The Edge of Graphicality

Elizabeth Moseman

in collaboration with
Brian Cloteaux, M. Drew LaMar, James Shook

February 5, 2013

Graphical Sequences

A sequence $\alpha = (\alpha_1, ..., \alpha_n)$ of positive integers is graphical if there is a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $d(v_i) = \alpha_i$.

Graphical Sequences

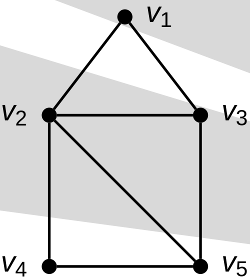
A sequence $\alpha = (\alpha_1, ..., \alpha_n)$ of positive integers is graphical if there is a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $d(v_i) = \alpha_i$.

Example: Let $\alpha = (2, 4, 3, 4, 3)$.

Graphical Sequences

A sequence $\alpha = (\alpha_1, ..., \alpha_n)$ of positive integers is graphical if there is a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $d(v_i) = \alpha_i$.

Example: Let $\alpha = (2, 4, 3, 4, 3)$.



Directed Graphs

A sequence $\alpha = \big((\alpha_1^+, \alpha_1^-), \ldots, (\alpha_n^+, \alpha_n^-)\big)$ of positive integer pairs is **digraphical** if there is a digraph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $d^+(v_i) = \alpha_i^+$, $d^-(v_i) = \alpha_i^-$.

Directed Graphs

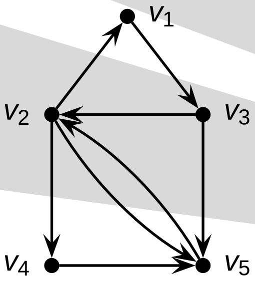
A sequence $\alpha = \big((\alpha_1^+, \alpha_1^-), \dots, (\alpha_n^+, \alpha_n^-)\big)$ of positive integer pairs is **digraphical** if there is a digraph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $d^+(v_i) = \alpha_i^+$, $d^-(v_i) = \alpha_i^-$.

Example: Let $\alpha = \big((1,1), (3,2), (2,1), (1,1), (1,3)\big)$

Directed Graphs

A sequence $\alpha = ((\alpha_1^+, \alpha_1^-), \ldots, (\alpha_n^+, \alpha_n^-))$ of positive integer pairs is **digraphical** if there is a digraph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $d^+(v_i) = \alpha_i^+$, $d^-(v_i) = \alpha_i^-$.

Example: Let $\alpha = ((1,1),(3,2),(2,1),(1,1),(1,3))$



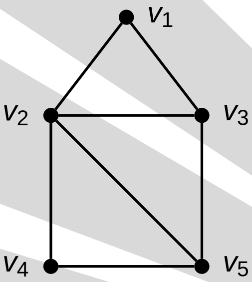
Adjacency Matrix

A zero-one matrix A is the **adjacency matrix** for a graph G if $a_{ij} = 1$ if and only if there is an edge (v_i, v_j).

Adjacency Matrix

A zero-one matrix A is the **adjacency matrix** for a graph G if $a_{ij} = 1$ if and only if there is an edge (v_i, v_j).

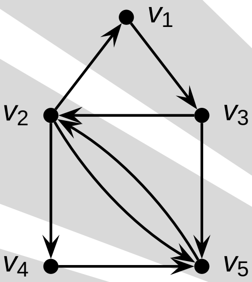
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$



Adjacency Matrix

A zero-one matrix A is the **adjacency matrix** for a graph G if $a_{ij} = 1$ if and only if there is an edge (v_i, v_j).

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



Sequence Order

- There is no order to the vertices of a graph, but all our sequences have order. We define canonical orders.

Sequence Order

- There is no order to the vertices of a graph, but all our sequences have order. We define canonical orders.
- For an undirected graph, the usual order is **non-increasing**.

Sequence Order

- There is no order to the vertices of a graph, but all our sequences have order. We define canonical orders.
- For an undirected graph, the usual order is **non-increasing**.
- An integer pair sequence α is in **positive lexicographical order** if $\alpha_i^+ \geq \alpha_{i+1}^+$ with $\alpha_i^- \geq \alpha_{i+1}^-$ when $\alpha_i^+ = \alpha_{i+1}^+$

Sequence Order

- There is no order to the vertices of a graph, but all our sequences have order. We define canonical orders.
- For an undirected graph, the usual order is **non-increasing**.
- An integer pair sequence α is in **positive lexicographical order** if $\alpha_i^+ \geq \alpha_{i+1}^+$ with $\alpha_i^- \geq \alpha_{i+1}^-$ when $\alpha_i^+ = \alpha_{i+1}^+$
- In this order, our example $\alpha = \big((1,1),(3,2),(2,1),(1,1),(1,3)\big)$ becomes $\big((3,2),(2,1),(1,3),(1,1)\big)$.

When is a sequence realizable?

Theorem (Erdős, Gallai (1960))

A non-increasing non-negative integer sequence $(d_1, ..., d_n)$ *is graphic if and only if the sum is even and the sequence satisfies*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\} \quad for\ 1 \leq k \leq n. \tag{1}$$

When is a sequence realizable?

Theorem (Fulkerson (1960), Chen (1966))

Let $\alpha = \left((\alpha_1^+, \alpha_1^-), \ldots, (\alpha_n^+, \alpha_n^-)\right)$ *be a non-negative integer sequence in positive lexicographic order. There is a digraph G that realizes* α *if and only if* $\sum \alpha_i^+ = \sum \alpha_i^-$ *and for every* k *with* $1 \le k < n$

$$\sum_{i=1}^{k} \min(\alpha_i^-, k-1) + \sum_{i=k+1}^{n} \min(\alpha_i^-, k) \ge \sum_{i=1}^{k} \alpha_i^+. \quad (2)$$

Unique labeled Realizations

Theorem (Collected Results)

Let α be a graphical sequence and G a realization of α. The following are equivalent:

The degree sequences and graphs satisfying any of the above are called **threshold graphs**.

Unique labeled Realizations

Theorem (Collected Results)

Let α be a graphical sequence and G a realization of α. The following are equivalent:

- *G is the unique labeled realization of α.*

The degree sequences and graphs satisfying any of the above are called **threshold graphs**.

Unique labeled Realizations

Theorem (Collected Results)

Let α be a graphical sequence and G a realization of α. The following are equivalent:

- *G is the unique labeled realization of α.*
- *There are no alternating four cycles in G.*

The degree sequences and graphs satisfying any of the above are called **threshold graphs**.

Unique labeled Realizations

Theorem (Collected Results)

Let α be a graphical sequence and G a realization of α. The following are equivalent:

- *G is the unique labeled realization of α.*
- *There are no alternating four cycles in G.*
- *G can be formed from a one vertex graph by adding a sequence of vertices, where each added vertex is either empty or dominating.*

The degree sequences and graphs satisfying any of the above are called **threshold graphs**.

Unique labeled Realizations

Theorem (Collected Results)

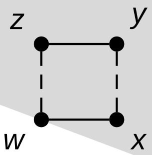
Let α be a graphical sequence and G a realization of α. The following are equivalent:

- *G is the unique labeled realization of α.*
- *There are no alternating four cycles in G.*
- *G can be formed from a one vertex graph by adding a sequence of vertices, where each added vertex is either empty or dominating.*
- *α satisfies the Erdős–Gallai conditions with equality.*

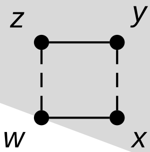
The degree sequences and graphs satisfying any of the above are called **threshold graphs**.

Alternating four cycle: Four distinct vertices so that (w, x) and (y, z) are edges and (w, z) and (x, y) are not.

Alternating four cycle: Four distinct vertices so that (w, x) and (y, z) are edges and (w, z) and (x, y) are not.



Alternating four cycle: Four distinct vertices so that (w, x) and (y, z) are edges and (w, z) and (x, y) are not.



Distinct integers i, j, k, l so that $a_{ik} = a_{jl} = 1$ and $a_{il} = a_{jk} = 0$

Adding vertices to form a threshold graph:

Adding vertices to form a threshold graph:

Adding vertices to form a threshold graph:

Adding vertices to form a threshold graph:

Adding vertices to form a threshold graph:

Adding vertices to form a threshold graph:

Adding vertices to form a threshold graph:

Unique realizations of Digraphs

Forbidden Configurations

Theorem (Rao, Jana and Bandyopadhyayl (1996))

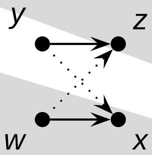
A digraph G is the unique realization of its degree sequence if and only if it has neither a two-switch nor an induced directed three-cycle.

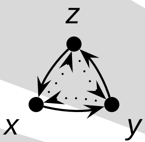
Unique realizations of Digraphs

Forbidden Configurations

Theorem (Rao, Jana and Bandyopadhyayl (1996))

A digraph G is the unique realization of its degree sequence if and only if it has neither a two-switch nor an induced directed three-cycle.





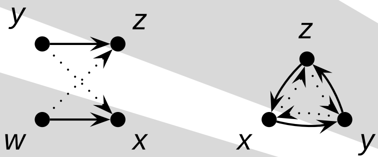
A two-switch and an induced directed three-cycle. Solid arcs must appear in the digraph and dashed arcs must not appear in the digraph. If an arc is not listed, then it may or may not be present.

Unique realizations of Digraphs

Forbidden Configurations

Theorem (Rao, Jana and Bandyopadhyayl (1996))

A digraph G is the unique realization of its degree sequence if and only if it has neither a two-switch nor an induced directed three-cycle.



A two-switch and an induced directed three-cycle.

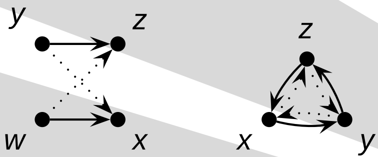
Distinct integers i, j, k, l so that $a_{ik} = a_{jl} = 1$ and $a_{il} = a_{jk} = 0$ form a two-switch.

Unique realizations of Digraphs

Forbidden Configurations

Theorem (Rao, Jana and Bandyopadhyayl (1996))

A digraph G is the unique realization of its degree sequence if and only if it has neither a two-switch nor an induced directed three-cycle.



A two-switch and an induced directed three-cycle.

Distinct integer i, j, k so that $a_{ij} = a_{jk} = a_{ki} = 1$ and $a_{ik} = a_{kj} = a_{ji} = 0$ form an induced directed three cycle.

Characterization

Theorem (2012)

Let G be a digraph, A its adjacency matrix, and α the degree sequence in positive lexicographical order. The following are equivalent:

1. *G is the unique labeled realization of the degree sequence α.*

Characterization

Theorem (2012)

Let G be a digraph, A its adjacency matrix, and α the degree sequence in positive lexicographical order. The following are equivalent:

1. *G is the unique labeled realization of the degree sequence α.*
2. *There are no 2-switches or induced directed 3-cycles in G.*

Characterization

Theorem (2012)

Let G be a digraph, A its adjacency matrix, and α the degree sequence in positive lexicographical order. The following are equivalent:

1. *G is the unique labeled realization of the degree sequence α.*
2. *There are no 2-switches or induced directed 3-cycles in G.*
3. *For every triple of distinct indices i, j and k with $i < j$, if $a_{jk} = 1$, then $a_{ik} = 1$.*

Characterization

Theorem (2012)

Let G be a digraph, A its adjacency matrix, and α the degree sequence in positive lexicographical order. The following are equivalent:

1. *G is the unique labeled realization of the degree sequence α.*
2. *There are no 2-switches or induced directed 3-cycles in G.*
3. *For every triple of distinct indices i, j and k with $i < j$, if $a_{jk} = 1$, then $a_{ik} = 1$.*
4. *The Fulkerson-Chen inequalities are satisfied with equality. In other words, for $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \min(\alpha_i^-, k-1) + \sum_{i=k+1}^{n} \min(\alpha_i^-, k) = \sum_{i=1}^{k} \alpha_i^+$$

Construction of Threshold Digraphs

- Let $\alpha^- = (\alpha_1^-, \ldots, \alpha_n^-)$ be a sequence of integers from $\{0, \ldots, n-1\}$.

$\alpha^- = (3, 4, 2, 1, 1)$

Construction of Threshold Digraphs

- Let $\alpha^- = (\alpha_1^-, \ldots, \alpha_n^-)$ be a sequence of integers from $\{0, \ldots, n-1\}$.
- Form a matrix by placing α_i^- ones in column i so that they are upper justified, skipping the diagonal.

$\alpha^- = (3, 4, 2, 1, 1)$

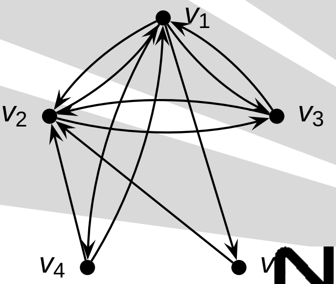
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Construction of Threshold Digraphs

- Let $\alpha^- = (\alpha_1^-, \ldots, \alpha_n^-)$ be a sequence of integers from $\{0, \ldots, n-1\}$.
- Form a matrix by placing α_i^- ones in column i so that they are upper justified, skipping the diagonal.
- This matrix is the adjacency matrix of a threshold digraph.

$$\alpha^- = (3, 4, 2, 1, 1)$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



Applications

Using our Theorem we can obtain a short constructive proof of the Fulkerson-Chen Inequalities.

Applications

Using our Theorem we can obtain a short constructive proof of the Fulkerson-Chen Inequalities.

Note that there are other construction algorithms for digraphs. Most notably the results of Kleitman and Wang in 1973.

Applications

Using our Theorem we can obtain a short constructive proof of the Fulkerson-Chen Inequalities.
Note that there are other construction algorithms for digraphs. Most notably the results of Kleitman and Wang in 1973.

Theorem

Let $\alpha = \left((\alpha_1^+, \alpha_1^-), \ldots, (\alpha_n^+, \alpha_n^-)\right)$ be a degree sequence in positive lexicographic order. There is a digraph G which realizes α if and only if $\sum \alpha_i^+ = \sum \alpha_i^-$ and for every k with $1 \leq k < n$

$$\sum_{i=1}^{k} \min(\alpha_i^-, k-1) + \sum_{i=k+1}^{n} \min(\alpha_i^-, k) \geq \sum_{i=1}^{k} \alpha_i^+.$$