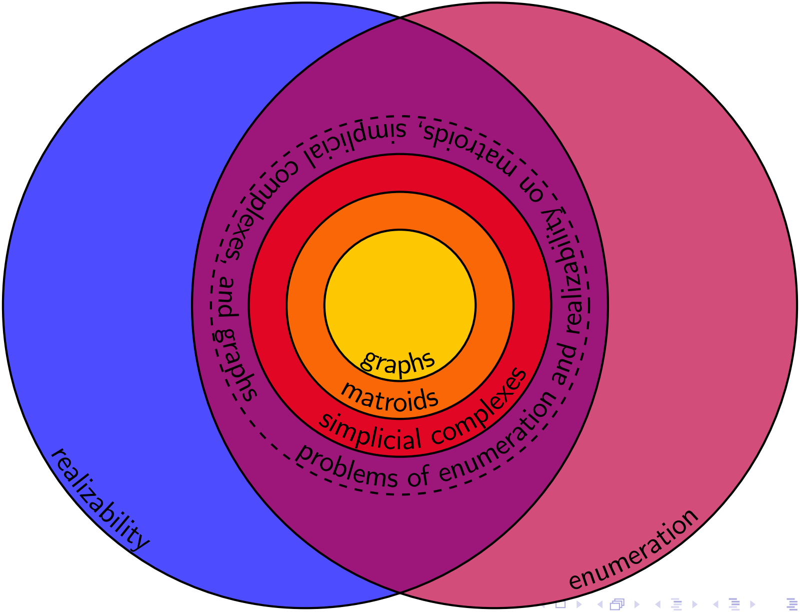

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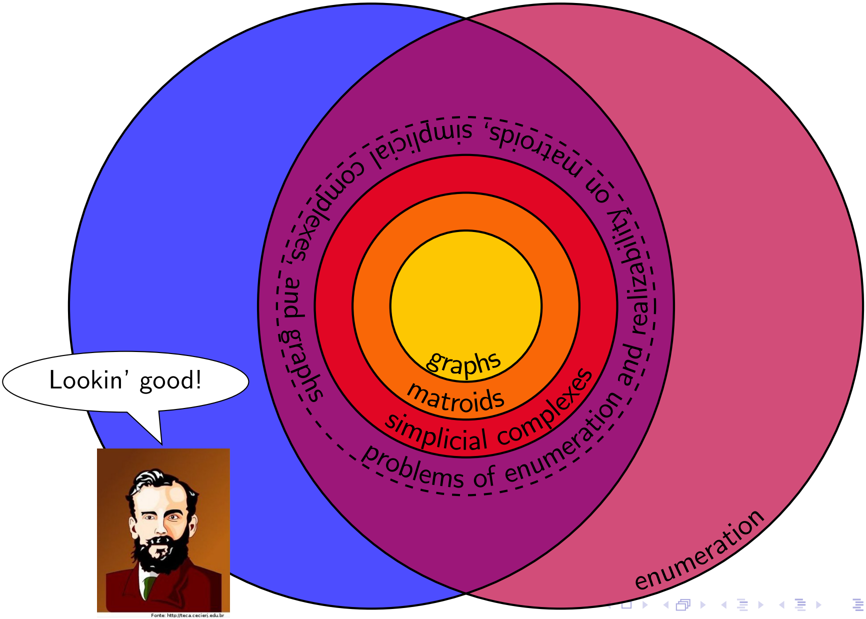

The Problems Three

- h-Vectors of Small Matroids
- Flows on Simplicial Complexes
- Polytopal Embeddings of Cayley Graphs

h-Vectors of Small Matroids

Simplicial Complexes: a Few More Definitions

Definition

The **dimension** of a face F is $|F| - 1$, and the **dimension** of Δ is $d = \max\{|F| : F \in \Delta\} - 1$.

Definition

A simplicial complex is **pure** if all maximal elements of Δ have the same cardinality.
In this case, a **facet** is a maximal face, a **ridge** is a face of one dimension lower.

Faces: A Natural Quantity to Measure

- The ***f*-vector** of a simplicial complex Δ, $\dim \Delta = d-1$, is

$$f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta)),$$

where $f_i(\Delta) := |\{F \in \Delta : \dim F = i\}|$.

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where $f_i(\Delta) := |\{F \in \Delta : \dim F = i\}|$.
- The ***h*-vector**, $h(\Delta) := (h_0(\Delta), \dots, h_d(\Delta))$, is given by:
$$\sum_{j=0}^{d} h_j(\Delta)\lambda^j = \sum_{i=0}^{d} f_{i-1}(\Delta)\lambda^i(1-\lambda)^{d-i}.$$

Characterizations of f- and h-Vectors

Definition

Given two integers $k, i > 0$, write

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$k^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \cdots + \binom{n_j}{j+1}.$$

Theorem (Schützenberger, Kruskal, Katona)

A vector $(1, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ *is the* f*-vector of some* $(d-1)$*-dimensional simplicial complex* Δ *if and only if*

$$0 < f_{i+1} \leq f_i^{(i+1)}, \qquad 0 \leq i \leq d-2.$$

Other Characterizations?

Question

Can we characterize subclasses of simplicial complexes?

Example

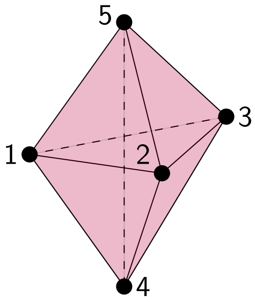
- Cohen-Macaulay complexes
- Flag complexes
- Shifted complexes
- Independence complexes of matroids

Matroid Complexes: An Example

Let M be given by:

$$\begin{aligned}\mathcal{I}(M) \quad = \quad & \{\emptyset, \\ & 1, 2, 3, 4, 5, \\ & 12, 13, 14, 15, 23, \\ & 24, 25, 34, 35, 45, \\ & 124, 125, 134, 135, \\ & 145, 234, 235, 245\}\end{aligned}$$

The corresponding complex:



Then: $f(M) = (1, 5, 10, 8)$ and $h(M) = (1, 2, 3, 2)$.

O-Sequences

- A non-empty set of monomials $\mathcal{M}$ is a **multicomplex** if

$$m \in \mathcal{M} \text{ and } n | m \Rightarrow n \in \mathcal{M}.$$

- A sequence $h = (h_0, h_1, \ldots, h_d)$ of integers is an **O-sequence** if there exists a multicomplex with precisely h_i monomials of degree i.
- An O-sequence is **pure** if all maximal elements have the same degree.

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Example

Let $M = \{1, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1x_2^2, x_1^2x_2\}$. Then, the corresponding (pure) O-sequence is:

$$O(M) = (1, 2, 3, 2).$$

Stanley's Conjecture

Conjecture (Stanley, 1977)

The h-vector of a matroid complex is a pure O-sequence.

Little progress was made for twenty years, but since 1997, the conjecture has been proved for matroids which are:

- of rank 4 (Klee, Samper),
- of rank less than or equal to 3 (Stokes, Há et al.),
- cographic (Biggs, Merino),
- lattice-path (Schweig),
- cotransversal (Oh),
- paving (Merino, et al.).

Results

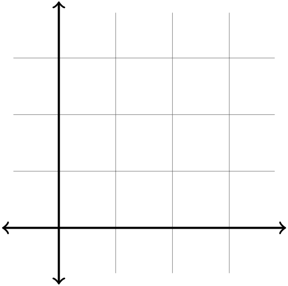
Theorem (De Loera, K., Klee)

- *Let M be a matroid of rank 2. Then $h(M)$ is a pure O-sequence.*
- *Let M be a matroid of corank 2. Then $h(M)$ is a pure O-sequence.*
- *Let M be a matroid of rank $d \geq 4$. Then, the subsequence $(1, h_1(M), h_2(M), h_3(M))$ of $h(M)$ is a pure O-sequence.*
  - *Let M be a matroid of rank 3. Then $h(M)$ is a pure O-sequence.*
- *Let M be a matroid on at most 9 elements. Then $h(M)$ is a pure O-sequence.*

An Experimental Result: Matroids on at Most Nine Elements

- Royle and Mayhew generated list of all matroids on at most nine elements - why not check them all?
- Used database to generate all h-vectors for these matroids.
- Generated list of all possible O-sequences of multicomplexes (up to maximal degree 9 on at most 9 variables), then checked that every h-vector appeared on this list.

Example: A Multicomplex in Two Variables



1. Pick a point $(a, b) \in \mathbb{Z}^2$ on the hyperplane $x + y = r$, where r is the rank of the matroid.

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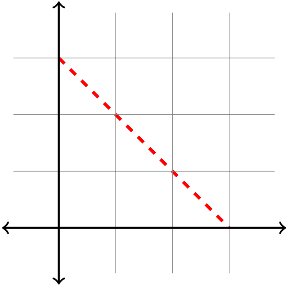

Let's say $r = 3$.

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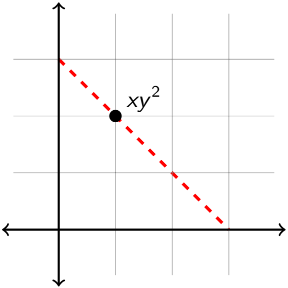

(a, b) corresponds
to the monomial $x^a y^b$

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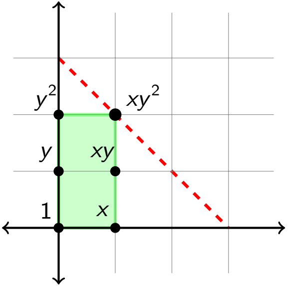

$$\mathcal{M} = \{1, x, y, xy, y^2, xy^2\}$$

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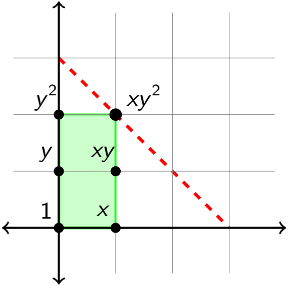

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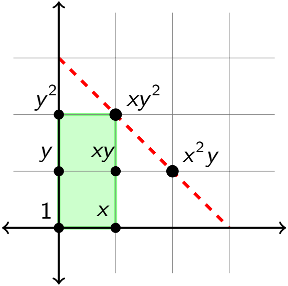

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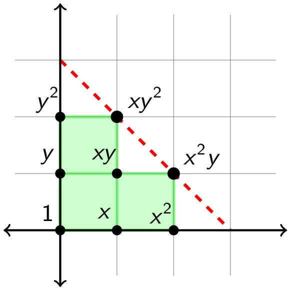

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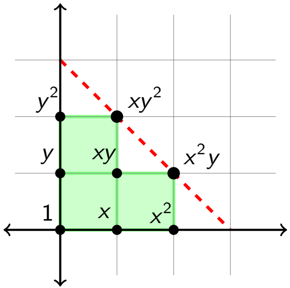

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The corresponding O-sequence is: $(1, 2, 3, 2)$.

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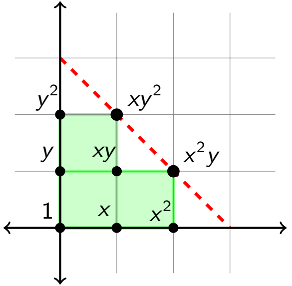

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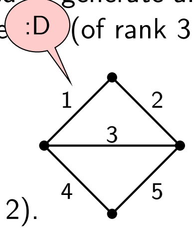

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Future Questions and Directions

- Cannot extend results directly, but can we use the geometric viewpoint to verify the conjecture for further classes of matroid complexes?
- Use PS-ear decomposability of matroid complexes?
- Characterize f- and h-vectors for further classes of simplicial complexes, such as matroid polytopes?
- There is a list of all matroids on at most 10 elements – can we find a counterexample here?

Flows on Simplicial Complexes

Flows on Graphs

Definition

A **$\mathbb{Z}_q$-flow** on an oriented graph G is a vector $\mathbf{x} \in \mathbb{Z}_q^E$ such that

$$\sum_{h(e)=v} x_e \equiv \sum_{t(e)=v} x_e \bmod q,$$

for all $v \in V$. A $\mathbb{Z}_q$-flow is **nowhere-zero** if it is fully supported.

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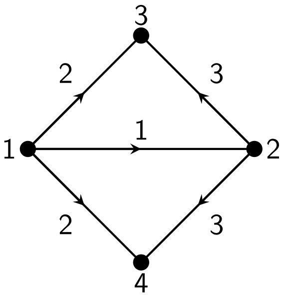
Equivalently, a nowhere-zero $\mathbb{Z}_q$-flow is a fully-supported element of the kernel mod q of the signed incidence matrix of the graph.

An Example



	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

$(1, 2, 2, 3, 3)$ is a nowhere-zero $\mathbb{Z}_5$-flow and an element of the kernel (mod 5) of the incidence matrix.

Graph Flows: Origins and Open Questions

- Flows originally defined in the context of electric circuits and networks
- Some Previous Work:
  - The number of nowhere-zero $\mathbb{Z}_q$-flows on a graph is a polynomial in q.
  - For a planar graph G, $\chi_G(k) = k^{c(G)}\phi_{G^*}(k)$.
  - The Max-Flow/Min-Cut problem of optimization.
- Open Questions:
  - 5-flow conjecture
  - Volumes of flow polytopes

Boundary/Incidence Matrices

Definition

Let ∂ be a boundary map on a $(d-1)$-dimensional complex Δ given by:

$$\partial[v_{i_0} \cdots v_{i_r}] = \sum_{j=0}^{r} (-1)^j [v_{i_0} \cdots \widehat{v_{i_j}} \cdots v_{i_r}],$$

where $0 \leq r \leq d$. The **boundary matrix** of Δ is given by the signs of the ridges in the boundary maps of the facets. We denote this matrix $\partial\Delta$.

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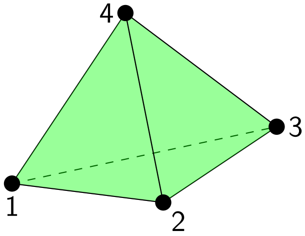
Definition

A **$\mathbb{Z}_q$-flow** on a pure simplicial complex Δ is an element of the kernel mod q of the boundary matrix of Δ.

Example

The surface of a tetrahedron.



	123	124	134	234
12	1	1	0	0
13	-1	0	1	0
14	0	-1	-1	0
23	1	0	0	1
24	0	1	0	-1
34	0	0	1	1

An example of a $\mathbb{Z}_4$-flow is: $(1, 3, 1, 3)$.

Results

Proposition (Beck, K.)

Let Δ be a triangulation of a manifold, and let $\phi_\Delta(q)$ be the number of nowhere-zero $\mathbb{Z}_q$-flows on Δ. Then,

$$\phi_\Delta(q) = \begin{cases} 0 & \textit{if } \Delta \textit{ has boundary;} \\ q-1 & \textit{if } \Delta \textit{ is without boundary, } \mathbb{Z}\textit{-orientable;} \\ 0 & \textit{if } \Delta \textit{ is without boundary, non-}\mathbb{Z}\textit{-orientable, } q \textit{ odd;} \\ 1 & \textit{if } \Delta \textit{ is without boundary, non-}\mathbb{Z}\textit{-orientable, } q \textit{ even.} \end{cases}$$

Results

Definition

A function ϕ in an integer variable t is a **quasipolynomial** if there exists an integer $k > 0$ and polynomials $p_0(t), \ldots, p_{k-1}(t)$ such that

$$\phi(t) = p_j(t) \qquad \text{if} \qquad t \equiv j \bmod k.$$

The minimal such k is the **period** of φ.

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The minimal such k is the **period** of φ.

Example

Let $\phi(t)$ be defined for $t \in \mathbb{Z}$ as follows:

$$\phi(t) = \begin{cases} t^2 + 1 & \text{if } t \equiv 0 \bmod 5 \\ t - 4 & \text{if } t \equiv 1, 3 \bmod 5 \\ 3t^3 + \frac{1}{2}t & \text{if } t \equiv 2 \bmod 5 \\ 0 & \text{if } t \equiv 4 \bmod 5. \end{cases}$$

Then $\phi(t)$ is a quasipolynomial with period 5.

Results

Theorem (Beck, K.)

The number $\phi_\Delta(q)$ *of nowhere-zero* $\mathbb{Z}_q$*-flows on* Δ *is a quasipolynomial in* q*. Furthermore, there exists a polynomial* $p(x)$ *such that* $\phi_\Delta(k) = p(k)$ *for all integers* k *that are relatively prime to the period of* $\phi_\Delta(q)$*.*

Theorem (Beck, K.)

- *Let* q *be a sufficiently large prime number, and let* Δ *be a simplicial complex of dimension* d*. Then the number* $\phi_\Delta(q)$ *of nowhere-zero* $\mathbb{Z}_q$*-flows on* Δ *is a polynomial in* q *of degree* $\dim_{\mathbb{Q}}(\widetilde{H}_d(\Delta;\mathbb{Q}))$.
- *In particular,* $\phi_\Delta(q) = (-1)^{|E(M)|-rk(M)} T_M(0, 1-q)$*, where* M *is the matroid given by the columns of* $\partial\Delta$*.*

Definitely Quasipolynomials: The Klein Bottle

Example

Let K be the Klein bottle. Then:

$$H_2(K;\mathbb{Z}_q) = \begin{cases} \mathbb{Z}_2 & q \text{ even} \\ 0 & q \text{ odd.} \end{cases}$$

Therefore:

$$\phi_K(q) = \begin{cases} 1 & q \text{ even} \\ 0 & q \text{ odd;} \end{cases}$$

$\phi_K(q)$ is a quasipolynomial with period 2.

The Period of the Flow Quasipolynomial

Definition

A matrix is **totally unimodular** (TU) iff every subdeterminant is $0, 1$, or -1.

Fact

If the boundary matrix of a simplicial complex Δ is TU, then $\phi_\Delta(q)$ has period 1.

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Theorem (Dey, Hirani, Krishnamoorthy)

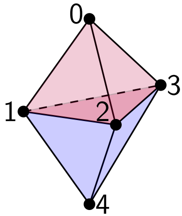
For a finite simplicial complex Δ of dimension greater than $d-1$, the boundary matrix $[\partial_d]$ is totally unimodular if and only if $H_{d-1}(L, L_0)$ is torsion-free for all pure subcomplexes L_0, L in Δ of dimensions $d-1$ and d respectively, where $L_0 \subset L$.

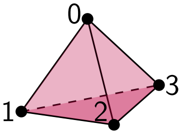
Definition

A **convex ear decomposition** of a pure rank-d simplicial complex Δ is an ordered sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ (the **ears**) of pure rank-d subcomplexes of Δ such that

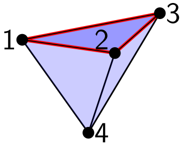
(1) Σ_1 is the boundary complex of a simplicial d-polytope, while for each $i = 2, \ldots, n$, Σ_i is a $(d-1)$-ball which is a (proper) sub-complex of the boundary complex of a simplicial d-polytope, and

(2) For $i \geq 2$, $\Sigma_i \cap \left(\bigcup_{j=1}^{i-1} \Sigma_j\right) = \partial\Sigma_i$.



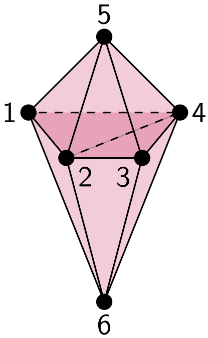


$\boldsymbol{\Sigma_1}$



$\boldsymbol{\Sigma_2}$

CED But Not TU



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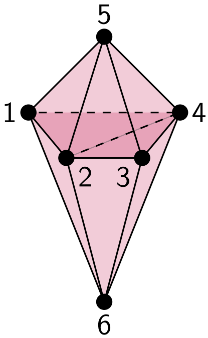
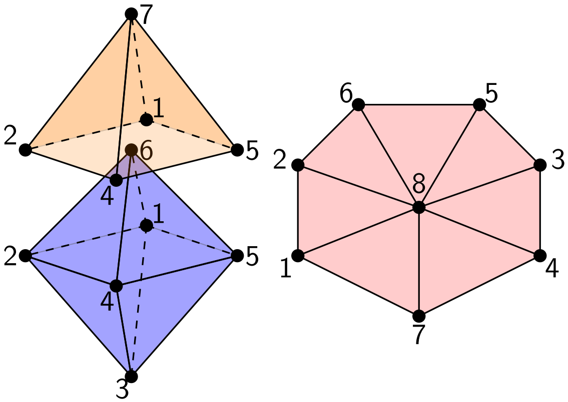


$$\phi_{\Delta}(q) = (q-1)(q-2)$$

BUT the period is still equal to 1 – perhaps there is hope...?

CED, Not TU, and $p > 1$



$$\phi_{\Delta}(q) = q^3 - 7q^2 + 15q - 8 - \gcd(2, q)$$

Open Questions and Future Directions

- Topological Conditions
  - Necessary and/or sufficient topological conditions for period equal to 1?
  - Necessary and/or sufficient topological conditions for period *greater than* 1?
- The Period of the Flow Quasipolynomial
  - Is there a bound for quasipolynomials from modular flows?
  - Can we find a subcomplex that guarantees a period greater than 1 – or is there always the possibility of period collapse?
- Constructions preserving/leading to polynomiality
- Families of simplicial complexes with $p = 1$?
- Relationship between $\phi_G(q)$ of a graph G and $\phi_{\Delta(G)}(q)$?

Polytopal Embeddings of Cayley Graphs

Cayley Graphs

Definition

Let Γ be a group, and Δ a set of generators of Γ. The **Cayley color graph**, $C(\Gamma, \Delta)$, of (Γ, Δ) is a directed, edge-colored graph such that:

- its vertices are the elements of Γ, and
- there is directed edge colored h from g_1 to g_2 if there exists a generator $h \in \Delta$ such that $g_1 h = g_2$.

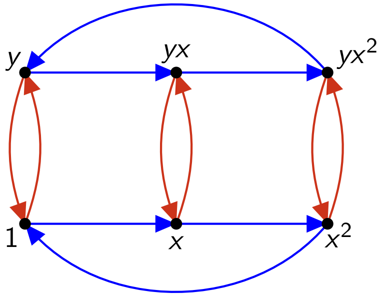
If we forget the colors and directions of the edges of $C(\Gamma, \Delta)$, we have the **Cayley graph**, $G(\Gamma, \Delta)$.

Remark

A group will have many Cayley graphs, which depend on the representation that is used.

An Example

Say we have a representation of a group
$\Gamma = \langle x, y \mid xy = yx,\ x^3 = y^2 = 1\rangle$. The Cayley color graph is:



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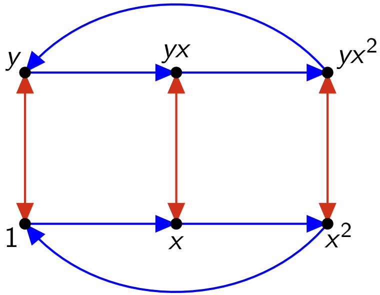

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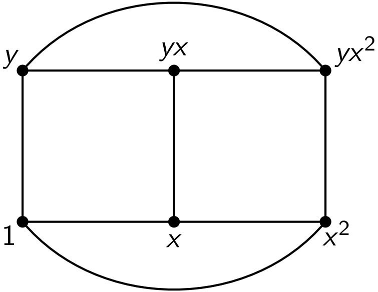

Definition

The **genus** of a graph is the minimal genus of all orientable surfaces in which G can be embedded.

Definition

The **genus** of a group Γ, $\gamma(\Gamma)$, is the minimal genus among the genera of all possible Cayley graphs of Γ.

The Genus of a Cayley Graph

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Open Problem!

Classify all finite groups of a particular genus γ, for all $\gamma > 2$.

Finite Groups of Genus 0, 1, and 2

- **Genus 0:** Classified by Maschke (1896)
- **Genus 1:** Classified by Proulx in her thesis (1978)
- **Genus 2:** Just one of them, found by Tucker (1984)

$$\begin{aligned}\langle x, y, z \quad | \quad & x^2 = y^2 = z^2 = 1, \\ & (xy)^2 = (yz)^3 = (xz)^8 = 1, \\ & y(xz)^4 y(xz)^4 = 1 \rangle\end{aligned}$$

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Finite Groups of Genus 0

Question

When is there a **polyhedral** *embedding of a planar group?*

Hold Up: Connectivity? and Other Definitions

Definition

- A **separator** S of a graph G is a subset of the vertices V such that $V \setminus S$ has at least two **components**.
- A ***k*-separator** is a separator of cardinality k.
- A graph is ***k*-connected** if there exist no separators of cardinality $\leq k - 1$.

Finite Groups of Genus 0

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Finite Groups of Genus 0

Question

When is there a **polyhedral** *embedding of a planar group?*

- **Fact 1:** (Steinitz, 1922) A graph is the 1-skeleton of a polyhedron iff it is 3-connected and planar.

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- **Fact 1+:** (Mani, 1971) Every 3-connected, planar graph G is the 1-skeleton of a polyhedron P such that every automorphism of G is induced by a symmetry of P.

Finite Groups of Genus 0

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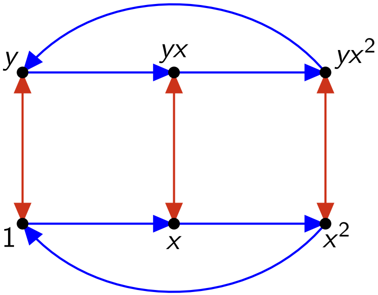
Proposition (De Loera)

Let $G(\Gamma, \Delta)$ *be a planar Cayley graph for the group* Γ*. Then* $G(\Gamma, \Delta)$ *can be embedded as the* 1*-skeleton of a polytope.*

An Example

Say we have a representation of a graph
$\Gamma = \langle x, y \mid xy = yx,\ x^3 = y^2 = 1 \rangle$. The polytonal embedding of the Cayley color graph is...?



It is 3-connected, and it is clearly planar...

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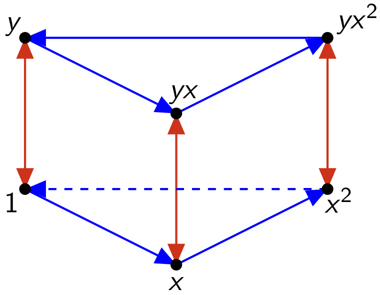

A Natural Question

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For any group Γ*, and any representation* Δ*, can we always find a convex polytope such that* $G(\Gamma, \Delta)$ *is its 1-skeleton?*

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Let's find out...

The Quaternions: Q_8

One presentation of Q_8 is:

$$\Delta = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle.$$

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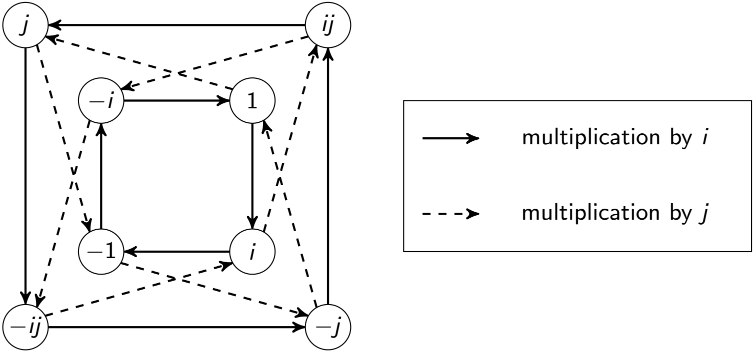
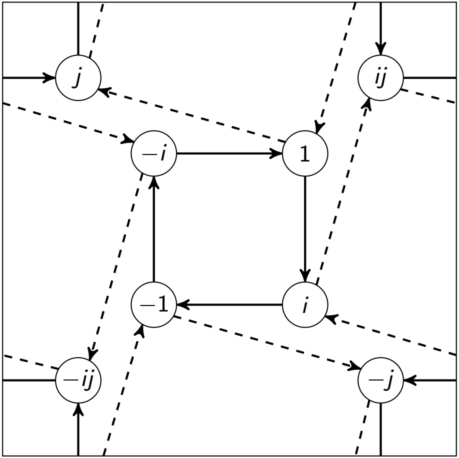
$$\Delta = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle.$$

This has the corresponding Cayley color graph:



Q_8 has Genus 1

We can embed this Cayley color graph on a Torus:



A Natural Question: Let's be specific, here.

Question

Can Q_8 be embedded as the 1-skeleton of some convex polytope?

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Theorem

Nope!

A Natural Question: Let's be specific, here.

Question

Can Q_8 be embedded as the 1-skeleton of some convex polytope?

Theorem

Nope!

A More Honest Theorem

There exists no convex polytope P with $G(P)$ equal to the Cayley graph of a **minimal** *presentation of the quaternion group.*

What's next?

- Are there (infinite) families of groups the minimal presentations of which cannot be embedded as the graphs of convex d-polytopes?
- Can we use group theory to characterize the embeddability of Cayley graphs?
  - Characterize subgroups that "block" the embedding of the Cayley graphs
  - Show that there exist no such subgroups
- Are there forbidden minor characterizations for the embeddability of Cayley graphs?
- Can we develop constructions that give d-polytopes with graphs equal to Cayley graphs?

Thank you!