Fourier, Gegenbauer, and Jacobi expansions related to a fundamental solution of the polyharmonic equation

Howard S. Cohl[⋆†]

⋆ Information Technology Laboratory, National Institute of Standards and Technology, Gaithersburg, Maryland, U.S.A.

Applied & Computational Mathematics Division Seminar Series, ITL, NIST

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Outline

Introduction – Special Functions & Fundamental Solutions

- **Main question of my research:**

  What can be learned about the **properties** of **special functions** by studying **separable solutions** of **linear homogeneous partial differential equations** which admit a **fundamental solution.**

- **Categories of PDEs:**
  - Variety/Type (focus: **elliptic – Laplace**)
  - Order **(polynomial generalizations)**
  - Differentiable **Riemannian manifolds**

The gamma function – Essential building block

- **Gamma function**: For Re $z > 0$, $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$$

- **Factorial:** For $n \in \mathbf{N}_0 := \{0, 1, 2, 3, \ldots\} \quad \Gamma(n+1) = n!$
- **Euler reflection formula:** $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$
- **Double factorial**, $\cdot !! : \{-1, 0, 1, \ldots\} \to \mathbf{N} := \{1, 2, 3, \ldots\}$, defined as

$$n!! := \begin{cases} n \cdot (n-2) \cdots 2 & \text{if } n \text{ even} \geq 2, \\ n \cdot (n-2) \cdots 1 & \text{if } n \text{ odd} \geq 1, \\ 1 & \text{if } n = -1, 0. \end{cases}$$

- **Pochhammer symbol (rising factorial),** $(\cdot)_n : \mathbf{C} \to \mathbf{C}$, defined as

$$(z)_0 := 1, \quad (z)_n := (z)(z+1) \cdots (z+n-1),$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (n \in \mathbf{N}_0)$$

The Gauss hypergeometric function

- **Gauss hypergeometric function**,

  ${}_2F_1 : \mathbf{C} \times \mathbf{C} \times (\mathbf{C} \setminus -\mathbf{N}_0) \times \{z \in \mathbf{C} : |z| < 1\} \to \mathbf{C}$, defined as
  $$ {}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

- **Symmetry**
  $$ {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z) $$

- **Unit value** (only the first term survives)
  $$ {}_2F_1(0, b; c; z) = {}_2F_1(a, 0; c; z) = {}_2F_1(a, b; c; 0) = 1 $$

- **Unit argument**: For $\mathrm{Re}(c - a - b) > 0$ and $c \notin -\mathbf{N}_0$
  $$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} $$

- **Hypergeometric polynomial**: For $n \in \mathbf{N}_0$: ${}_2F_1(-n, b; c; z)$ is a **polynomial** in z of degree n.

Important hypergeometric polynomials

- **Jacobi polynomials**, $P_n^{(\alpha,\beta)} : [-1,1] \to \mathbf{R}$

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n(-\alpha-n)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right)$$

- **Gegenbauer polynomials**, $C_n^\mu : [-1,1] \to \mathbf{R}$

$$C_n^\mu(x) := \frac{(2\mu)_n}{(\mu+\frac{1}{2})_n} P_n^{(\mu-1/2,\mu-1/2)}(x)$$

- **Chebyshev polynomials of the first kind**, $T_n : [-1,1] \to \mathbf{R}$

$$T_n(x) = \frac{1}{\epsilon_n} \lim_{\mu\to 0} \frac{n+\mu}{\mu} C_n^\mu(x),$$

where $T_n(\cos\phi) := \cos(n\phi)$, and ϵ_n is the **Neumann factor**:

$$\epsilon_n := 2 - \delta_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, 2, 3, \ldots \end{cases}$$

- **Legendre polynomials**, $P_n : [-1,1] \to [-1,1]$, $P_n(x) := C_n^{1/2}(x)$.

Transformations of the Gauss hypergeometric function

- **Transformations of the Gauss hypergeometric function**
  - **Kummer's** 24 **solutions:** 2 **exponents** at each of the 3 **possible singular points,** each of which appears 4 times due to **Euler's** and **Pfaff's linear transformations** of the **Gauss hypergeometric function**

    Euler $${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1\left(c-a,c-b;c;z\right)$$

    Pfaff 1 $${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

    Pfaff 2 $${}_2F_1(a,b;c;z) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right)$$

  - **Quadratic transformations** of the **Gauss hypergeometric function (Legendre functions)**

Associated Legendre functions

- **Associated Legendre differential equation**

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right] w = 0,$$

- **Ferrers function of the first kind:** $\mathrm{P}_\nu^\mu : (-1,1) \to \mathbf{C}$
(associated Legendre function of the first kind on the cut)

$$\mathrm{P}_\nu^\mu(x) := \frac{1}{\Gamma(1-\mu)}\left[\frac{1+x}{1-x}\right]^{\frac{\mu}{2}} {}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-x}{2}\right)$$

- **Legendre function of the first kind**: $P_\nu^\mu : \mathbf{C}\setminus(-\infty,1] \to \mathbf{C}$

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)}\left[\frac{z+1}{z-1}\right]^{\frac{\mu}{2}} {}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\right)$$

Ferrers function of the second kind

- **Ferrers function of the second kind**, $\mathrm{Q}_\nu^\mu : (-1,1) \to \mathbf{C}$

$$\mathrm{Q}_\nu^\mu(x) := \sqrt{\pi} 2^\mu \cos\left[\frac{\pi}{2}(\nu+\mu)\right] \frac{\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} x(1-x^2)^{-\mu/2}$$

$$\times {}_2F_1\left(\frac{1-\nu-\mu}{2}, \frac{\nu-\mu+2}{2}; \frac{3}{2}; x^2\right)$$

$$-\sqrt{\pi} 2^{\mu-1} \sin\left[\frac{\pi}{2}(\nu+\mu)\right] \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+2}{2}\right)} (1-x^2)^{-\mu/2}$$

$$\times {}_2F_1\left(\frac{-\nu-\mu}{2}, \frac{\nu-\mu+1}{2}; \frac{1}{2}; x^2\right)$$

Associated Legendre function of the 2nd kind

- **Legendre function of the second kind**, $Q_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi} e^{i\pi\mu} \Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) z^{\nu+\mu+1}}$$

$$\times {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

- For instance, **Magnus, Oberhettinger & Soni (1966)** tabulates a list of **36 different ways** to describe **associated Legendre functions of the first and second kind** in terms of **Gauss hypergeometric functions**.

Fundamental solutions for homogeneous PDEs in $\mathbf{R}^d$

- **Linear inhomogeneous partial differential equation (PDE)**

$$L(\mathbf{x})\Phi(\mathbf{x}) = f(\mathbf{x})$$

  given in terms of a **partial differential operator**:

$$L(\mathbf{x}) = g\left(x_1, \ldots, x_d, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$$

- For certain **linear operators** a **solution** to the **PDE** can be obtained

$$\Phi(\mathbf{x}) = L^{-1}(\mathbf{x}, \mathbf{x}')\, f(\mathbf{x}')$$

  through an **integral inverse**, in terms of a **fundamental solution** $\mathcal{G}_L$

$$\Phi(\mathbf{x}) = L^{-1} f(\mathbf{x}') = \int \mathcal{G}_L(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}'$$

$$L(\mathbf{x})\, \mathcal{G}_L(\mathbf{x}, \mathbf{x}') = \delta^d(\mathbf{x} - \mathbf{x}')$$

- **Fundamental solutions** encapsulate the **influence of a partial differential operator** over the entire **space** $\mathbf{R}^d$.

PDEs which admit fundamental solutions and special function solutions via separation of variables

- **Laplace's** equation, **polyharmonic** equation, **Helmholtz** equation, **Hamilton-Jacobi** equation, **Wave** equation, **Heat** equation, **Schrödinger** equation, **Klein-Gordon** equation, **higher order extensions,** and on **certain differentiable manifolds**, are examples of operators which admit **fundamental solutions**
- **Investigation.** What can be **learned** about the **special functions** which **naturally arise** from **fundamental solutions** of **linear partial differential operators** and the **Fourier**, **Gegenbauer**, and **Jacobi analysis** of these **fundamental solutions**.
- **Main interest.** Explore the **properties** of the **classical special functions**, i.e. those which arise from the theory of **separation of variables** from the homogeneous linear **partial differential equations** of **mathematical physics** in real **Euclidean space** $\mathbf{R}^d$

Global analysis on d-dimensional Euclidean space $\mathbf{R}^d$

- Zero **curvature** d-dimensional **Euclidean** space $\mathbf{R}^d$, finite-dimensional vector space with the Euclidean **inner** (dot) product $(\cdot,\cdot):\mathbf{R}^d\times\mathbf{R}^d\to\mathbf{R}$ defined for $\mathbf{x},\mathbf{y}\in\mathbf{R}^d$ such that

$$(\mathbf{x},\mathbf{y}) := x_1y_1 + x_2y_2 + \ldots + x_dy_d$$

- **Euclidean inner product** for $\mathbf{x}\in\mathbf{R}^d$ induces the **Euclidean norm**

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})}$$

- **Geodesic distance** between two points $\mathbf{x},\mathbf{y}\in\mathbf{R}^d$ is given by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$$

- **Laplace-Beltrami operator (Laplacian)** $\Delta : C^p(\mathbf{R}^d)\to C^{p-2}(\mathbf{R}^d)$ for $p\ge 2$ is defined by

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$$

Fundamental solution of Laplace and polyharmonic equation

- **Laplace** equation, $\Phi : \mathbf{R}^d \to \mathbf{R}$, **harmonic**

$$-\Delta\Phi(\mathbf{x}) = 0,$$

- **Polyharmonic** equation, $\Phi : \mathbf{R}^d \to \mathbf{R}$, **polyharmonic**

$$(-\Delta)^k\Phi(\mathbf{x}) = 0,$$

where $k \in \mathbf{N} := \{1, 2, 3, \ldots\}$

- **Fundamental solution** of the **polyharmonic equation**

$$(-\Delta)^k\mathcal{G}_k^d(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

where $\mathcal{G}_k^d : \mathbf{R}^d \times \mathbf{R}^d \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$

Fundamental solution of Laplace equation in Euclidean space

$$-\Delta\mathcal{G}^d(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$$

Theorem. Fundamental solution of Laplace's equation in d-dimensional Euclidean space $\mathbf{R}^d$

Let $d = 1, 2, 3, \ldots$. Define $\mathcal{G}^d : \mathbf{R}^d \times \mathbf{R}^d \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$, such that

$$\mathcal{G}^d(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{\Gamma(d/2)}{2\pi^{d/2}(d-2)} \dfrac{1}{\|\mathbf{x}-\mathbf{x}'\|^{d-2}} & \text{if } d = 1 \text{ or } d \geq 3, \\ -\dfrac{1}{2\pi} \log \|\mathbf{x}-\mathbf{x}'\| & \text{if } d = 2, \end{cases}$$

then $\mathcal{G}^d$ is a fundamental solution for $-\Delta$, where Δ is the Laplacian.

Fundamental solution of the polyharmonic equation in $\mathbf{R}^d$

$$(-\Delta)^k \mathcal{G}_k^d(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$$

Theorem. Fundamental solution of the polyharmonic equation in $\mathbf{R}^d$

Let $d,k \in \mathbf{N}$. Define $\mathcal{G}_k^d : \mathbf{R}^d \times \mathbf{R}^d \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$, such that

$$\mathcal{G}_k^d(\mathbf{x},\mathbf{x}') = \begin{cases} \dfrac{(-1)^{k+d/2+1}\|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)!\,(k-d/2)!\,2^{2k-1}\pi^{d/2}}\left(\log\|\mathbf{x}-\mathbf{x}'\| - \beta_{k-d/2,d}\right) & \\ \qquad\qquad \text{if } d \text{ even}, \;\; k \ge d/2, & \\ \dfrac{\Gamma(d/2-k)\|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)!\;2^{2k}\pi^{d/2}} \qquad \text{otherwise}, & \end{cases}$$

where $\beta_{p,d} \in \mathbf{Q}$ such that $\beta_{p,d} := \frac{1}{2}\left[H_p + H_{d/2+p-1} - H_{d/2-1}\right]$, and $H_j \in \mathbf{Q}$ is the j-th harmonic number $H_0 := 0$, $H_j := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j}$, then $\mathcal{G}_k^d$ is a fundamental solution of the polyharmonic equation.

Theory of separation of variables for $(-\Delta)^k f = 0$

- Symmetry group of **polyharmonic equation** is **conformal group**
- **Symmetries** are **differential operators** which map solutions to solutions
- Conformal **symmetries** include **inversions, reflections, dilatations**
- These *extra* **symmetries** imply the existence of both **simple** and R-**separation of variables**
- R-**separation** yields extra **coordinate systems**
  - $\mathbf{R}^3$: **quadrics – 2nd order surfaces** which are two-dimensional generalization of the **conic sections**

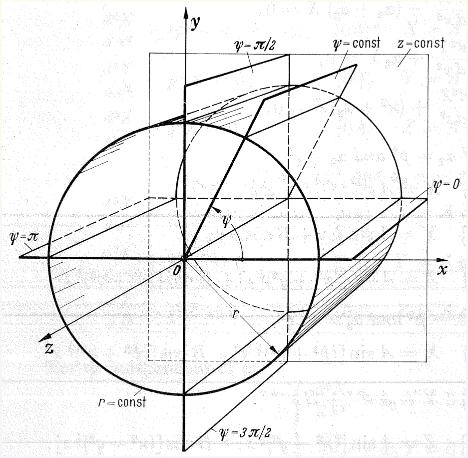
$$P_2(x, y, z) = 0$$

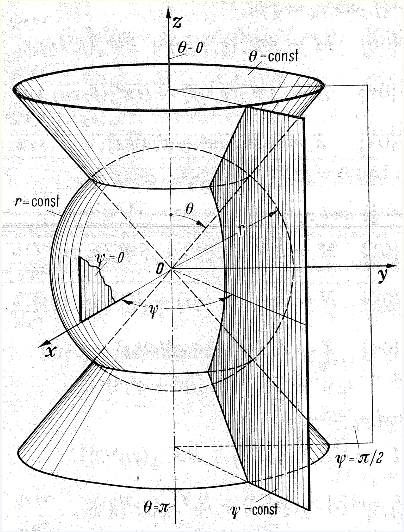
  - $\mathbf{R}^3$: **cyclides – 4th order surfaces**

$$c(x^2 + y^2 + z^2)^2 + P_2(x, y, z) = 0, \quad c \in \mathbf{R}$$

  - and their $(d-1)$-dimensional generalizations
- We look for **rotationally invariant** separable **coordinate systems**
  - all share **trigonometric** $e^{im\phi}$ **(poly-)harmonics**

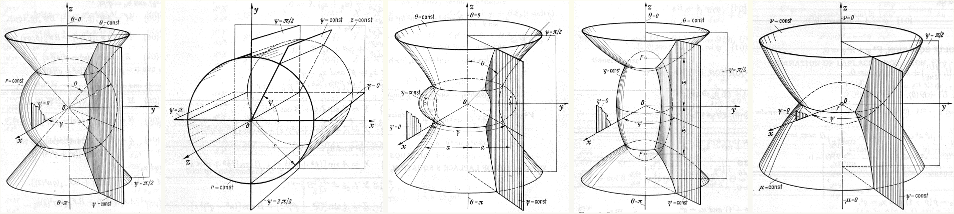
Subgroup-type rotationally-invariant coordinates



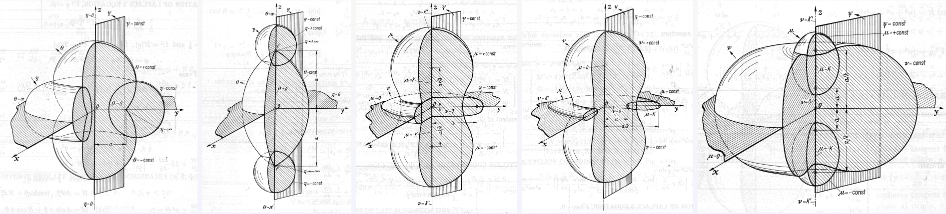


Separable rotationally invariant coordinate systems

$\mathbf{R}^3$: Rotational (**quadric**)



$\mathbf{R}^3$: Rotational (**cyclidic**)

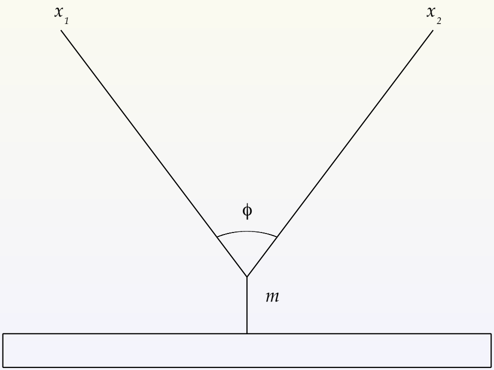


Vilenkin's polyspherical coordinates

- The **method of trees**: a method for constructing **subgroup-type coordinates** on the d-dimensional **hypersphere**
- Example: $\mathbf{R}^2$ - putting coordinates on $\mathbf{S}^1$

$$x_1 = r\cos\phi$$

$$x_2 = r\sin\phi$$



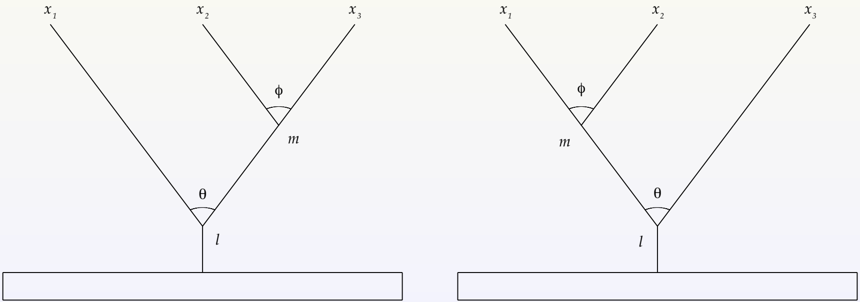
Vilenkin's polyspherical coordinates (cont.)

- Example: $\mathbf{R}^3$ - putting coordinates on $\mathbf{S}^2$

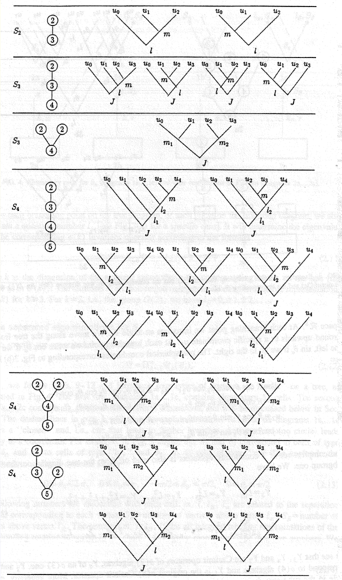
$$x_1 = r\cos\theta \qquad\qquad x_1 = r\cos\theta\cos\phi$$

$$x_2 = r\sin\theta\cos\phi \qquad\qquad x_2 = r\cos\theta\sin\phi$$

$$x_3 = r\sin\theta\sin\phi \qquad\qquad x_3 = r\sin\theta$$



Catalan numbers: 1,2,5,14,42,132,429,1430,4862,16796,58786,208012,. . .



No.	Subgroup chain	Subgroup diagram	Tree diagram
2	$O(2)$		
3	$O(3) \supset O(2)$		
4.1	$O(4) \supset O(3) \supset O(2)$		
4.2	$O(4) \supset O(2) \otimes O(2)$		
5.1	$O(5) \supset O(4) \supset O(3) \supset O(2)$		
5.2	$O(5) \supset O(4) \supset O(2) \otimes O(2)$		
5.3	$O(5) \supset O(3) \otimes O(2) \supset O(2)$		

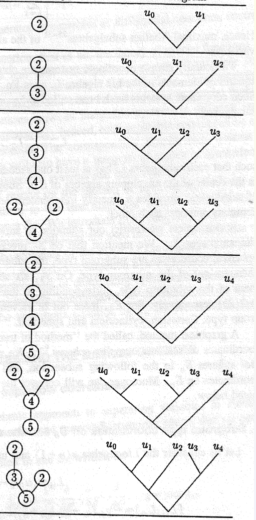


FIG. 1. Subgroup and tree diagrams for S_n.

Wedderburn-Ethernington numbers: 1,1,2,3,6,11,23,46,98,207,451,983,. . .

Fourier $e^{in\phi}$, Gegenbauer $C_n^\mu(\cos\theta)$, and Jacobi $P_n^{(a,b)}(\cos\theta)$ analysis of a fundamental solution for $(-\Delta)^k\Phi(\mathbf{x}) = 0$

- We would like to perform **Fourier** $e^{im\phi}$, **Gegenbauer** $C_l^\mu(\cos\theta)$, and **Jacobi** $P_l^{a,b}(\cos\theta)$ analysis of a **fundamental solution** for the **polyharmonic equation**
- Perform an **eigenfunction expansion** of a fundamental solution of the polyharmonic equation in a **basis** given by **Chebyshev polynomials of the first kind**

$$T_n(\cos\phi) = \cos(n\phi)$$

  and **Gegenbauer polynomials**
- **Gegenbauer polynomials** with argument given in terms of the $\cos\gamma$ are simply **hyperspherical harmonics** on $\widehat{\mathbf{x}}, \widehat{\mathbf{x}}' \in \mathbf{S}^{d-1}$

Separable rotationally invariant coordinate systems

- **Separable curvilinear coordinate systems** are those which transform $(-\Delta)^k\Phi(\mathbf{x}) = 0$ into a set of d-**uncoupled ODEs**, each **separately** in terms of ξ_i such that $i = 1, 2, \ldots, d$, with $(d-1)$-**separation constants.**
- Consider the following **rotationally invariant** coordinate system:

$$
\begin{aligned}
x_1 &= R(\xi_1, \xi_2, \ldots, \xi_{d-1}) \cos\phi \\
x_2 &= R(\xi_1, \xi_2, \ldots, \xi_{d-1}) \sin\phi \\
x_3 &= x_3(\xi_1, \xi_2, \ldots, \xi_{d-1}) \\
&\vdots \\
x_d &= x_d(\xi_1, \xi_2, \ldots, \xi_{d-1})
\end{aligned}
$$

- **Parametrize** points on the $(d-1)$-dimensional **half-hyperplane** given by $\phi = const$ and $R > 0$ using separable **curvilinear coordinates** $(\xi_1, \xi_2, \ldots, \xi_{d-1})$.

Fourier expansions for a fundamental solution of $(-\Delta)^k$

- **Expand**, over the $(d-1)$-**separation constants**, a **fundamental solution** for $(-\Delta)^k$ in terms of separable **poly-harmonics** in each **rotationally invariant coordinate system**.
- Since all **rotationally invariant coordinate systems** share ϕ as an **azimuthal angle,** one can always expand a fundamental solution of the polyharmonic equation as a **azimuthal Fourier cosine series** over $\cos(m(\phi-\phi'))$ with $m \in \mathbf{N}_0$
- In a (κ, d) **rotationally invariant coordinate system,** we can write

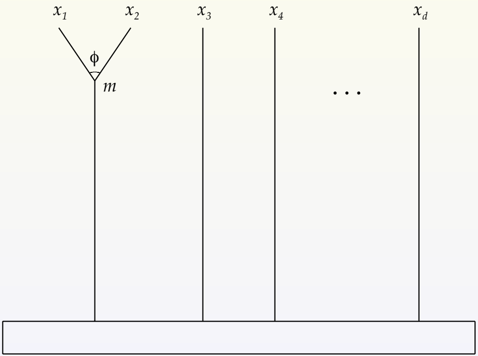
$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2RR'}\sqrt{\chi_\kappa^d - \cos(\phi-\phi')}$$

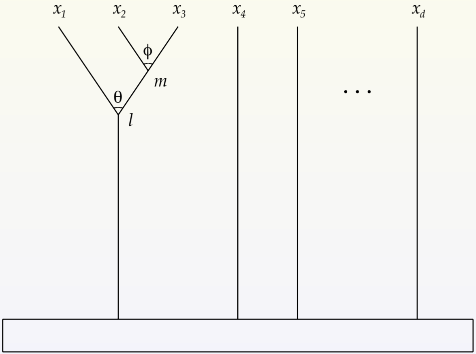
where $\chi_\kappa^d > 1$ is defined by

$$\chi_\kappa^d := \frac{R^2 + {R'}^2 + (x_3 - x_3')^2 + \ldots + (x_d - x_d')^2}{2RR'}$$

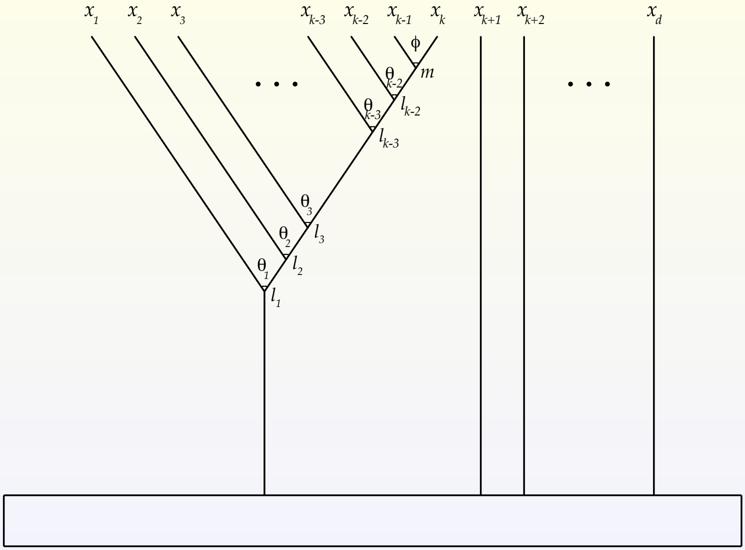
Example: $(2,d)$ and $(3,d)$ Euclidean-hyperspherical

Embedded 2 and 3 dimensional hyperspherical coordinates





Example: (κ, d) Euclidean-hyperspherical coordinates



Addition theorem for hyperspherical harmonics

- **Addition theorem for hyperspherical harmonics**

$$C_l^{d/2-1}(\cos\gamma) = \frac{2\pi^{d/2}(d-2)}{(2l+d-2)\Gamma(d/2)} \sum_K Y_l^K(\widehat{\mathbf{x}})\overline{Y_l^K(\widehat{\mathbf{x}}')},$$

where $\mathbf{x}, \mathbf{x}' \in \mathbf{S}^{d-1}$, and K is a set of **quantum numbers** which label **representations** for l in **subgroup-type coordinates** which **parametrize points** on $\mathbf{S}^{d-1}$.

Fourier and Gegenbauer expansions: powers of the distance

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \sqrt{\frac{2}{\pi}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)}\left(2rr'\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i{}'\right)^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}$$
$$\times\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi)$$

$$\chi := \chi_d^d = \frac{r^2+r'^2-2rr'\sum_{i=1}^{d-2}\cos\theta_i\cos\theta_i{}'\prod_{j=1}^{i-1}\sin\theta_j\sin\theta_j{}'}{2rr'\prod_{i=1}^{d-2}\sin\theta_i\sin\theta_i{}'}$$

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \frac{e^{i\pi(\nu+d-1)/2}\Gamma\left(\frac{d-2}{2}\right)}{2\sqrt{\pi}\Gamma\left(-\frac{\nu}{2}\right)}\frac{\left(r_>^2-r_<^2\right)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}}$$
$$\times\sum_{\lambda=0}^{\infty}(2\lambda+d-2)\,Q_{\lambda+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_\lambda^{d/2-1}(\cos\gamma)$$

Example: Spherical coordinates on $\mathbf{R}^3$

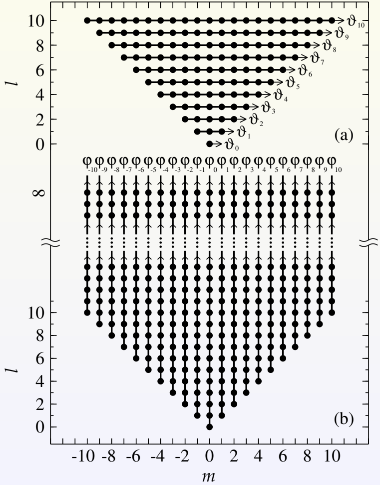
$$\|\mathbf{x}-\mathbf{x}'\|^\nu = \sqrt{\frac{2}{\pi}}\frac{e^{i\pi(\nu+1)/2}}{\Gamma\left(-\nu/2\right)}\left(2rr'\sin\theta\sin\theta'\right)^{\nu/2}\left(\chi^2-1\right)^{(\nu+1)/4}$$
$$\times\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}Q_{m-1/2}^{-(\nu+1)/2}(\chi),$$

$$\chi = \frac{r^2+r'^2-2rr'\cos\theta\cos\theta'}{2rr'\sin\theta\sin\theta'}$$

$$\|\mathbf{x}-\mathbf{x}'\|^\nu = -\frac{e^{i\pi\nu/2}}{2\Gamma(-\nu/2)}\frac{(r_>^2-r_<^2)^{(\nu+2)/2}}{rr'}\sum_{l=0}^{\infty}(2l+1)Q_l^{-(\nu+2)/2}\left(\frac{r^2+r'^2}{2rr'}\right)$$
$$\times\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')e^{im(\phi-\phi')}$$

Sum over configuration space

Alternative double summations in (l,m) space (a) first over m at fixed l to form partial sums ϑ_l (b) first over l at fixed m to form partial sums φ_m



Addition theorems in $\mathbf{R}^3$

- General **addition theorem**

$$Q_{m-1/2}^{-(\nu+1)/2}(\chi) = \frac{i\sqrt{\pi}2^{-(\nu+3)/2}(\sin\theta\sin\theta')^{-\nu/2}}{(\chi^2-1)^{(\nu+1)/4}}\left(\frac{r_>^2-r_<^2}{rr'}\right)^{(\nu+2)/2}$$

$$\times\sum_{l=|m|}^{\infty}(2l+1)\frac{(l-m)!}{(l+m)!}Q_l^{-(\nu+2)/2}\left(\frac{r^2+r'^2}{2rr'}\right)P_l^m(\cos\theta)P_l^m(\cos\theta')$$

- **Addition theorem** for $\nu=-1$

$$Q_{m-1/2}(\chi) = \pi\sqrt{\sin\theta\sin\theta'}\sum_{l=|m|}^{\infty}\frac{(l-m)!}{(l+m)!}\left(\frac{r_<}{r_>}\right)^{l+1/2}P_l^m(\cos\theta)P_l^m(\cos\theta')$$

Generating functions for orthogonal polynomials

- **Chebyshev polynomials of the 1st kind**: $T_n(\cos\theta) = \cos(n\theta)$

$$\frac{1 - x\rho}{1 + \rho^2 - 2\rho x} = \sum_{n=0}^{\infty} T_n(x)\rho^n$$

- **Chebyshev polynomials of the 2nd kind**: $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$

$$\frac{1}{1 + \rho^2 - 2\rho x} = \sum_{n=0}^{\infty} U_n(x)\rho^n$$

- **Gegenbauer polynomials**

$$\frac{1}{(1 + \rho^2 - 2\rho x)^\mu} = \sum_{n=0}^{\infty} C_n^\mu(x)\rho^n$$

- **Jacobi polynomials**

$$\frac{2^{\alpha+\beta}}{R(1-\rho+R)^\alpha(1+\rho+R)^\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\rho^n$$

where $R = \sqrt{1 + \rho^2 - 2\rho x}$. **Should we study these?**

The alg. functions $\sqrt{1+\rho^2-2\rho x}$, $\sqrt{z-x}$ from geometry

- The **distance between two points** $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$ in a **pure hyperspherical coordinate system** is given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{r^2 + r'^2 - 2rr'\cos\gamma}, \tag{1}$$

where $r = \|\mathbf{x}\|$, $r' = \|\mathbf{x}'\|$, and $\cos\gamma = \dfrac{(\mathbf{x},\mathbf{x}')}{rr'}$. If you define $r_{\lessgtr} := {\min \atop \max}\{r, r'\}$, then you can rewrite (1) as

$$\|\mathbf{x}-\mathbf{x}'\| = r_>\sqrt{1 + \left(\frac{r_<}{r_>}\right)^2 - 2\frac{r_<}{r_>}\cos\gamma},$$

or with $\rho := \dfrac{r_<}{r_>}$, and $x := \cos\gamma$ we have

$$\|\mathbf{x}-\mathbf{x}'\| = r_>\sqrt{1+\rho^2 - 2\rho\cos\gamma},$$

where $\rho \in (0,1)$. The other option is:

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2rr'}\sqrt{z-x}, \quad \text{where} \quad z = \frac{1+\rho^2}{2\rho} \in (1,\infty)$$

Generalizations of generating functions

- The **Chebyshev polynomials** can also be defined in terms of **Gegenbauer polynomials**
  - **Chebyshev polynomial of the second kind**
  $$U_n(x) := C_n^1(x)$$
  - **Chebyshev polynomial of the first kind**
  $$T_n(x) := \frac{1}{\epsilon_n}\lim_{\mu\to 0}\frac{n+\mu}{\mu}C_n^\mu(x),$$
- Generalization of **generating function** for **Gegenbauer polynomials**

$$\frac{(z^2-1)^{(\nu-\mu)/2-1/4}}{(z-x)^\nu} = \frac{2^{\mu+1/2}\Gamma(\mu)e^{i\pi(\mu-\nu+1/2)}}{\sqrt{\pi}\,\Gamma(\nu)}\sum_{n=0}^\infty (n+\mu)Q_{n+\mu-1/2}^{\nu-\mu-1/2}(z)C_n^\mu(x)$$

- Generalization of **generating function** for **Chebyshev polynomials**

$$\frac{(z^2-1)^{\nu/2-1/4}}{(z-x)^\nu} = \sqrt{\frac{2}{\pi}}\frac{e^{-i\pi(\nu-1/2)}}{\Gamma(\nu)}\sum_{n=0}^\infty \epsilon_n T_n(x)Q_{n-1/2}^{\nu-1/2}(z)$$

Super expansion for C_n^μ through various limiting procedures

- The **generalization** of the **generating function for Gegenbauer polynomials** produces the **generalized Heine's identity** in the limit as $\mu \to 0$.
- **Produces** an **analogous generalized identity** for **Chebyshev polynomials of the second kind.**
- **Produces Fourier** and **hyperspherical harmonic** expansions for arbitrary **powers of the distance between two points** between two points in Euclidean space.
- Through **limit-derivative** technique, produces **Fourier** and **Gegenbauer** expansions for **logarithmic fundamental solutions** of the polyharmonic equation
- Produces **expansions** for $(1-x)^\mu$ and (cf. Szmytkowski (2011)) for $(y-x)^\mu$ where $x, y \in (-1,1)$ and $y > x$.

Super expansion for $P_n^{(\alpha,\beta)}(x)$?

- **Super expansion** for **Gegenbauer polynomials** is a consequence of expanding the **generating function for Gegenbauer polynomials** $C_n^\mu(x)$ in terms of the complete set $C_n^\nu(x)$
- This expansions is consequence of **connection formulae** for **orthogonal polynomials** which illustrate how one expands an orthogonal polynomial of **one index** in terms of a finite-sum over the same orthogonal polynomial of a **different index.**
- One must identify the coefficients of this expansion, which for **Gegenbauer** and **Jacobi polynomials** are given in terms of a ${}_3F_2$ hypergeometric function of **unit argument,** which can be expressed in terms of a gamma functions using **Watson's formula** and **Chu's (2011) recent extension.**

Generating functions for Jacobi polynomials

- The **Gegenbauer polynomial** can be defined in terms of the **Jacobi polynomial.**

$$C_n^\mu(x) := \frac{(2\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\mu-1/2,\mu-1/2)}(x)$$

- Relevant **generating functions** for **generalization**

$$\sum_{n=0}^\infty P_n^{(\alpha,\beta)}(x)\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^n = \frac{1}{(1+\rho)^{\alpha+\beta+1}} \times {}_2F_1\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2};\beta+1;\frac{2\rho(1+x)}{(1+\rho)^2}\right)$$

and its **companion**

$$\sum_{n=0}^\infty P_n^{(\alpha,\beta)}(x)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^n = \frac{1}{(1-\rho)^{\alpha+\beta+1}} \times {}_2F_1\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2};\alpha+1;\frac{-2\rho(1-x)}{(1-\rho)^2}\right)$$

Legendre function representations of generating functions

- **Associated Legendre function representation**

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^n = \left(\frac{2}{\rho(1+x)}\right)^{\beta/2} \times \frac{\Gamma(\beta+1)}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}} P_\alpha^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

- **Ferrers function representation**

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^n = \left(\frac{2}{\rho(1-x)}\right)^{\alpha/2} \times \frac{\Gamma(\alpha+1)}{(1+\rho^2-2\rho x)^{(\beta+1)/2}} \mathrm{P}_\beta^{-\alpha}\left(\frac{1-\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

Extensions

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^n = \frac{(\alpha+\beta+1)(1-\rho)}{(1+\rho)^{\alpha+\beta+2}} \times {}_2F_1\left(\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2};\beta+1;\frac{2\rho(1+x)}{(1+\rho)^2}\right)$$

and its **companion**

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^n = \frac{(\alpha+\beta+1)(1+\rho)}{(1-\rho)^{\alpha+\beta+2}} \times {}_2F_1\left(\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2};\alpha+1;\frac{-2\rho(1-x)}{(1-\rho)^2}\right)$$

Legendre function representations of extensions

- **Associated Legendre function representation** of extension

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^n = \left(\frac{2}{\rho(1+x)}\right)^{\beta/2} \times \frac{(\alpha+\beta+1)(1-\rho)\Gamma(\beta+1)}{(1+\rho^2-2\rho x)^{(\alpha+2)/2}} P_{\alpha+1}^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

- **Ferrers function representation** of extension

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^n = \left(\frac{2}{\rho(1-x)}\right)^{\alpha/2} \times \frac{(\alpha+\beta+1)(1+\rho)\Gamma(\alpha+1)}{(1+\rho^2-2\rho x)^{(\beta+2)/2}} \mathrm{P}_{\beta+1}^{-\alpha}\left(\frac{1-\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

Generalization of extension

- **Hypergeometric generalization**

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n+\alpha+\beta+1)_m \frac{(\alpha+\beta+1)_n}{(\beta+1)_n}$$

$$\times {}_2F_1\left(m,-m+1;\alpha+\beta+2n+2;\frac{-\rho}{1-\rho}\right)\rho^n$$

$$= \frac{(\alpha+\beta+1)_m(1-\rho)^m}{(1-\rho)^{\alpha+\beta+m+1}}$$

$$\times {}_2F_1\left(\frac{\alpha+\beta+m+1}{2},\frac{\alpha+\beta+m+2}{2};\beta+1;\frac{2\rho(1+x)}{(1+\rho)^2}\right)$$

Generalizations of extensions + Jacobi generating functions

- **Associated Legendre generalization**

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n+\alpha+\beta+1)(\alpha+\beta+m+1)_{2n}\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\beta+n+1)}$$
$$\times P_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1+\rho}{1-\rho}\right)$$
$$=\frac{\rho^{(\alpha+1)/2}(1-\rho)^m}{(1+\rho^2-2\rho x)^{(\alpha+m+1)/2}}\left(\frac{2}{1+x}\right)^{\beta/2}$$
$$\times P_{\alpha+m}^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

- **And its companions**

Generating function for products of Gegenbauer polynomials

- **Generating function for product of Gegenbauer polynomials**

$$\sum_{k=0}^{\infty}\frac{k!}{(2\alpha)_k}C_k^\alpha(x)C_k^\alpha(y)\rho^k={}_2F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\alpha+\frac{1}{2};\frac{4(1-x^2)(1-y^2)\rho^2}{(1+\rho^2-2xy\rho)^2}\right)$$

$$=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha)\left(\rho\sin\theta\sin\phi\right)^\alpha}Q_{\alpha-1}\left(\frac{1+\rho^2-2\rho\cos\theta\cos\phi}{2\rho\sin\theta\sin\phi}\right)$$

Generating function for Jacobi polynomials

- **Generating function for Jacobi polynomials**

$$\sum_{n=0}^{\infty} P_{2n}^{(\alpha-2n,\beta-2n)}(x)\rho^n = {}_2F_1\left(\frac{-\alpha}{2}, \frac{-\alpha+1}{2}; \frac{1}{2}; \xi\right) {}_2F_1\left(\frac{-\beta}{2}, \frac{-\beta+1}{2}; \frac{1}{2}; \eta\right)$$

$$-\frac{1}{4}\alpha\beta(1-x^2)\rho\, {}_2F_1\left(\frac{-\alpha+1}{2}, \frac{-\alpha+2}{2}; \frac{3}{2}; \xi\right) {}_2F_1\left(\frac{-\beta+1}{2}, \frac{-\beta+2}{2}; \frac{3}{2}; \eta\right)$$

$$= \frac{(1-\xi)^{\alpha/2}(1-\eta)^{\beta/2}}{4}\left[\left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{\alpha} + \left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{-\alpha}\right]\left[\left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{\beta} + \left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{-\beta}\right]$$

$$-\frac{\rho(1-x^2)}{4\sqrt{\xi\eta}}\left[\left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{\alpha} - \left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{-\alpha}\right]\left[\left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{\beta} - \left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{-\beta}\right]$$

where

$$\xi = \frac{(1+x)^2\rho}{4}, \text{ and } \eta = \frac{(1-x)^2\rho}{4}.$$

Generating function for Jacobi polynomials

- **Generating function for Jacobi polynomials**

$$
\begin{aligned}
\sum_{n=0}^{\infty} & P_{2n+1}^{(\alpha-2n,\beta-2n)}(x)\rho^n \\
&= \frac{(1-\xi)^{(\alpha+1)/2}(1-\eta)^{(\beta+1)/2}}{8\sqrt{\xi}}(x-1)\left[\left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{\alpha}+\left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{-\alpha}\right] \\
&\qquad \times\left[\left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{\beta}+\left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{-\beta}\right] \\
&+\frac{(1-\xi)^{(\alpha+1)/2}(1-\eta)^{(\beta+1)/2}}{8\sqrt{\eta}}(x+1)\left[\left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{\alpha}-\left(\tfrac{1+\sqrt{\xi}}{\sqrt{1-\xi}}\right)^{-\alpha}\right] \\
&\qquad \times\left[\left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{\beta}-\left(\tfrac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)^{-\beta}\right]
\end{aligned}
$$

Ongoing Investigations

- **A whole host** of other **Jacobi generating functions** can be expressed in terms of **associated Legendre functions** and **Ferrers functions,** and in terms of the **second kind.**
- **Multi-summation** (power-law and logarithmic) **addition theorems**
- **Analysis of special functions** and **fundamental solutions** on **highly symmetric Riemannian manifolds**