Clique Number and Chromatic Number of Graphs defined by Convex Geometries

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Erdős – Szekeres conjecture

Conjecture

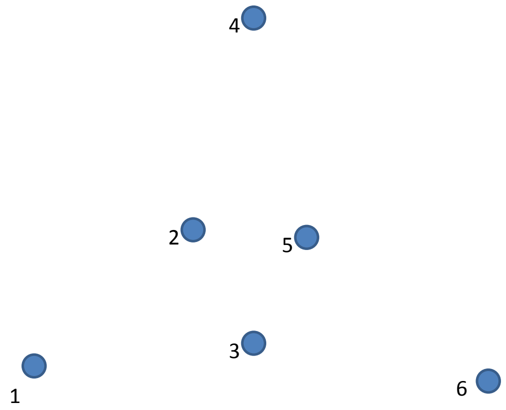
(Erdős and Szekeres, 1935): If X is a set of points in $\mathbb{R}^2$, with no three on a line, and $|X| \geq 2^{n-2} + 1$, then X contains the vertex set of a convex n-gon.

Known:

- If $|X| \geq \binom{2n-5}{n-3} + 1$, then X contains the vertex set of a convex n-gon. (Toth, Valtr 2004)
- If $|X| \geq 17$, then X contains the vertex set of a convex 6-gon. (Szekeres, Peters 2005)
- For all n, there exists a point set X with $|X| = 2^{n-2}$ and with no vertex set of a convex n-gon. (Erdős, Szekeres 1961).

In order to prove that $|X| > 17$ implies that X contains the vertex set of a convex 6-gon, Szekeres and Peters created an integer program with $\binom{17}{3} = 680$ binary variables for which infeasibility implied that no set of 17 points not containing the vertex set of a convex 6-gon exists. The proof of infeasibility used a clever branching strategy.

The analogous integer program for showing that every 33-point set contains the vertex set of a convex 7-gon would have $\binom{33}{3} = 5456$ variables.


4
2
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6

Closed sets

Let X be a finite set of points in $\mathbb{R}^2$.

A subset A of the point set X is called *closed* if $X \cap conv(A) = A$.

If $x \in X$ then a maximal closed subset of $X \backslash \{x\}$ is called a *copoint* attached to x.

The *copoint graph* $\mathcal{G}(X)$ has as its vertices the copoints of X, with copoint A attached to point a adjacent to copoint B attached to b iff $a \in B$ and $b \in A$.

(5,46) (4,12356) (2,14)

(5,1236) (2,1356)

(5,1234) (2,3456)

(6,12345) (1,23456)

(3,1245) (3,2456)

(3,16)

Induced 9-Antihole

If X is a set of points in $\mathbb{R}^2$, with no three on a line, then a subset A of X is the vertex set of a convex n-gon if and only if there is a clique of size n in the copoint graph of X, consisting of copoints attached to the vertices of A.

Conjecture

(Erdős and Szekeres, 1935): If X is a set of points in $\mathbb{R}^2$, with no three on a line, and $|X| \geq 2^{n-2} + 1$, then the clique number of the copoint graph of X is at least n.

A related coloring theorem

Theorem

(Morris, 2006): If X is a set of points in $\mathbb{R}^2$, with no three on a line, and $|X| \geq 2^{n-2} + 1$, then the chromatic number of the copoint graph of X is at least n.

Idea of Proof

Given a proper coloring of the copoint graph of X, each point of X can be labelled by an *odd* subset of the set of colors. No two elements of X get the same label.

empirical evidence

If X is a set of at most 8 points in $\mathbb{R}^2$, with no three on a line, then the clique number and the chromatic number of the copoint graph differ by at most 1.

This can be proved by going through the list of order types of planar point sets compiled by Aichholzer et.al.

Can we find a sequence of point sets for which the chromatic number of the copoint graph is much larger than the clique number?

We look for such examples in an abstract setting.

Alignments

Let X be a finite set. A collection $\mathscr{L}$ of subsets of X is an *alignment* on X if

- $\emptyset \in \mathscr{L}$ and $X \in \mathscr{L}$
- If $A, B \in \mathscr{L}$, then $A \cap B \in \mathscr{L}$.

Following Edelman and Jamison, we will also use $\mathscr{L}$ to denote a closure operator: For $A \subseteq X$, $\mathscr{L}(A) = \cap\{C \in \mathscr{L} : A \subseteq C\}$

Convex Geometries

Let $\mathscr{L}$ be an alignment on X. The following are equivalent:

- For all $C \in \mathscr{L}$, there exists $x \in X \backslash C$ so that $C \cup \{x\} \in \mathscr{L}$.
- If $C \in \mathscr{L}$, $p \neq q \in X \backslash C$, $p \in \mathscr{L}(C \cup \{q\})$, then $q \notin \mathscr{L}(C \cup \{p\})$.

If $\mathscr{L}$ satisfies these conditions, it is called a *convex geometry* on X.

Example: Let $\mathscr{L} = \{\emptyset, X\}$. Then $\mathscr{L}$ is an alignment, but it is not a convex geometry when $|X| \geq 2$.

Convex Geometries from point sets in $\mathbb{R}^d$

If X is a finite set of points in $\mathbb{R}^d$, then

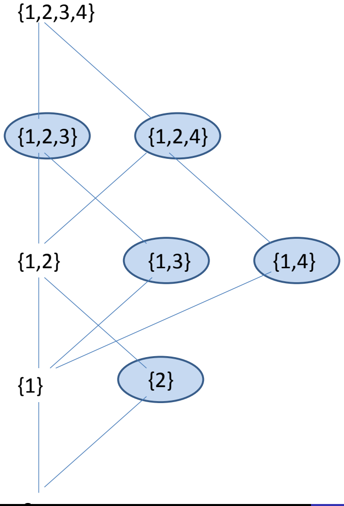
$$\mathscr{L} := \{C \subseteq X : C = X \cap conv(C)\}$$

is a convex geometry on X. We call $\mathscr{L}$ the convex geometry *realized* by X.

Copoints

An element C of a convex geometry $\mathscr{L}$ is called a *copoint* of $\mathscr{L}$ if there exists exactly one element $x \in X \backslash C$ so that $C \cup \{x\}$ is in $\mathscr{L}$.

In this case we say that C is *attached* to x and write $x = \alpha(C)$.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Independent Sets

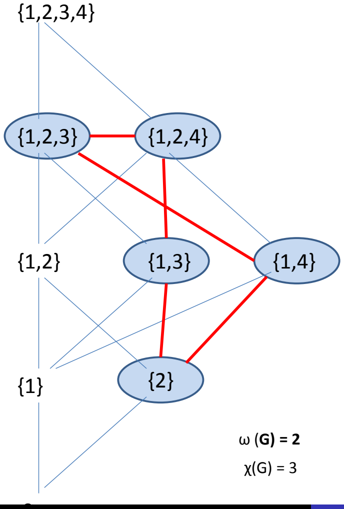
Let $\mathscr{L}$ be a convex geometry on X and let $A \subseteq X$. A is called *independent* if $a \notin \mathscr{L}(A \backslash \{a\})$ for all $a \in A$.

If $\mathscr{L}$ is the convex geometry realized by a set of points X in $\mathbb{R}^2$, not all on a line, then a subset A of X is independent iff it is the vertex set of a convex polygon.

Copoint Graph

Let $\mathscr{L}$ be a convex geometry on X. We define a graph $\mathcal{G}(\mathscr{L}) = (V, E)$ of where V is the set of copoints of $\mathscr{L}$ and copoints C and D are adjacent if $\alpha(C) \in D$ and $\alpha(D) \in C$.

A subset K of V is a clique in $\mathcal{G}(\mathscr{L})$ if $\{\alpha(C) : C \in K\}$ is an independent set in $\mathscr{L}$. Conversely, if $A \subseteq X$ is independent in $\mathscr{L}$, one can find a collection K of copoints so that $A = \{\alpha(C) : C \in K\}$.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ω (G) = 2
χ(G) = 3

Not every graph is a copoint graph

The 6-cycle is not the copoint graph of any convex geometry.

We do not know if every graph is an induced subgraph of the copoint graph of a convex geometry.

Definitions from Graph Theory

Let $G = (V, E)$ be a graph. A *proper coloring* of G is a function f from V to some set R, so that $f(x) \neq f(y)$ whenever $(x, y) \in E$. The *chromatic number* of G is the size of the smallest set R for which there exists a proper coloring of G from V to R.

We denote by $\omega(G)$ the size of the largest clique of G, and by $\chi(G)$ the chromatic number of G.

It is true for every graph G that $\omega(G) \leq \chi(G)$.

A graph is called *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G.

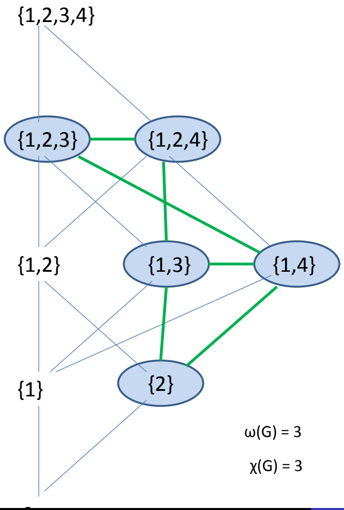
Dilworth's Theorem

Let $\mathcal{P} = (P, \leq)$ be a finite partially ordered set. Dilworth's theorem states that the maximum size of an antichain in $\mathcal{P}$ is equal to the minimum number of chains needed to cover P.

Define the *incomparability graph* $G = (P, E)$ to have an edge between two elements of P when the two elements are incomparable.

Dilworth's theorem has the equivalent statement: $\omega(G) = \chi(G)$.

In fact, incomparability graphs of finite partially ordered sets are perfect.



$\omega(G) = 3$

$\chi(G) = 3$

Strong Perfect Graph Theorem

Theorem

(Chudnovsky, Robertson, Seymour, Thomas, 2002) Let G be a graph. If $\omega(G) < \chi(G)$ then G contains a cycle of length n or the complement of a cycle of length n, for some odd $n \geq 5$, as an induced subgraph.

Sequence of Examples

Let $X = \{1, 2, \ldots, n\}$ Let $\mathscr{L}$ consist of all sets of the following two types:

- $\{1, 2, \ldots, i\}$ for $i = 0, 1, \ldots, n$
- $\{1, 2, \ldots, i\} \cup \{j\}$ for $0 \leq i < j \leq n$

$\mathscr{L}$ is a convex geometry. Every set of the form $\{1, 2, \ldots, i\} \cup \{j\}$ for $i + 1 < j$ is a copoint attached to $i + 1$. The only other copoint is $\{1, 2, \ldots, n - 1\}$, attached to n.

Clique number is constant for examples

Suppose that C_1, C_2, C_3 are three copoints attached to distinct elements of X, e.g. $\alpha(C_1) < \alpha(C_2) < \alpha(C_3)$. There is at most one element of C_1 larger than $\alpha(C_1)$. Therefore C_1 cannot contain both of $\alpha(C_2)$ and $\alpha(C_3)$. Thus C_1 cannot be adjacent to both C_2 and C_3 in $\mathcal{G}(\mathscr{L})$.

Corollary

$\omega(\mathcal{G}(\mathscr{L})) = 2$ *for all* $n \geq 2$.

Labelling of elements by color sets

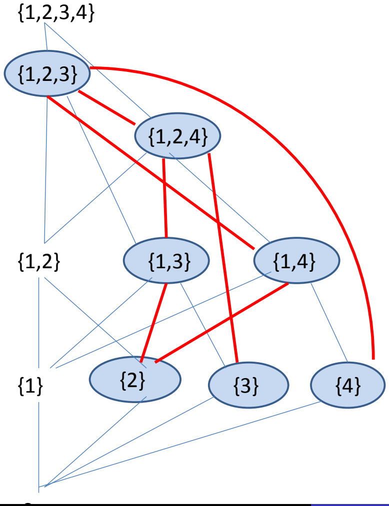
Suppose that there is a proper coloring of $\mathcal{G}(\mathscr{L})$ to a set of colors R. For each $i = 1, 2, \ldots, n$, let A_i be the subset of R to which copoints attached to element i have been assigned.

Lemma

If $1 \leq i < j \leq n$ there is a copoint attached to i that is adjacent in $\mathcal{G}(\mathscr{L})$ to every copoint attached to j.

Corollary

The sets A_i, $i = 1, 2, \ldots, n$, are distinct.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Chromatic number grows without bound

Corollary

$\chi(\mathcal{G}(\mathscr{L})) \geq \lceil log_2(n+1) \rceil$

This sequence of examples is closely related to a set of graphs called *shift graphs*, which have been attributed to "folklore."

The technique of labelling the elements of X by the sets of colors of copoints attached to the elements appears to be very useful for getting lower bounds for the chromatic number.

For any k one can get a similar sequence of examples so that the clique number is fixed at k and the chromatic number grows without bound. For these examples, the number of elements of an X in the sequence is an exponential function of the chromatic number. This we can prove using a result found in the survey paper "A survey of binary covering arrays," by J. Lawrence, M. Forbes, R. Kacker, R. Kuhn, and Y. Lei.

Lower bound for the chromatic number of convex geometries

Theorem

Suppose that $\mathscr{L}$ contains every two-element subset of the set X. If $|X|$ is larger than the number of maximal intersecting families of subsets of an n-element set, then the chromatic number of $\mathcal{G}(\mathscr{L})$ is more than n.

Idea of Proof

Suppose that we have a proper n-coloring of the graph $\mathcal{G}(\mathscr{L})$. Then we can label each element x of X and each element each element $y \neq x$ of X, we can define the set S_{yx} to be the set of colors used on copoints containing y attached to x. The collection of S_{yx} for $y \neq x$ is an intersecting family, and no two such collections can be the same for distinct elements of X.

The number of maximal intersecting families of subsets of an n-element set is $2^{O\left(\binom{n}{\lfloor n/2 \rfloor}\right)}$

References

- Edelman, P. and Jamison, R. “The Theory of Convex Geometries” *Geometriae Dedicata* **19** (1985) pp. 247 - 270
- Morris, W. “Coloring Copoints of Planar Point sets” *Discrete Applied Mathematics* **154**(2006) pp. 1742 – 1752.