The Analysis of Periodic Point Processes

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- Simulations for the MEA algorithm is joint work with Brian Sadler of the Army Research Laboratories.

- Simulations for the EQUIMEA algorithm is joint work with Kevin Duke of American University. Special thanks to Kevin for allowing us to experimentally verify the EQUIMEA.
1. Motivation: Signal and Image Signatures

2. $\pi$, the Primes, and Probability

3. The Modified Euclidean Algorithm (MEA)

4. Deinterleaving Multiple Signals (EQUIMEA)

5. Epilogue: The Riemann Zeta Function
Assumption – noisy signal data is set of event times (TOA’s) $s(t) + \eta(t)$ with (large) gaps in the data.
Data Models

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Questions – $s(t)$ periodic? period $\tau = ?$ Are there multiple periods $\tau_k = ?$ If so, what are they? How do we deinterleave the signals?
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Examples –

- radar or sonar
- bit synchronization in communications
- unreliable measurements in a fading communications channel
- biomedical applications
- times of a pseudorandomly occurring change in the carrier frequency of a “frequency hopping” radio, where the change rate is governed by a shift register output. In this case it is desired to find the underlying fundamental period $\tau$
Mathematical Models – Single Period

Finite set of real numbers

\[ S = \{s_j\}_{j=1}^n, \text{ with } s_j = k_j \tau + \varphi + \eta_j, \]

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where

- \( \tau \) (the period) is a fixed positive real number to be determined
- \( k_j \)'s are non-repeating positive integers (natural numbers)
- \( \varphi \) (the phase) is a real random variable uniformly distributed over the interval \([0, \tau)\)
- \( \eta_j \)'s (the noise) are zero-mean independent identically distributed (iid) error terms. We assume that the \( \eta_j \)'s have a symmetric probability density function (pdf), and that

\[ |\eta_j| \leq \eta_0 \leq \frac{\tau}{2} \text{ for all } j , \]

where \( \eta_0 \) is an \textit{a priori} noise bound.
Approaches to the Analysis

The data can be thought of as a set of event times of a periodic process, which generates a zero-one time series or delta train with additive jitter noise $\eta(t)$ –

$$s(t) = \sum_{j=1}^{n} \delta(t - ((k_j \tau + \varphi) + \eta(t))).$$
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- Another model – Let $f(t) = \sin(\frac{\pi}{\tau}(t - \varphi))$ and $S = \{\text{occurrence time of noisy zero-crossings of } f \text{ with missing observations}\}$.
- The $k_j$’s determine the best procedure for analyzing this data.
- Given a sequence of consecutive $k_j$’s, use least squares.
- Fourier analytic methods, e.g., Wiener’s periodogram, work with some missing observations, but when the percentage of missing observations is too large ($> 50\%$), they break down.
- Number theoretic methods can work with very sparse data sets ($> 90\%$ missing observations). Trade-off – low noise – number theory vs. higher noise – combine Fourier with number theory.
The Structure of Randomness over $\mathbb{Z}$
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**Theorem**

Given $n$ ($n \geq 2$) “randomly chosen” positive integers $\{k_1, \ldots, k_n\}$,

$$P\{\gcd(k_1, \ldots, k_n) = 1\} \longrightarrow 1^- \text{ quickly! as } n \longrightarrow \infty.$$
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Given $n$ ($n \geq 2$) “randomly chosen” positive integers $\{k_1, \ldots, k_n\}$,

$$P\{\gcd(k_1, \ldots, k_n) = 1\} = [\zeta(n)]^{-1}.$$
An Algorithm for Finding $\tau$

$$S = \{s_j\}_{j=1}^n, \text{ with } s_j = k_j \tau + \varphi + \eta_j$$

Let $\hat{\tau}$ denote the value the algorithm gives for $\tau$, and let “$\leftarrow$” denote replacement, e.g., “$a \leftarrow b$” means that the value of the variable $a$ is to be replaced by the current value of the variable $b$.

**Initialize:** Sort the elements of $S$ in descending order. Set $\text{iter} = 0$.

1.) [Adjoin 0 after first iteration.] If $\text{iter} > 0$, then $S \leftarrow S \cup \{0\}$.
2.) [Form the new set with elements $(s_j - s_{j+1})$.] Set $s_j \leftarrow (s_j - s_{j+1})$.
3.) [Sort.] Sort the elements in descending order.
4.) [Eliminate zero(s).] If $s_j = 0$, then $S \leftarrow S \setminus \{s_j\}$.
5.) The algorithm terminates if $S$ has only one element $s_1$. Declare $\hat{\tau} = s_1$. If not, $\text{iter} \leftarrow (\text{iter} + 1)$. Go to 1.)
Simulation Results

“To err is human. To really screw up, you need a computer.”
The Murphy Institute

Assume $\tau = 1$.

- Estimates and their standard deviations are based on averaging over 100 Monte-Carlo runs
- $n =$ number of data points, $iter =$ average number of iterations required for convergence, and $\%miss =$ average number of missing observations
- Estimates of $\tau$ are labeled $\hat{\tau}$, and $std(\hat{\tau})$ is the experimental standard deviation
- Threshold value of $\eta_0 = 0.35\tau = 0.35$ was used
1.)  *Noise-free estimation.*

Results from simulating noise-free estimation of $\tau$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M$</th>
<th>%miss</th>
<th>iter</th>
<th>$\tau$</th>
<th>$2\tau$</th>
<th>$3\tau$</th>
<th>$&gt;3\tau$</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>$10^1$</td>
<td>81.69</td>
<td>3.3</td>
<td>100%</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$10^2$</td>
<td>97.92</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
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<tr>
<td>4</td>
<td>$10^2$</td>
<td>97.84</td>
<td>15.2</td>
<td>82%</td>
<td>12</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>$10^2$</td>
<td>97.82</td>
<td>14.2</td>
<td>97</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
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<td>10.2</td>
<td>98</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
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<td>10.2</td>
<td>99</td>
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<td>0</td>
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<tr>
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<td>7.4</td>
<td>100</td>
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<td>0</td>
<td>0</td>
</tr>
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</table>
2.) *Uniformly distributed noise.*

Results from estimation of $\tau$ from noisy measurements.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M$</th>
<th>$\Delta$</th>
<th>%miss</th>
<th>iter</th>
<th>$\hat{\tau}$</th>
<th>std($\hat{\tau}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$10^1$</td>
<td>$10^{-3}$</td>
<td>81.37</td>
<td>4.35</td>
<td>0.9987</td>
<td>0.0005</td>
</tr>
<tr>
<td>10</td>
<td>$10^2$</td>
<td>$10^{-3}$</td>
<td>97.88</td>
<td>9.67</td>
<td>0.9980</td>
<td>0.0010</td>
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<tr>
<td>50</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
<td>99.80</td>
<td>16.0</td>
<td>0.9969</td>
<td>0.0028</td>
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<tr>
<td>10</td>
<td>$10^1$</td>
<td>$10^{-2}$</td>
<td>80.85</td>
<td>4.38</td>
<td>0.9888</td>
<td>0.0046</td>
</tr>
<tr>
<td>10</td>
<td>$10^1$</td>
<td>$10^{-2}$</td>
<td>81.94</td>
<td>4.45</td>
<td>0.9883</td>
<td>0.0051</td>
</tr>
<tr>
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<td>$10^1$</td>
<td>$10^{-1}$</td>
<td>81.05</td>
<td>4.33</td>
<td>0.8857</td>
<td>0.0432</td>
</tr>
</tbody>
</table>
Let $\mathbb{P} = \{p_1, p_2, p_3, \ldots\} = \{2, 3, 5, \ldots\}$ be the set of all prime numbers.
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“God gave us the integers. The rest is the work of man.”

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"God gave us the integers. The rest is the work of man."

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"... the Euler formulae (1736)

$$\zeta(n) = \sum_{n=1}^{\infty} n^{-z} = \prod_{j=1}^{\infty} \frac{1}{1 - (p_j)^{-z}}, \Re(z) > 1$$

was introduced to us at school, as a joke."

Littlewood
“Euclid’s algorithm is found in Book 7, Proposition 1 and 2 of his Elements (c.300 B.C.). We might call it the grand daddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day.”  Knuth
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The Euclidean algorithm is a division process for the integers \( \mathbb{Z} \). The algorithm is based on the property that, given two positive integers \( a \) and \( b \), \( a > b \), there exist two positive integers \( q \) and \( r \) such that \( a = q \cdot b + r \), \( 0 \leq r < b \). If \( r = 0 \), we say that \( b \) divides \( a \).
\[ \pi \text{, the Primes, and Probability, Cont'd} \]

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The Euclidean algorithm yields the greatest common divisor of two (or more) elements of \( \mathbb{Z} \). The greatest common divisor of two integers \( a \) and \( b \), denoted by \( \gcd(a, b) \), is the the largest integer that evenly divides both integers.
Theorem (Fundamental Theorem of Arithmetic)

Every positive integer can be written uniquely as the product of primes, with the prime factors in the product written in the order of nondecreasing size.
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*Every positive integer can be written uniquely as the product of primes, with the prime factors in the product written in the order of nondecreasing size.*

- If \( \gcd(a, b) = 1 \), we say that the numbers are relatively prime. This means that \( a \) and \( b \) share no common prime factors in their prime factorization.
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- $\gcd(3, 6) = 3$
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Examples
- \( \gcd(3, 7) = 1 \)
- \( \gcd(3, 6) = 3 \)
- \( \gcd(35, 21) = 7 \)
- \( \gcd(35, 21, 15) = 1 \)
Theorem

Given \( n \) (\( n \geq 2 \)) “randomly chosen” positive integers \( \{k_1, \ldots, k_n\} \),

\[
P\{\gcd(k_1, \ldots, k_n) = 1\} = [\zeta(n)]^{-1}.
\]

\( \zeta(n) \) denotes the Riemann zeta function.
Heuristic argument for this “theorem.” Given randomly distributed positive integers, by the Law of Large Numbers, about 1/2 of them are even, 1/3 of them are multiples of three, and 1/p are a multiple of some prime p. Thus, given n independently chosen positive integers,

\[
P\{p|k_1, p|k_2, \ldots, \text{and } p|k_n\} = \\
\text{(Independence)} \\
P\{p|k_1\} \cdot P\{p|k_2\} \cdot \ldots \cdot P\{p|k_n\} = \\
1/(p) \cdot 1/(p) \cdot \ldots \cdot 1/(p) = \\
1/(p)^n.
\]

Therefore,

\[
P\{p \nmid k_1, p \nmid k_2, \ldots, \text{and } p \nmid k_n\} = 1 - 1/(p)^n.
\]
By the Fundamental Theorem of Arithmetic, every integer has a unique representation as a product of primes. Combining that theorem with the definition of gcd, we get

\[ P\{\gcd(k_1, \ldots, k_n) = 1\} = \prod_{j=1}^{\infty} 1 - 1/(p_j)^n, \]

where \( p_j \) is the \( j^{th} \) prime.
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\[ P\{\gcd(k_1, \ldots, k_n) = 1\} = \prod_{j=1}^{\infty} 1 - \frac{1}{(p_j)^n}, \]

where \(p_j\) is the \(j^{th}\) prime.

But, by Euler’s formula,

\[ \zeta(z) = \prod_{j=1}^{\infty} \frac{1}{1 - (p_j)^{-z}}, \Re(z) > 1. \]

Therefore,

\[ P\{\gcd(k_1, \ldots, k_n) = 1\} = \frac{1}{\zeta(n)}. \]
This argument breaks down on the first line. Any uniform distribution on the positive integers would have to be identically zero. The merit in the argument lies in the fact that it gives an indication of how the zeta function plays a role in the problem.
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Let \( \text{card}\{\cdot\} \) denote cardinality of the set \( \{\cdot\} \), and let \( \{1, \ldots, \ell\}^n \) denote the sublattice of positive integers in \( \mathbb{R}^n \) with coordinates \( c \) such that \( 1 \leq c \leq \ell \). Therefore, \( N_n(\ell) = \text{card}\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\} \) is the number of relatively prime elements in \( \{1, \ldots, \ell\}^n \).
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**Theorem (MEA Theorem, C (1998), ...)**

Let \( N_n(\ell) = \text{card}\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\} \). For \( n \geq 2 \), we have that

\[
\lim_{\ell \to \infty} \frac{N_n(\ell)}{\ell^n} = [\zeta(n)]^{-1}.
\]
Brief Discussion of Proof: Let \( \lfloor x \rfloor \) denote the floor function of \( x \), namely

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\lfloor x \rfloor = \max\{k : k \in \mathbb{Z}, k \leq x\}.
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\[
\lfloor x \rfloor = \max\{k : k \in \mathbb{Z} \text{ and } k \leq x\}.
\]

\[
N_n(\ell) = \ell^n - \sum_{p_i} \left( \left\lfloor \frac{\ell}{p_i} \right\rfloor \right)^n + \sum_{p_i < p_j} \left( \left\lfloor \frac{\ell}{p_i \cdot p_j} \right\rfloor \right)^n - \sum_{p_i < p_j < p_k} \left( \left\lfloor \frac{\ell}{p_i \cdot p_j \cdot p_k} \right\rfloor \right)^n + \cdots
\]
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\[
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\]

Convergence is demonstrated by a sequence of careful estimates, use of Möbius Inversion, and more careful estimates.
The counting formula is seen as follows. Choose a prime number $p_i$. The number of integers in $\{1, \ldots, \ell\}$ such that $p_i$ divides an element of that set is $\left\lfloor \frac{\ell}{p_i} \right\rfloor$. (Note that it is possible to have $p_i > \ell$, because $\left\lfloor \frac{\ell}{p_i} \right\rfloor = 0$.)

Therefore, the number of $n$-tuples $(k_1, \ldots, k_n)$ contained in the lattice $\{1, \ldots, \ell\}^n$ such that $p_i$ divides every integer in the $n$-tuple is

$$\left(\left\lfloor \frac{\ell}{p_i} \right\rfloor\right)^n.$$
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$$\left( \left\lfloor \frac{\ell}{p_i} \right\rfloor \right)^n.$$ 

Next, if $p_i \cdot p_j$ divides an integer $k$, then $p_i | k$ and $p_j | k$. Therefore, the number of $n$-tuples $(k_1, \ldots, k_n)$ contained in the lattice $\{1, \ldots, \ell\}^n$ such that $p_i$ or $p_j$ or their product divide every integer in the $n$-tuple is 

$$\left( \left\lfloor \frac{\ell}{p_i} \right\rfloor \right)^n + \left( \left\lfloor \frac{\ell}{p_j} \right\rfloor \right)^n - \left( \left\lfloor \frac{\ell}{p_i \cdot p_j} \right\rfloor \right)^n,$$

where the last term is subtracted so that we do not count the same numbers twice (in a simple application of the inclusion-exclusion principle).
Each term is convergent –

\[
\frac{1}{\ell^n} \sum_{p_i < \ldots < p_k} \left( \frac{\ell}{p_i \cdot \ldots \cdot p_k} \right)^n \leq \frac{1}{\ell^n} \sum_{p_i < \ldots < p_k \leq \ell} \left( \frac{\ell}{p_i \cdot p_j \cdot \ldots \cdot p_k} \right)^n
\]

\[
= \sum_{p_i < \ldots < p_k \leq \ell} \left( \frac{1}{p_i \cdot \ldots \cdot p_k} \right)^n = \left( \sum_{p \leq \ell} \frac{1}{p^n} \right)^k
\]

\[
\leq \left( \sum_{p \text{ prime}} \frac{1}{p^n} \right)^k \leq \left( \sum_{j=2}^{\infty} \frac{1}{j^n} \right)^k.
\]

Since \( n \geq 2 \), this series is convergent.
Now, let

$$M_k = \left( \sum_{j=2}^{\infty} \frac{1}{j^n} \right)^k,$$

for $k = 2, 3, \ldots$.

By noting that since $n \geq 2$ and the sum is over $j \in \mathbb{N} \setminus \{1\}$, we get

$$0 < \sum_j \frac{1}{j^n} \leq \left( \frac{\pi^2}{6} - 1 \right) < 1.$$

Since the $k^{\text{th}}$ term in the expansion of $N_n(\ell)/\ell^n$ is dominated by $M_k$ and since

$$\sum_{k=0}^{\infty} M_k \leq \sum_{k=0}^{\infty} \left( \frac{\pi^2}{6} - 1 \right)^k = \frac{6}{12 - \pi^2},$$

is convergent, the series converges absolutely.
Euler showed that

\[
1 - \sum_{p_i} \frac{1}{p_i^n} + \sum_{p_i < p_j} \frac{1}{(p_i \cdot p_j)^n} - \sum_{p_i < p_j < p_k} \frac{1}{(p_i \cdot p_j \cdot p_k)^n} + \cdots
\]

\[
= \sum_m \frac{\mu(m)}{m^n} = [\zeta(n)]^{-1}.
\]

where the last sum is over \( m \in \mathbb{N} \). For \( n \geq 2 \), this series is absolutely convergent. \( \square \)
Theorem

Let $\omega \in (1, \infty)$. Then $\lim_{\omega \to \infty} [\zeta(\omega)]^{-1} = 1$, converging to 1 from below faster than $1/(1 - 2^{1-\omega})$. 
Theorem

Let \( \omega \in (1, \infty) \). Then \( \lim_{\omega \to \infty} [\zeta(\omega)]^{-1} = 1 \), converging to 1 from below faster than \( 1/(1 - 2^{1-\omega}) \).

Proof: Since \( \zeta(\omega) = \sum_{n=1}^{\infty} n^{-\omega} \) and \( \omega > 1 \),

\[
1 \leq \zeta(\omega) = 1 + \frac{1}{2^\omega} + \frac{1}{3^\omega} + \frac{1}{4^\omega} + \frac{1}{5^\omega} + \cdots \leq 1 + \frac{1}{2^\omega} + \frac{1}{2^\omega} + \frac{1}{4^\omega} + \cdots + \frac{1}{4^\omega} + \frac{1}{8^\omega} + \cdots + \frac{1}{8^\omega} + \cdots \\
= \sum_{k=0}^{\infty} \left( \frac{2}{2^\omega} \right)^k = \frac{1}{1 - \frac{2}{2^\omega}} = \frac{1}{1 - 2^{1-\omega}}.
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Theorem

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\]

\[
= \sum_{k=0}^{\infty} \left( \frac{2}{2^\omega} \right)^k = \frac{1}{1 - \frac{2}{2^\omega}} = \frac{1}{1 - 2^{1-\omega}}.
\]

As \( \omega \to \infty \), \( (1 - 2^{1-\omega}) \to 1^+ \). Thus, \( [\zeta(\omega)]^{-1} \to 1^- \) as \( \omega \to \infty \).
The Modified Euclidean Algorithm (MEA)

$S = \{s_j\}_{j=1}^n$, with $s_j = k_j\tau + \varphi + \eta_j$

Let $\hat{\tau}$ denote the value the algorithm gives for $\tau$, and let “←” denote replacement.

Initialize: Sort the elements of $S$ in descending order. Set $\text{iter} = 0$.

1.) [Adjoin 0 after first iteration.] If $\text{iter} > 0$, then $S \leftarrow S \cup \{0\}$.
2.) [Form the new set with elements $(s_j - s_{j+1})$.] Set $s_j \leftarrow (s_j - s_{j+1})$.
3.) [Sort.] Sort the elements in descending order.
4.) [Eliminate zero(s).] If $s_j = 0$, then $S \leftarrow S \setminus \{s_j\}$.
5.) The algorithm terminates if $S$ has only one element $s_1$. Declare $\hat{\tau} = s_1$. If not, $\text{iter} \leftarrow (\text{iter} + 1)$. Go to 1.)
The Modified Euclidean Algorithm (MEA), Cont’d

- Euclidean algorithm for \( \{k_j\}_{j=1}^n \subset \mathbb{N}, \tau > 0 \)

**Lemma**

\[
\gcd(k_1\tau, \ldots, k_n\tau) = \tau \gcd(k_1, \ldots, k_n).
\]
The Modified Euclidean Algorithm (MEA), Cont’d

- Euclidean algorithm for \( \{k_j\}_{j=1}^n \subset \mathbb{N}, \tau > 0 \) –

**Lemma**

\[ \gcd(k_1\tau, \ldots, k_n\tau) = \tau \gcd(k_1, \ldots, k_n). \]

- What if “integers are noisy?”
The Modified Euclidean Algorithm (MEA), Cont’d

- Euclidean algorithm for \( \{k_j\}_{j=1}^n \subset \mathbb{N}, \tau > 0 \) –

**Lemma**

\[
\gcd(k_1\tau, \ldots, k_n\tau) = \tau \gcd(k_1, \ldots, k_n).
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- What if “integers are noisy?”
- Remainder terms could be noise, and thus could be non-zero numbers arbitrarily close to zero. Subsequent steps in the procedure may involve dividing by such numbers, which would result in arbitrarily large numbers. The standard algorithm is unstable under perturbation by noise.
The Modified Euclidean Algorithm (MEA), Cont’d

- Euclidean algorithm for \( \{k_j\}_{j=1}^n \subset \mathbb{N} \), \( \tau > 0 \)

**Lemma**

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- What if “integers are noisy?”
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**Solution**: Replace division with subtraction, and threshold/average/filter/transform to eliminate noise.
Lemma

\[ \gcd(k_1, \ldots, k_n) = \gcd((k_1 - k_2), (k_2 - k_3), \ldots, (k_{n-1} - k_n), k_n). \]
Lemma

\[ \gcd(k_1, \ldots, k_n) = \gcd((k_1 - k_2), (k_2 - k_3), \ldots, (k_{n-1} - k_n), k_n). \]

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\[ \gcd((k_1 - k_2), (k_2 - k_3), \ldots, (k_{n-1} - k_n)) = \gcd((k_1 - k_n), \ldots, (k_{n-1} - k_n)). \]
The Modified Euclidean Algorithm (MEA), Cont’d

Combining the MEA Theorem with the Lemmas above gives the theoretical underpinnings of the Modified Euclidean Algorithm.
Combining the MEA Theorem with the Lemmas above gives the theoretical underpinnings of the Modified Euclidean Algorithm.

**Corollary**

Let \( n \geq 2 \). Given a randomly chosen \( n \)-tuple of positive integers \((k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n\),

\[
gcd(k_1\tau, \ldots, k_n\tau) \to \tau,
\]

with probability \( [\zeta(n)]^{-1} \) as \( \ell \to \infty \).
Combining the MEA Theorem with the Lemmas above gives the theoretical underpinnings of the Modified Euclidean Algorithm.

**Corollary**

Let \( n \geq 2 \). Given a randomly chosen \( n \)-tuple of positive integers \((k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n\),

\[
gcd(k_1\tau, \ldots, k_n\tau) \longrightarrow \tau,
\]

with probability \([\zeta(n)]^{-1}\) as \(\ell \longrightarrow \infty\).

Moreover, the estimate

\[
(1 - 2^{1-\omega})^{-1} \leq [\zeta(\omega)]^{-1} \leq 1
\]

shows that the algorithm very likely produces this value in the noise-free case or with minimal noise with as few as 10 data elements.
The Modified Euclidean Algorithm (MEA), Cont’d


The MEA can work with very sparse data sets (> 95% missing observations). Trade-off – low noise – use MEA vs. higher noise – combine spectral analysis with MEA theory.
Mathematical Models – Multiple Periods

Our data model is the union of $M$ copies of $S = \{s_{i,j}\}_{j=1}^{n_i}$ with $s_j = k_j \tau + \varphi + \eta_j$, each with different periods or “generators” $\Gamma = \{\tau_i\}$, $k_{ij}$’s and phases. Let $\tau_M = \max_i\{\tau_i\}$ and $\tau_m = \min_i\{\tau_i\}$. Then our data is

$$S = \bigcup_{i=1}^{M} \left\{ \varphi_i + k_{ij} \tau_i + \eta_{ij} \right\}_{j=1}^{n_i}$$
Mathematical Models – Multiple Periods

Our data model is the union of $M$ copies of $S = \{s_{i,j}\}^{n_i}_{j=1}$ with $s_j = k_j \tau + \varphi + \eta_j$, each with different periods or “generators” $\Gamma = \{\tau_i\}$, $k_{ij}$’s and phases. Let $\tau_M = \max_i \{\tau_i\}$ and $\tau_m = \min_i \{\tau_i\}$. Then our data is

$$S = \bigcup_{i=1}^{M} \left\{ \varphi_i + k_{ij} \tau_i + \eta_{ij} \right\}^{n_i}_{j=1},$$

- where $n_i$ is the number of elements from the $i^{th}$ generator
- the different periods or “generators” are $\Gamma = \{\tau_i\}$
- $\{k_{ij}\}$ is a linearly increasing sequence of natural numbers with missing observations
- $\varphi_i$ (the phases) are random variables uniformly distributed in $[0, \tau_i)$
- $\eta_{ij}$’s are zero-mean iid Gaussian with standard deviation $3\sigma_{ij} < \tau/2$
- We think of the data as events from $M$ periodic processes, and represent it, after reindexing, as $S = \{\alpha_i\}^{N}_{i=1}$, where $N = \sum_i n_i$. 

Stephen Casey

The Analysis of Periodic Point Processes
The Structure of Randomness over $[0, T)$

Theorem (Weyl's Equidistribution Theorem)

Let $\phi$ be an irrational number, $j \in \mathbb{N}$. Let $\langle j \phi \rangle = j \phi - \lfloor j \phi \rfloor$.

Then given $a, b, 0 \leq a < b < 1$,

$$\lim_{n \to \infty} \frac{\text{card} \{ 1 \leq j \leq n : \langle j \phi \rangle \in [a, b) \}}{n} = b - a.$$

Stephen Casey

The Analysis of Periodic Point Processes
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Then given $a, b$, $0 \leq a < b < 1$,

$$\frac{1}{n} \text{card} \left\{ 1 \leq j \leq n : \langle j\phi \rangle \in [a, b] \right\} \longrightarrow (b - a)$$

as $n \longrightarrow \infty$. 
The Structure of Randomness over \([0, T]\)

Assuming only minimal knowledge of the range of \(\{\tau_i\}\), namely bounds \(T_L, T_U\) such that \(0 < T_L \leq \tau_i \leq T_U\), we phase wrap the data by the mapping

\[
\Phi_\rho(\alpha_l) = \left\langle \frac{\alpha_l}{\rho} \right\rangle = \frac{\alpha_l}{\rho} - \left\lfloor \frac{\alpha_l}{\rho} \right\rfloor,
\]

where \(\rho \in [T_L, T_U]\), and \(\lfloor \cdot \rfloor\) is the floor function. Thus \(\langle \cdot \rangle\) is the fractional part, and so \(\Phi_\rho(\alpha_l) \in [0, 1]\).
Assuming only minimal knowledge of the range of \( \{\tau_i\} \), namely bounds \( T_L, T_U \) such that \( 0 < T_L \leq \tau_i \leq T_U \), we phase wrap the data by the mapping

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\Phi_{\rho}(\alpha_l) = \left\langle \frac{\alpha_l}{\rho} \right\rangle = \frac{\alpha_l}{\rho} - \left\lfloor \frac{\alpha_l}{\rho} \right\rfloor,
\]

where \( \rho \in [T_L, T_U] \), and \( \lfloor \cdot \rfloor \) is the floor function. Thus \( \langle \cdot \rangle \) is the fractional part, and so \( \Phi_{\rho}(\alpha_l) \in [0, 1) \).

**Definition**

A sequence of real random variables \( \{x_j\} \subset [0, 1) \) is essentially uniformly distributed in the sense of Weyl if given \( a, b, 0 \leq a < b < 1 \),

\[
\frac{1}{n} \text{card}\left\{ 1 \leq j \leq n : x_j \in [a, b] \right\} \longrightarrow (b - a)
\]

as \( n \longrightarrow \infty \) almost surely.
We assume that for each \( i \), \( \{ k_{ij} \} \) is a linearly increasing infinite sequence of natural numbers with missing observations such that

\[
k_{ij} \rightarrow \infty \text{ as } j \rightarrow \infty.
\]

Weyl’s Theorem applies asymptotically.
Applying Weyl’s Theorem

We assume that for each $i$, $\{k_{ij}\}$ is a linearly increasing infinite sequence of natural numbers with missing observations such that

$$k_{ij} \to \infty \text{ as } j \to \infty.$$  

Weyl’s Theorem applies asymptotically.

**Theorem (C (2014))**

For almost every choice of $\rho$ (in the sense of Lebesgue measure) $\Phi_{\rho}(\alpha_1)$ is essentially uniformly distributed in the sense of Weyl.
Moreover, the set of $\rho$’s for which this is not true are rational multiples of $\{\tau_i\}$. Therefore, except for those values, $\Phi_{\rho}(\alpha_{ij})$ is essentially uniformly distributed in $[T_L, T_U]$. The values at which $\Phi_{\rho}(\alpha_{ij}) = 0$ almost surely are $\rho \in \{\tau_i/n : n \in \mathbb{N}\}$. These values of $\rho$ cluster at zero, but spread out for lower values of $n$. 

We phase wrap the data by computing modulus of the spectrum, i.e., compute $|S_{\text{iter}}(\tau)| = \left|\sum_{j=1}^{N} e^{(2\pi i (j/\tau))}\right|$. The values of $|S_{\text{iter}}(\tau)|$ will have peaks at the periods $\tau_j$ and their harmonics ($\tau_j/k$). 

Stephen Casey

The Analysis of Periodic Point Processes
Moreover, the set of \( \rho \)'s for which this is not true are rational multiples of \( \{\tau_i\} \). Therefore, except for those values, \( \Phi_\rho(\alpha_{ij}) \) is essentially uniformly distributed in \( [T_L, T_U) \). The values at which \( \Phi_\rho(\alpha_{ij}) = 0 \) almost surely are \( \rho \in \{\tau_i/n : n \in \mathbb{N}\} \). These values of \( \rho \) cluster at zero, but spread out for lower values of \( n \).

We phase wrap the data by computing modulus of the spectrum, i.e., compute

\[
|S_{iter}(\tau)| = \left| \sum_{j=1}^{N} e^{(2\pi is(j)/\tau)} \right|.
\]
Moreover, the set of $\rho$’s for which this is not true are rational multiples of $\{\tau_i\}$. Therefore, except for those values, $\Phi_\rho(\alpha_{ij})$ is essentially uniformly distributed in $[T_L, T_U)$. The values at which $\Phi_\rho(\alpha_{ij}) = 0$ almost surely are $\rho \in \{\tau_i/n : n \in \mathbb{N}\}$. These values of $\rho$ cluster at zero, but spread out for lower values of $n$.

We phase wrap the data by computing modulus of the spectrum, i.e., compute

$$|S_{iter}(\tau)| = \left| \sum_{j=1}^{N} e^{(2\pi is(j)/\tau)} \right|.$$ 

The values of

$$|S_{iter}(\tau)|$$

will have peaks at the periods $\tau_j$ and their harmonics $(\tau_j)/k$. 
The EQUIMEA Algorithm – One Period

\[ S = \{s_j\}_{j=1}^n, \text{ with } s_j = k_j \tau + \varphi + \eta_j \]

**Initialize:** Sort the elements of \( S \) in descending order. Form the new set with elements \((s_j - s_{j+1})\). Set \( s_j \leftarrow (s_j - s_{j+1}) \). (Note, this eliminates the phase \( \varphi \).) Let \( \hat{\tau} \) denote the value the algorithm gives for \( \tau \), and let “\( \leftarrow \)” denote replacement.
The EQUIMEA Algorithm – One Period

1.) [Adjoin 0 after first iteration.] \( S_{iter} \leftarrow S \cup \{0\}. \)
2.) [Sort.] Sort the elements of \( S_{iter} \) in descending order.
3.) [Compute all differences.] Set \( S_{iter} = \bigcup (s_j - s_k) \) for all \( j, k \) with \( s_j > s_k \).
4.) [Eliminate zero(s).] If \( s_j = 0 \), then \( S_{iter} \leftarrow S_{iter} \setminus \{s_j\} \).
5.) [Adjoin previous iteration.] Form \( S_{iter} \leftarrow S_{iter} \cup S_{iter-1} \).
6.) [Compute spectrum.] Compute

\[
|S_{iter}(\tau)| = \left| \sum_{j=1}^{N} e^{(2 \pi is(j)/\tau)} \right|
\]

7.) [Threshold.] Choose the largest peak. Label it as \( \tau_{iter} \)
8.) The algorithm terminates if \( |\tau_{iter} - \tau_{iter-1}| < \text{Error} \). Declare \( \hat{\tau} = \tau_{iter} \). If not, \( \text{iter} \leftarrow (\text{iter} + 1) \). Go to 1.)
The EQUIMEA Algorithm – One Period, Cont’d

Figure: EQUIMEA One Period Tau – Original Data
The EQUIMEA Algorithm – One Period, Cont’d

Figure: EQUIMEA One Period Tau – One Iteration
The EQUIMEA Algorithm – One Period, Cont’d

Figure: EQUIMEA One Period Tau – One Iteration – Spectrum
The EQUIMEA Algorithm – One Period, Cont’d

Figure: EQUIMEA One Period Tau – Third Iteration
The EQUIMEA Algorithm – One Period, Cont’d

Figure: EQUIMEA One Period Tau – Third Iteration – Spectrum
Deinterleaving Multiple Signals

![Diagram of signal processing](image-url)
The EQUIMEA Algorithm – Multiple Periods

Our data model is the union of $M$ copies of $S = \{s_{i,j}\}_{j=1}^{n_i}$ with $s_j = k_j \tau + \varphi + \eta_j$, each with different periods or “generators” $\Gamma = \{\tau_i\}$, $k_{ij}$’s and phases. Let $\tau_M = \max_i \{\tau_i\}$ and $\tau_m = \min_i \{\tau_i\}$. Then our data is

$$S = \bigcup_{i=1}^{M} \left\{\varphi_i + k_{ij} \tau_i + \eta_{ij}\right\}_{j=1}^{n_i},$$

Let $\hat{\tau}$ denote the value the algorithm gives for $\tau$, and let “$\leftarrow$” denote replacement.

After reindexing, $S = \{\alpha_i\}_{i=1}^{N}$, where $N = \sum_i n_i$.

**Initialize:** Sort the elements of $S$ in descending order. Form the new set with elements $(s_i - s_{i+1})$. Set $s_i \leftarrow (s_i - s_{i+1})$. (Note, this eliminates the phase $\varphi$.) Set $\text{iter} = 1$, $i = 1$, and $\text{Error}$. Go to 1.)
The EQUIMEA Algorithm – Multiple Periods

1.) [Adjoin 0 after first iteration.] $S_{iter} \leftarrow S \cup \{0\}$.

2.) [Sort.] Sort the elements of $S_{iter}$ in descending order.

3.) [Compute all differences.] Set $S_{iter} = \bigcup (s_j - s_k)$ with $s_j > s_k$.

4.) [Eliminate zero(s).] If $s_j = 0$, then $S_{iter} \leftarrow S_{iter} \setminus \{s_j\}$.

5.) [Adjoin previous iteration.] Form $S_{iter} \leftarrow S_{iter} \cup S_{iter-1}$.

6.) [Compute spectrum.] Compute $|S_{iter}(\tau)| = \left| \sum_{j=1}^{N} e^{(2\pi is(j)/\tau)} \right|$.

7.) [Threshold.] Choose the largest peak. Label it as $\tau_{iter}$.

8.) If $|\tau_{iter} - \tau_{iter-1}| < \text{Error}$, Declare $\hat{\tau}_i = \tau_{iter}$. If not, $\text{iter} \leftarrow \text{iter} + 1$. Go to 1.)

9.) Given $\tau_i$, frequency notch $|S_{iter}(\tau)|$ for $\hat{\tau}_i/m$, $m \in \mathbb{N}$. Let $i \leftarrow i + 1$.

10.) [Compute spectrum.] Compute $|S_{iter}(\tau)| = \left| \sum_{j=1}^{N} e^{(2\pi is(j)/\tau)} \right|$.

11.) [Threshold.] Choose the largest peak. Label it as $\tau_{i+1}$. Algorithm terminates when there are no peaks. Else, go to 9.)
The EQUIMEA Algorithm – Two Periods

Figure: Two Periods – OriginalData
The EQUIMEA Algorithm – Two Periods, Cont’d

Figure: Spectrum of Two Period Data
The EQUIMEA Algorithm – Two Periods, Cont’d

Figure: EQUIMEA – Two Periods – Iter1
The EQUIMEA Algorithm – Two Periods, Cont’d

Figure: EQUIMEA – Two Periods – Iter1 – Spectrum
The EQUIMEA Algorithm – Two Periods, Cont’d

Figure: EQUIMEA – Two Periods – Iter2
The EQUIMEA Algorithm – Two Periods, Cont’d

Figure: EQUIMEA – Two Periods – Iter2 – Spectrum
The EQUIMEA Algorithm – Three Periods

Figure: Three Periods – Original Data
The EQUIMEA Algorithm – Three Periods, Cont’d

Figure: Spectrum of Three Period Data
The EQUIMEA Algorithm – Three Periods, Cont’d

Figure: EQUIMEA – Three Periods – Iter1
The EQUIMEA Algorithm – Three Periods, Cont’d

Figure: EQUIMEA – Three Periods – Iter1 – Spectrum
The EQUIMEA Algorithm – Three Periods, Cont’d

Figure: EQUIMEA – Three Periods – Iter2
The EQUIMEA Algorithm – Three Periods, Cont’d

Figure: EQUIMEA – Three Periods – Iter2 – Spectrum
Estimating the $\varphi_i$'s

For a good estimate $\hat{\tau}_i$ of $\tau_i$, $\exp(2\pi i \frac{k_j \tau_i}{\hat{\tau}_i}) \approx 1$. For $\eta_j \ll \tau_i \approx \hat{\tau}_i$, $\eta_j/\hat{\tau}_i \ll 1$, and so $\exp(2\pi i \eta_j/\hat{\tau}_i) \approx \exp(0) = 1$. 
Estimating the $\varphi_i$'s

For a good estimate $\hat{\tau}_i$ of $\tau_i$, $\exp(2\pi i \frac{k_j \tau_i}{\hat{\tau}_i}) \approx 1$. For $\eta_j \ll \tau_i \approx \hat{\tau}_i$, $\eta_j/\hat{\tau}_i \ll 1$, and so $\exp(2\pi i \eta_j/\hat{\tau}_i) \approx \exp(0) = 1$.

Therefore,

$$\frac{\hat{\tau}_i}{2\pi} \arg \left\{ \sum_{j=1}^{n} \exp(2\pi i \frac{S_{ij}}{\hat{\tau}_i}) \right\}$$

$$= \frac{\hat{\tau}_i}{2\pi} \arg \left\{ \sum_{j=1}^{n} \exp(2\pi i \frac{k_{ij} \tau_i}{\hat{\tau}_i}) \exp(2\pi i \frac{\eta_{ij}}{\hat{\tau}_i}) \exp(2\pi i \frac{\varphi_i}{\hat{\tau}_i}) \right\}$$

$$\approx \frac{\hat{\tau}_i}{2\pi} \arg \left\{ \sum_{j=1}^{n} \exp(2\pi i \frac{\varphi_i}{\hat{\tau}_i}) \right\} = \frac{\hat{\tau}_i}{2\pi} \arg \left\{ n \cdot \exp(2\pi i \frac{\varphi_i}{\hat{\tau}_i}) \right\}$$

$$= \frac{\hat{\tau}_i}{2\pi} \left( \arg \left\{ n \right\} + \arg \left\{ \exp(2\pi i \frac{\varphi_i}{\hat{\tau}_i}) \right\} \right) = \frac{\hat{\tau}_i}{2\pi} \arg \left\{ \exp(2\pi i \frac{\varphi_i}{\hat{\tau}_i}) \right\}$$

$$= \frac{\hat{\tau}_i}{2\pi} \frac{2\pi \varphi_i}{\hat{\tau}_i} = \varphi_i$$
Estimating the $k_{ij}$’s’s

We present two methods of getting an estimate on the set of $k_{ij}$’s. Let $\text{round}(\cdot)$ denotes rounding to the nearest integer. Given a good estimate $\hat{\phi}_i$, the first is to form the set

$$\sigma = \{k_{ij} \tau_i + \varphi_i + \eta_{ij} - \hat{\phi}\}_{j=1}^n.$$  

Given the estimate $\hat{\tau}_i$, estimate $k_{ij}$ by

$$\hat{k}_{ij} = \text{round} \left( \frac{k_{ij} \tau_i + \varphi_i + \eta_{ij} - \hat{\phi}_i}{\hat{\tau}_i} \right).$$
Estimating the $k_{ij}$'s's

- We present two methods of getting an estimate on the set of $k_{ij}$'s. Let $\text{round}(\cdot)$ denotes rounding to the nearest integer. Given a good estimate $\hat{\varphi}_i$, the first is to form the set
  \[ \sigma = \{ k_{ij} \tau_i + \varphi_i + \eta_{ij} - \hat{\varphi} \}_{j=1}^n. \]
  Given the estimate $\hat{\tau}_i$, estimate $k_{ij}$ by
  \[ \hat{k}_{ij} = \text{round} \left( \frac{k_{ij} \tau_i + \varphi_i + \eta_{ij} - \hat{\varphi}_i}{\hat{\tau}_i} \right). \]

- Let $\sigma' = \{ K_{ij} \tau - i + \eta'_{ij} \}_{j=1}^{n-1} \cup \{ k(i,n_i) \tau_i + \varphi + \eta_{in} - \hat{\varphi}_i \}$, where
  $K_{ij} = k_{ij} - k(i,j+1)$ and $\eta'_{ij} = \eta_{ij} - \eta(i,j+1)$. Given the estimate $\hat{\tau}_i$, estimate $k(i,n_i)$ by
  \[ \hat{k}(i,n_i) = \text{round} \left( \frac{k(i,n_i) \tau_i + \varphi + \eta_{i,n_i} - \hat{\varphi}_i}{\hat{\tau}_i} \right) \]
  and $K_{ij}$ by
  \[ \hat{K}_{ij} = \text{round} \left( \frac{K_{ij} \tau_i + \eta'_{ij}}{\hat{\tau}_i} \right). \]

Then, $\hat{k}(i,n_i-1) = \hat{K}(i,n_i-1) + \hat{k}(i,n_i)$, $\hat{k}(i,n_i-2) = \hat{K}(i,n_i-2) + \hat{k}(i,n_i-1)$, and so on.
Epilogue: The Riemann Zeta Function

A nice “byproduct” of the MEA work is a novel way to compute values of the Riemann Zeta Function.
A nice “byproduct” of the MEA work is a novel way to compute values of the Riemann Zeta Function.

**Definition**

Riemann Zeta Function: For \( \{ z \in \mathbb{C} : z = x + iy, x > 1 \} \),

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\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.
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\]

But first, let us see how the number \( \pi \) surprisingly appears in some known series values.
π and Series

- \( x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots = \int_0^x \frac{1}{1+y^2} \, dy = \arctan(x) \).

- Letting \( x \to 1 \), we get \( 1 - \frac{1}{3} + \frac{1}{5} + \ldots + (-1)^n \frac{1}{2n+1} + \ldots = \frac{\pi}{4} \).
\( \pi \) and Series

- \( x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots = \int_0^x \frac{1}{1+y^2} \, dy = \arctan(x) \).
- Letting \( x \uparrow 1 \), we get \( 1 - \frac{1}{3} + \frac{1}{5} + \ldots + (-1)^n \frac{1}{2n+1} + \ldots = \frac{\pi}{4} \).

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} , \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} , \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} , \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450} .
\]
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\[
\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} + \ldots \longrightarrow \frac{\pi^3}{something}. \]
\( \pi \text{ and Series} \)

- \[ x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots = \int_0^x \frac{1}{1+y^2} \, dy = \arctan(x). \]

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\]

**Calculus Joke:** \( \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi^3}{something} \). I’ll give you an “A” in Calculus 2 if you can tell me the exact value of something.
The Riemann Zeta Function: Euler’s Sums

**Theorem**

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}, \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450},
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and the formula for \( k = 1, 2, 3, \ldots \)

\[
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \text{ where } B_{2k} = 2k^{\text{th}} \text{ Bernoulli Number}.
\]
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“The result is due to Euler (circa 1736) and constitutes one of his most remarkable computations.”

K. Ireland and M. Rosen.
Definition

The Bernoulli numbers are given by the generating series

\[ \frac{z}{e^z - 1} + \frac{1}{2}z = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}, \quad |z| < 2\pi. \]
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\]

We need the Weierstrass product representation of \(\sin(z)\), namely

\[
\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).
\]
We may take the logarithmic derivative of both sides in the annular region \(\{0 < |z| < \pi\}\), getting

\[
\frac{\pi \cos(\pi z)}{\sin(\pi z)} = \pi \cot(\pi z) = \frac{1}{z} - \sum_{n=1}^{\infty} \left( \frac{2z}{n^2 - z^2} \right).
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Euler’s Sums, Cont’d

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Writing the cot in terms of exponentials and simplifying gives

\[
\pi z \cot(\pi z) = \pi iz \left[ \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right] = \frac{2\pi iz}{e^{2\pi iz} - 1} + \pi iz.
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Substituting in the generating function of the Bernoulli numbers gives

\[
\pi z \cot(\pi z) = \frac{2\pi iz}{e^{\pi iz} - 1} + \pi iz = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi)^{2n} B_{2n}}{(2n)!} z^{2n}.
\]
We have that

\[
\pi z \cot(\pi z) = 1 - \sum_{n=1}^{\infty} \left( \frac{2z^2}{n^2 - z^2} \right).
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Euler’s Sums, Cont’d

We have that

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Since \( n \geq 2 \), if \( \{0 < |z| < 2\} \), we have that

\[ \frac{z^2}{n^2 - z^2} = \frac{z^2}{1 - \frac{z^2}{n^2}} = \sum_{m=1}^{\infty} \left( \frac{z^2}{n^2} \right)^m. \]
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Since both of the two series in the previous formulas are absolutely convergent, we may reverse the order of summation and get
\[ 1 - 2 \sum_{n=1}^{\infty} \left( \frac{z^2}{n^2 - z^2} \right) = 1 - 2 \sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2m}} \right) z^{2m}. \]
Matching indexes we have that

\[ 1 - 2 \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m^{2n}} \right) z^{2n} = \pi z \cot(\pi z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi)^{2n} B_{2n}}{(2n)!} z^{2n}. \]
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Equating coefficients and dividing by 2 gives the result.
Euler’s Sums, Cont’d

Table: Some values of the Zeta Function $\zeta(n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\zeta(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{\pi^2}{6}$</td>
</tr>
</tbody>
</table>

Motivation: Signal and Image Signatures
\[ \pi, \text{ the Primes, and Probability} \]
The Modified Euclidean Algorithm (MEA)
Deinterleaving Multiple Signals (EQUIMEA)
Epilogue: The Riemann Zeta Function

Stephen Casey
The Analysis of Periodic Point Processes
Some Sum History

- The infinite series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is an example of one of the first infinite series to be summed in the history of mathematics; it was used by Archimedes circa 250-200 BC.
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- The infinite series $\sum_{n=1}^{\infty} 1/4^n$ is an example of one of the first infinite series to be summed in the history of mathematics; it was used by Archimedes circa 250-200 BC.
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- For $\zeta(2k+1)$, we only have that Apery proved that $\zeta(3)$ is irrational in 1978. The determination of the irrationality of $\zeta(5), \zeta(7), \ldots$ is still open. The lack of the formulae is certainly not the result of a lack of effort, e.g., see papers by Borwein$^3$, Bradley, and Crandall.
Theorem (Asymptotic Estimates, C (2013), ...)

Let

\[ N_n(\ell) = \text{card}\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\}, \]

For \( \ell \geq 2 \), we have that

\[ N_2(\ell) = \frac{\ell^2}{\zeta(2)} + \mathcal{O}(\ell \log(\ell)), \]

and for \( n > 2 \),

\[ N_n(\ell) = \frac{\ell^n}{\zeta(n)} + \mathcal{O}(\ell^{n-1}). \]
Numerical Computations

Numerical Computations


- “To err is human. To really foul things up requires a computer.”
  
  *The Computer Maxim from The Murphy Institute*
Numerical Computations


- “To err is human. To really foul things up requires a computer. ”
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- “The beauty of a computer is that it can take human error and compound it millions of times per second.” *Anon.*
Some Observations About Computers and Mathematics.

“To err is human. To really foul things up requires a computer.”
*The Computer Maxim from The Murphy Institute*

“The beauty of a computer is that it can take human error and compound it millions of times per second.” *Anon.*

“To err is human - and to blame it on a computer is even more so.” *Robert Orben*
Numerical Computations, Cont’d

\[
\frac{N_2(\ell) \cdot \pi^2}{\ell^2} = 6 + \mathcal{O}\left(\frac{\log(\ell)}{\ell}\right),
\]

\[
\frac{N_3(\ell) \cdot \pi^3}{\ell^3} = \Lambda_3 + \mathcal{O}\left(\frac{1}{\ell}\right),
\]

\[
\frac{N_4(\ell) \cdot \pi^4}{\ell^4} = 90 + \mathcal{O}\left(\frac{1}{\ell}\right),
\]

\[
\frac{N_5(\ell) \cdot \pi^5}{\ell^5} = \Lambda_5 + \mathcal{O}\left(\frac{1}{\ell}\right),
\]

\[
\frac{N_6(\ell) \cdot \pi^6}{\ell^6} = 945 + \mathcal{O}\left(\frac{1}{\ell}\right).
\]
I created a list of primes, and used the formula for $N_n(\ell)$ given above. The advantage of this approach is that once we have a list of primes $p_i \leq \ell$, we can generate numerical approximations of $\zeta(n)$ by changing the exponents.

**Table:** Some values of $(N_2(\ell) \cdot \pi^2)/\ell^2$ and $\log(\ell)/\ell$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$(N_2(\ell) \cdot \pi^2)/\ell^2$</th>
<th>$\log(\ell)/\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>6.0000300909036373</td>
<td>$\approx 0.00001151292546$</td>
</tr>
<tr>
<td>1,000,000</td>
<td>6.0000000289078077</td>
<td>$\approx 0.0000001151292546$</td>
</tr>
</tbody>
</table>
Table: Some values of \( \left( N_n(\ell) \cdot \pi^n \right)/\ell^n \).

<table>
<thead>
<tr>
<th>\ell</th>
<th>( (N_3(\ell) \cdot \pi^3)/\ell^3 )</th>
<th>( (N_4(\ell) \cdot \pi^4)/\ell^4 )</th>
<th>( (N_5(\ell) \cdot \pi^5)/\ell^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>25.794384413862879</td>
<td>90.0000456705</td>
<td>295.121570196</td>
</tr>
<tr>
<td>1,000,000</td>
<td>25.794351968305143</td>
<td>90.0000037099</td>
<td>295.121515514</td>
</tr>
</tbody>
</table>
To compute $\zeta(3)$ we have a relatively fast formula thanks to Ramanujan. The Ramanujan formula –

$$
\zeta(3) = \frac{8}{7} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \frac{1}{k^3} - \frac{4}{21} \log^3 2 + \frac{2}{21} \pi^2 \log 2 .
$$

My student Andreas Wiede programmed these computations in both Python and Julia.

<table>
<thead>
<tr>
<th>$n = 1,000,000$</th>
<th>Julia</th>
<th>Python</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brute Force</td>
<td>1.202056903159</td>
<td>1.202056903159</td>
</tr>
<tr>
<td>Ramanujan Method</td>
<td>1.202056903159594</td>
<td>1.2020569031595942</td>
</tr>
</tbody>
</table>
Numerical Computations, Cont’d

<table>
<thead>
<tr>
<th>Euler Product</th>
<th>Julia</th>
<th>Python</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\zeta(3))</td>
<td>1.202056903159594</td>
<td>1.202056903159594</td>
</tr>
<tr>
<td>(\zeta(5))</td>
<td>1.036927755143369</td>
<td>1.036927755143369</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MEA Method</th>
<th>(\ell = 10^6)</th>
<th>(\ell = 10^8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\zeta(3))(^{-1}\cdot \pi^3)</td>
<td>25.79435196830</td>
<td>25.7943501926105</td>
</tr>
<tr>
<td>(\zeta(5))(^{-1}\cdot \pi^5)</td>
<td>295.121515513789</td>
<td>295.121509986379</td>
</tr>
</tbody>
</table>

\[
\frac{10\pi^3}{258} < \zeta(3) < \frac{4\pi^3}{103},
\]
\[
20\pi^5 \frac{5903}{5903} < \zeta(5) < \frac{25\pi^5}{7378}.
\]
References


Signal to Noise Ratio (SNR)

\[ SNR = 10 \cdot \log_{10} \left( \frac{S}{N} \right) \text{ DB} \]

- Algorithm worst case – 3 DB
- Algorithm noise range – 3 DB – 30 DB
- AM signal from a distant radio station – 10 DB
- TV picture gets “snowy” – 20 DB
- AM signal from a local radio station, 8-track tape – 30 DB
- FM signal from a local radio station – 40–45 DB
- Cassette tape with Dolby – 45–50 DB
- Background noise in department store amplifier – 55–60 DB
- Quantization noise in my CD – 72.247 DB
- Background noise in my amp – 80 DB