PBIBD and its applications in Cryptology

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In this talk ...

We will first describe the combinatorial framework of PBIBD

And then proceed to show its applications in Cryptology

1. **Key Predistribution** in Wireless Sensor Networks
2. **Traitor Tracing** in schemes with restricted access
3. **Secret Sharing** schemes using Visual Cryptography

Partially Balanced Incomplete Block Design (PBIBD)

Combinatorial Designs

A set system or *design* is a pair $(X, \mathcal{A})$, where

- X is the main set of elements
- $\mathcal{A}$ is a set of subsets of X, called *blocks*

Balanced Incomplete Block Design

$BIBD(v, b, r, k; \lambda)$ is a design which satisfy

- $|X| = v$ and $|\mathcal{A}| = b$
- Each block in $\mathcal{A}$ contains exactly k elements
- Each element in X occurs in r blocks
- Each pair of elements in X occurs in exactly λ blocks

Example: $BIBD(7, 7, 3, 3; 1)$ on set $X = \{0, 1, 2, 3, 4, 5, 6\}$
$\mathcal{A} = \{(1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 0), (5, 6, 1), (6, 0, 2), (0, 1, 3)\}$

PBIBD: Partially Balanced Incomplete Block Design

$PB\,[k; \lambda_1, \lambda_2, \ldots, \lambda_m; v]$ is a design such that

- There are b blocks, each of size k, on a v-set X
- It is an association scheme with m associate classes
- Each element of X has exactly n_i number of i-th associates
- Two i-th associate elements occur together in λ_i blocks

	Associates		
	1-st	2-nd	3-rd
1	2, 3	4	5, 6
2	1, 3	5	4, 6
3	1, 2	6	4, 5
4	5, 6	1	2, 3
5	4, 6	2	1, 3
6	4, 5	3	1, 2

Example: $PB[3; 2, 2, 1; 6]$

$X = \{1, 2, 3, 4, 5, 6\}$
$v = 6,\ b = 8,\ r = 4,\ k = 3$

$$\mathcal{A} = \{(1,2,4), (1,3,4), (1,2,5), (1,3,6), (2,3,5), (2,3,6), (4,5,6), (4,5,6)\}$$

PBIBD: Another example

2-associate class PBIBD

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	7	*	10
4	7	9	10	*

1-st associates : Same row or column
2-nd associates: Rest of the elements

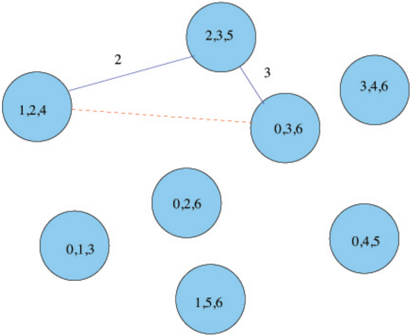
1-st associate of 6 : 1, 5, 7, 3, 8, 10
2-nd associate of 6: 2, 4, 9

Block 1: (2, 3, 4, 5, 6, 7)
Block 2: (1, 3, 4, 5, 8, 9)
Block 3: (1, 2, 4, 6, 8, 10)
Block 4: (1, 2, 3, 7, 9, 10)
Block 5: (1, 2, 6, 7, 8, 9)
Block 6: (1, 3, 5, 7, 8, 10)
Block 7: (1, 4, 5, 6, 9, 10)
Block 8: (2, 3, 5, 6, 9, 10)
Block 9: (2, 4, 5, 7, 8, 10)
Block 10: (3, 4, 6, 7, 8, 9)

Application of PBIBD in Key Predistribution

Key Predistribution

- Security of the WSN depends on efficient key distribution
- PKC and ECC are too computation intensive for WSNs
- Thus we need distribution of keys in nodes prior to deployment



Problem: Distribute node keys from key-pool $\{0, 1, 2, 3, 4, 5, 6\}$.

Metrics to evaluate Key Predistribution schemes

General metrics:

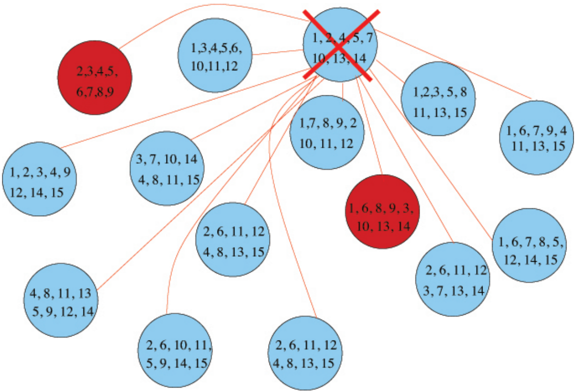
- Scalability: Allow post-deployment increase in network size
- Efficiency: Time taken for communication between nodes
- Storage: Amount of memory required to store the keys
- Computation: No. of cycles needed for key agreement
- Communication: No. of messages sent for key agreement

Security metrics:

- Key Connectivity: The probability that two nodes share one/more keys should be high
- Resiliency: Even if a number of nodes are compromised and the keys contained are revealed, the whole network should not fail, i.e., only a part of the network should get affected

Resiliency - an example

$V(s)$ = Fraction of nodes disconnected for s nodes compromised
$E(s)$ = Fraction of links broken for s nodes compromised



$V(2) = 1/13 = 0.0769$ and $E(2) = (14 + 13 + 12)/105 = 0.371$

Mapping PBIBD to Key Predistribution

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	7	*	10
4	7	9	10	*

2-associate class PBIBD

1: (2, 3, 4, 5, 6, 7)
2: (1, 3, 4, 5, 8, 9)
3: (1, 2, 4, 6, 8, 10)
4: (1, 2, 3, 7, 9, 10)
5: (1, 2, 6, 7, 8, 9)
6: (1, 3, 5, 7, 8, 10)
7: (1, 4, 5, 6, 9, 10)
8: (2, 3, 5, 6, 9, 10)
9: (2, 4, 5, 7, 8, 10)
10: (3, 4, 6, 7, 8, 9)

In this situation, we have $n = 5$, and

- Number of sensor nodes $= n(n-1)/2 = 10$
- Number of keys in key-pool $= n(n-1)/2 = 10$
- Number of keys in each node $= 2(n-2) = 6$
- Number of keys common to any two nodes $= 4$ or $(n-2) = 3$

Advantages of the Design

1. Number of keys per node is $2(n-2)$, i.e., just $O(\sqrt{N})$, when the size of the network is $N = n(n-1)/2$.

2. Any two nodes can communicate directly as they have at least one key shared among them.

3. Resiliency is increased in general, as follows.
   3.1 When two nodes in a row (or column) are compromised, then exactly one node will be disconnected ($n > 5$).
   3.2 Any two nodes compromised in different rows (or columns) will not disconnect any other node.
   3.3 If more than $\lceil n/2 \rceil + 1$ nodes are compromised in total, then at least one node will be disconnected.
   3.4 Maximum number of nodes disconnected when s nodes are compromised is $s(s-1)/2$ (when they are in a row/column).

Experimental Results

n	Network size N	Number of keys k	Captured nodes s	Affected nodes $V(s)$	Affected links $E(s)$
30	435	56	10	0.0753	0.3500
40	780	76	10	0.0351	0.2510
50	1225	96	10	0.0156	0.1800
60	1770	116	10	0.0085	0.1314
70	2415	136	10	0.0058	0.0724

The values of $V(s)$ and $E(s)$ in the table are experimental data.

Scope:

- Is it possible to reduce the number of keys, but still improve the resiliency of the network?
- How can we repeatedly apply the PBIBD schemes and increase the scalability of the network?

Application of PBIBD in Traitor Tracing

Traitor Tracing

Situation:

- Supplier distributes products for only authorized users to use.
- Malicious authorized users (traitors) create pirated copies and distribute them to unauthorized users.

Goal of Traitor Tracing:

- Prevent authorized users to produce unauthorized copies.
- Trace the source of piracy if unauthorized copies are created.
- Trace traitors without harming the innocent users.

Traitor Tracing - Setup

Setup: The distributor supplies each user U_i the following:

- A set of k personal keys denoted by $P(U_i)$.
- Enabling block to create session key s using personal keys.
- The plaintext message encrypted using the session key s.

Example: Number of users = 4, and Key pool = $\{000, 001, 010, 011, 100, 101\}$.

$P(U_1) = \{000, 010, 100\}$ $\quad$ $P(U_2) = \{000, 011, 101\}$
$P(U_3) = \{001, 011, 100\}$ $\quad$ $P(U_4) = \{001, 010, 101\}$

Session key = 110. (obtained by binary addition of the keys modulo 2)
No other combination of keys can generate the same session key upon binary addition.

$\{000, 001, 010\} \to 011$, $\{000, 001, 011\} \to 010$, $\{000, 001, 100\} \to 101$,
$\{000, 001, 101\} \to 100$, $\{000, 010, 011\} \to 001$, $\{000, 010, 101\} \to 111$,
$\{000, 011, 100\} \to 111$, $\{001, 010, 100\} \to 111$, $\{000, 100, 101\} \to 001$,
$\{001, 010, 011\} \to 000$, $\{001, 011, 101\} \to 111$, $\{001, 100, 101\} \to 000$,
$\{010, 011, 100\} \to 111$, $\{010, 011, 101\} \to 100$, $\{010, 100, 101\} \to 011$,
$\{011, 100, 101\} \to 010$.

Traitor Tracing - Action

Piracy: Some users pool in their keys to make another valid key.

Users U_1, U_2, $\cdots$, U_c can collude and create a *pirate decoder* F.

$$F \subseteq \bigcup_{i=1}^{c} P(U_i) \text{ and } |F| = k.$$

Tracing:

- If less than a certain number of authorized users collude, the distributor can trace them using the key distribution scheme.
- If more than this number of traitors collude, the distributor can not trace them without the risk of harming innocent users.

Problem: Design such a key distribution scheme for $P(U_i)$.

c-Traceability Scheme

Suppose there are b users U_i, each having a share of k personal keys $P(U_i)$. Let the size of the whole key pool be v.

c-TS(v, b, k) is a c-traceability scheme if *at least* one traitor can be identified when a coalition of c or less traitors collude.

c-FRTS(v, b, k) is a *fully resilient c-traceability scheme* if *all* the traitors can be identified when a coalition of c or less traitors collude.

Problem: Design c-TS(v, b, k) or c-FRTS(v, b, k) using PBIBD, such that is supports large number of users b, small number of personal keys k, and large margin c for tracing traitors.

Example: 2-Traceability

There are 25 users, and each is assigned 6 keys.
The pirated set of keys is $F = \{0, 1, 2, 3, 6, 8\}$.

$$
\begin{array}{llll}
P(B_1) = & \{\underline{0}, 1, \underline{6}, 18, 22, 29\}, & P(B_2) = & \{0, \underline{2}, 3, \underline{8}, 20, 24\}, \\
P(B_3) = & \{\underline{1, 3}, 4, 9, 21, 25\}, & P(B_4) = & \{2, 4, 5, 10, 22, 26\}, \\
P(B_5) = & \{3, 5, 6, 11, 23, 27\}, & P(B_6) = & \{4, 6, 7, 12, 24, 28\}, \\
P(B_7) = & \{5, 7, 8, 13, 25, 29\}, & P(B_8) = & \{0, 7, 9, 10, 15, 27\}, \\
P(B_9) = & \{1, 8, 10, 11, 16, 28\}, & P(B_{10}) = & \{2, 9, 11, 12, 17, 29\}, \\
P(B_{11}) = & \{0, 4, 11, 13, 14, 19\}, & P(B_{12}) = & \{1, 5, 12, 14, 15, 20\}, \\
P(B_{13}) = & \{2, 6, 13, 15, 16, 21\}, & P(B_{14}) = & \{3, 7, 14, 16, 17, 22\}, \\
P(B_{15}) = & \{4, 8, 15, 17, 18, 23\}, & P(B_{16}) = & \{5, 9, 16, 18, 19, 24\}, \\
P(B_{17}) = & \{6, 10, 17, 19, 20, 25\}, & P(B_{18}) = & \{7, 11, 18, 20, 21, 26\}, \\
P(B_{19}) = & \{8, 12, 19, 21, 22, 27\}, & P(B_{20}) = & \{9, 13, 20, 22, 23, 28\}, \\
P(B_{21}) = & \{10, 14, 21, 23, 24, 29\}, & P(B_{22}) = & \{0, 12, 16, 23, 25, 26\}, \\
P(B_{23}) = & \{1, 13, 17, 24, 26, 27\}, & P(B_{24}) = & \{2, 14, 18, 25, 27, 28\}, \\
P(B_{25}) = & \{3, 15, 19, 26, 28, 29\}. & &
\end{array}
$$

The 2 traitors B_1 and B_2 are uniquely traced.
For 3 traitors: Confusion between $\{B_1, B_2, B_3\}$ and $\{B_1, B_2, B_{13}\}$

Mapping PBIBD to Traitor Tracing

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	7	*	10
4	7	9	10	*

2-associate class PBIBD

1: (2, 3, 4, 5, 6, 7)	2: (1, 3, 4, 5, 8, 9)
3: (1, 2, 4, 6, 8, 10)	4: (1, 2, 3, 7, 9, 10)
5: (1, 2, 6, 7, 8, 9)	6: (1, 3, 5, 7, 8, 10)
7: (1, 4, 5, 6, 9, 10)	8: (2, 3, 5, 6, 9, 10)
9: (2, 4, 5, 7, 8, 10)	10: (3, 4, 6, 7, 8, 9)

In this situation, we have $n = 5$, and

- Number of total users: $b = n(n-1)/2 = 10$
- Number of keys for each user: $k = 2(n-2) = 6$
- Number of keys in key-pool: $v = n(n-1)(n-2)/2 = 30$

Identifiable collusion limit in this scheme is $c = \sqrt{2(n-2)} \approx 2$.

Our Result

A $\sqrt{2(n-2)} - FRTS(n(n-1)(n-2)/2, n(n-1)/2, 2(n-2))$ can be constructed from a $[2; 0, 1; n(n-1)/2]$-PBIBD, when $n \geq 5$.

Previous example was for a $2 - FRTS(30, 10, 6)$ scheme ($n = 5$).

Merit of the scheme:

- For a system with N users, each user having a set of $O(\sqrt{N})$ keys, a collusion of at most $O(\sqrt[4]{N})$ traitors can be traced.
- That is, for a set of 10,000 users, each user having a set of 100 keys, a collusion of at most 10 traitors can be traced.

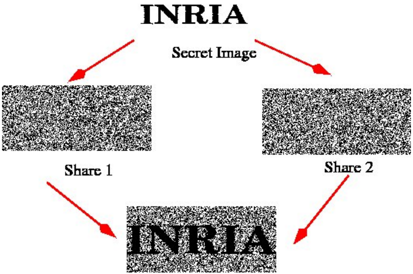
Scope: Improve bound of c compared to N (better than $O(\sqrt[4]{N})$).

Application of PBIBD in Secret Sharing

Secret Sharing in Visual Cryptography

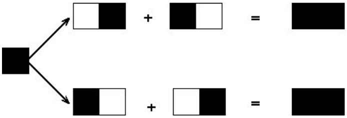
Visual Cryptography: Naor and Shamir, 1994

- Secret sharing scheme with *n* participants, 1 secret image
- Secret image to be split into *n* shadow images called shares
- Certain qualified subsets of participants can recover the secret
- Other forbidden sets of participants have no information

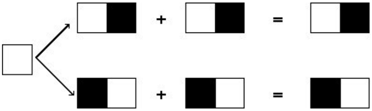


Example: (2, 2) Visual Cryptography Scheme

Number of shares is $n = 2$, and 2 shares can recover the secret.



Shares for Black pixel



Shares for White pixel

Construction of shares

$$S^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S^0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Problem Statement

Construct a (m, n) Visual Cryptography Scheme (VCS) such that

- There are n participants and 1 secret image
- Secret image to be split into n shadow images called shares
- Any m-subset of participants can recover the secret
- No t-subset of participants can recover the image if $t < m$

In particular, we will construct a $(2, n)$-VCS in this talk.

Metric: Relative Contrast

If $(2, n)$-VCS has basis matrices S^0, S^1 and pixel expansion m, then relative contrast *for participants in subset X is given by $\alpha_X(m) = \frac{1}{m}(w(S^1_X) - w(S^0_X))$.*

Mapping PBIBD to VCS

Suppose there exists an $(v, b, r, k, \lambda_1, \lambda_2)$-PBIBD.
It maps to a $(2, n)$-VCS with $n = v$, and pixel expansion $m = b$.

Relative contrast in a subset $X = \{\beta, \gamma\}$ of participants:

- If β, γ are 1-st associates, $\alpha_X(m) = \frac{1}{m}(r - \lambda_1)$
- If β, γ are 2-nd associates, $\alpha_X(m) = \frac{1}{m}(r - \lambda_2)$

Mapping:

1. Suppose N is the *incidence matrix* of the PBIBD.
2. Take share $S^1 = N$, which has r number of 1's in each row.
3. Construct share S^0 with all identical rows, with r 1's in each.
4. These shares S^0, S^1 will make a $(2, n)$-VCS with $n = v$.

Example: PBIBD to VCS

Let us have a $(v = 6, b = 4, r = 2, k = 3, \lambda_1 = 0, \lambda_2 = 1)$-PBIBD

- $X = \{1, 2, 3, 4, 5, 6\}$ and
- $\mathcal{A} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\}$

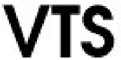
Construction of a $(2, 6)$-VCS

$$S^1 = N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } S^0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

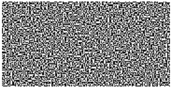
Pixel expansion is clearly $m = 4$, from the rows of the shares.
Relative contrast is either $\frac{1}{2}$ or $\frac{1}{4}$.

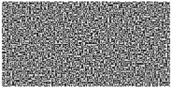
Example: PBIBD to VCS

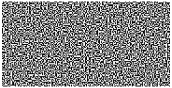
Visual outcome of $(6, 4, 2, 3, 0, 1)$-PBIBD to $(2, 6)$-VCS

Secret image: 

One Share

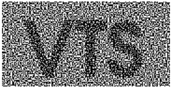
Share 1: 

Share 2: 

Share 6: 

Two Shares

Shares 1 & 6: 

Shares 1 & 2: 

Relative contrast is
$\frac{1}{2}$ for 1 & 6 and $\frac{1}{4}$ for 1 & 2

Thank You