Fourier, Gegenbauer, and Jacobi expansions related to a fundamental solution of the polyharmonic equation

Howard S. Cohl

* Information Technology Laboratory, National Institute of Standards and Technology, Gaithersburg, Maryland, U.S.A.

Applied & Computational Mathematics Division Seminar Series, ITL, NIST

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Outline

1 Introduction

2 Special Functions & Fundamental Solutions

3 Separation of Variables – Coordinate Systems on $\mathbb{R}^d$

4 Eigenfunction Expansions – Addition Theorems

5 Generating Functions, Simplifications and Generalizations

6 Ongoing Investigations
Main question of my research:
What can be learned about the properties of special functions by studying separable solutions of linear homogeneous partial differential equations which admit a fundamental solution.

Categories of PDEs:
- Variety/Type (focus: elliptic – Laplace)
- Order (polynomial generalizations)
- Differentiable Riemannian manifolds
The gamma function – Essential building block

- **Gamma function**: For \( \text{Re} \, z > 0 \), \( \Gamma(z + 1) = z\Gamma(z) \)

\[
\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt
\]

- **Factorial**: For \( n \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} \) \( \Gamma(n + 1) = n! \)

- **Euler reflection formula**: \( \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \)

- **Double factorial, \( \cdot !! \)**: \( \{−1, 0, 1, \ldots\} \rightarrow \mathbb{N} := \{1, 2, 3, \ldots\} \), defined as

\[
n!! := \begin{cases} 
n \cdot (n - 2) \cdots 2 & \text{if } n \text{ even } \geq 2, \\
n \cdot (n - 2) \cdots 1 & \text{if } n \text{ odd } \geq 1, \\
1 & \text{if } n = -1, 0.
\end{cases}
\]

- **Pochhammer symbol (rising factorial)**, \( (\cdot)_n : \mathbb{C} \rightarrow \mathbb{C} \), defined as

\[
(z)_0 := 1, \quad (z)_n := (z)(z + 1) \cdots (z + n - 1),
\]

\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \quad (n \in \mathbb{N}_0)
\]
The Gauss hypergeometric function

- **Gauss hypergeometric function**, 
  
  \[ 2F_1 : \mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus -\mathbb{N}_0) \times \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C}, \text{ defined as} \]
  
  \[ 2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \]

- **Symmetry**
  
  \[ 2F_1(a, b; c; z) = 2F_1(b, a; c; z) \]

- **Unit value** (only the first term survives)
  
  \[ 2F_1(0, b; c; z) = 2F_1(a, 0; c; z) = 2F_1(a, b; c; 0) = 1 \]

- **Unit argument**: For \( \text{Re}(c - a - b) > 0 \) and \( c \notin -\mathbb{N}_0 \)
  
  \[ 2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \]

- **Hypergeometric polynomial**: For \( n \in \mathbb{N}_0 \): \( 2F_1(-n, b; c; z) \) is a polynomial in \( z \) of degree \( n \).
Important hypergeometric polynomials

- **Jacobi polynomials**, $P_n^{(\alpha,\beta)} : [-1, 1] \to \mathbb{R}$

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n(-\alpha - n)n}{n!} \binom{-n, n + \alpha + \beta + 1; \alpha + 1; 1 - x}{2}$$

- **Gegenbauer polynomials**, $C_n^\mu : [-1, 1] \to \mathbb{R}$

$$C_n^\mu(x) := \frac{(2\mu)^n}{(\mu + \frac{1}{2})n} P_n^{(\mu-1/2,\mu-1/2)}(x)$$

- **Chebyshev polynomials of the first kind**, $T_n : [-1, 1] \to \mathbb{R}$

$$T_n(x) = \frac{1}{\epsilon_n} \lim_{\mu \to 0} \frac{n + \mu}{\mu} C_n^\mu(x),$$

where $T_n(\cos \phi) := \cos(n\phi)$, and $\epsilon_n$ is the Neumann factor:

$$\epsilon_n := 2 - \delta_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, 2, 3, \ldots \end{cases}$$

- **Legendre polynomials**, $P_n : [-1, 1] \to [-1, 1]$, $P_n(x) := C_n^{1/2}(x)$. 
Transformations of the Gauss hypergeometric function

- **Transformations of the Gauss hypergeometric function**
  
  - **Kummer’s 24 solutions:** 2 exponents at each of the 3 possible singular points, each of which appears 4 times due to Euler’s and Pfaff’s linear transformations of the Gauss hypergeometric function.

  - **Euler**
    \[
    _2F_1(a, b; c; z) = (1 - z)^{c-a-b} _2F_1 (c - a, c - b; c; z)
    \]

  - **Pfaff 1**
    \[
    _2F_1(a, b; c; z) = (1 - z)^{-a} _2F_1 \left(a, c - b; c; \frac{z}{z - 1}\right)
    \]

  - **Pfaff 2**
    \[
    _2F_1(a, b; c; z) = (1 - z)^{-b} _2F_1 \left(c - a, b; c; \frac{z}{z - 1}\right)
    \]

- **Quadratic transformations** of the Gauss hypergeometric function (Legendre functions)
Associated Legendre functions

- **Associated Legendre differential equation**

\[(1 - z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + \left[ \nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] w = 0,\]

- **Ferrers function of the first kind:** \(P^\mu_\nu : (-1, 1) \to \mathbb{C}\)  
  (associated Legendre function of the first kind on the cut)

\[
P^\mu_\nu(x) := \frac{1}{\Gamma(1 - \mu)} \left[ \frac{1 + x}{1 - x} \right]^{\frac{\mu}{2}} 2F_1 \left( -\nu, \nu + 1; 1 - \mu; \frac{1 - x}{2} \right)
\]

- **Legendre function of the first kind:** \(P^\mu_\nu : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}\)

\[
P^\mu_\nu(z) := \frac{1}{\Gamma(1 - \mu)} \left[ \frac{z + 1}{z - 1} \right]^{\frac{\mu}{2}} 2F_1 \left( -\nu, \nu + 1; 1 - \mu; \frac{1 - z}{2} \right)
\]
Ferrers function of the second kind, $Q^\mu_\nu : (-1, 1) \to \mathbb{C}$

$$Q^\mu_\nu(x) := \sqrt{\pi} 2^\mu \cos \left[ \frac{\pi}{2} (\nu + \mu) \right] \frac{\Gamma \left( \frac{\nu+\mu+2}{2} \right)}{\Gamma \left( \frac{\nu-\mu+1}{2} \right)} x (1 - x^2)^{-\mu/2}$$

$$\times _2 F_1 \left( \frac{1 - \nu - \mu}{2}, \frac{\nu - \mu + 2}{2}; \frac{3}{2}; x^2 \right)$$

$$- \sqrt{\pi} 2^{\mu-1} \sin \left[ \frac{\pi}{2} (\nu + \mu) \right] \frac{\Gamma \left( \frac{\nu+\mu+1}{2} \right)}{\Gamma \left( \frac{\nu-\mu+2}{2} \right)} (1 - x^2)^{-\mu/2}$$

$$\times _2 F_1 \left( \frac{-\nu - \mu}{2}, \frac{\nu - \mu + 1}{2}; \frac{1}{2}; x^2 \right)$$
Associated Legendre function of the 2nd kind

Legendre function of the second kind, \( Q_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \rightarrow \mathbb{C} \)

\[
Q_\nu^\mu(z) := \frac{\sqrt{\pi} e^{i\pi \mu} \Gamma(\nu + \mu + 1)(z^2 - 1)^{\mu/2}}{2^{\nu+1} \Gamma(\nu + \frac{3}{2}) z^{\nu+\mu+1}} \times _2F_1 \left( \frac{\nu + \mu + 2}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{z^2} \right)
\]

For instance, Magnus, Oberhettinger & Soni (1966) tabulates a list of 36 different ways to describe associated Legendre functions of the first and second kind in terms of Gauss hypergeometric functions.
Fundamental solutions for homogeneous PDEs in $\mathbb{R}^d$

- **Linear inhomogeneous partial differential equation (PDE)**
  \[ L(x)\Phi(x) = f(x) \]
  given in terms of a partial differential operator:
  \[ L(x) = g \left( x_1, \ldots, x_d, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right) \]

- For certain linear operators a solution to the PDE can be obtained
  \[ \Phi(x) = L^{-1}(x, x') f(x') \]
  through an integral inverse, in terms of a fundamental solution $G_L$
  \[ \Phi(x) = L^{-1} f(x') = \int G_L(x, x') f(x') \, dx' \]
  \[ L(x) G_L(x, x') = \delta^d(x - x') \]

- **Fundamental solutions** encapsulate the influence of a partial differential operator over the entire space $\mathbb{R}^d$. 
PDEs which admit fundamental solutions and special function solutions via separation of variables

- Laplace’s equation, polyharmonic equation, Helmholtz equation, Hamilton-Jacobi equation, Wave equation, Heat equation, Schrödinger equation, Klein-Gordon equation, higher order extensions, and on certain differentiable manifolds, are examples of operators which admit fundamental solutions.

- Investigation. What can be learned about the special functions which naturally arise from fundamental solutions of linear partial differential operators and the Fourier, Gegenbauer, and Jacobi analysis of these fundamental solutions.

- Main interest. Explore the properties of the classical special functions, i.e. those which arise from the theory of separation of variables from the homogeneous linear partial differential equations of mathematical physics in real Euclidean space $\mathbb{R}^d$. 
Global analysis on $d$-dimensional Euclidean space $\mathbb{R}^d$

- **Zero curvature** $d$-dimensional Euclidean space $\mathbb{R}^d$, finite-dimensional vector space with the Euclidean inner (dot) product $(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined for $x, y \in \mathbb{R}^d$ such that
  \[(x, y) := x_1y_1 + x_2y_2 + \ldots + x_dy_d\]

- **Euclidean inner product** for $x \in \mathbb{R}^d$ induces the Euclidean norm
  \[\|x\| = \sqrt{(x, x)}\]

- **Geodesic distance** between two points $x, y \in \mathbb{R}^d$ is given by
  \[d(x, y) = \|x - y\|\]

- **Laplace-Beltrami operator (Laplacian)** $\Delta : C^p(\mathbb{R}^d) \to C^{p-2}(\mathbb{R}^d)$ for $p \geq 2$ is defined by
  \[\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}\]
Fundamental solution of Laplace and polyharmonic equation

- **Laplace equation**, $\Phi : \mathbb{R}^d \to \mathbb{R}$, harmonic
  
  $$-\Delta \Phi(x) = 0,$$

- **Polyharmonic equation**, $\Phi : \mathbb{R}^d \to \mathbb{R}$, polyharmonic
  
  $$(-\Delta)^k \Phi(x) = 0,$$

  where $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$

- **Fundamental solution of the polyharmonic equation**
  
  $$(-\Delta)^k \mathcal{G}^d_k(x, x') = \delta(x - x')$$

  where $\mathcal{G}^d_k : \mathbb{R}^d \times \mathbb{R}^d \backslash \{(x, x) : x \in \mathbb{R}^d\} \to \mathbb{R}$
Fundamental solution of Laplace equation in Euclidean space

\[-\Delta G^d(x, x') = \delta(x - x')\]

**Theorem. Fundamental solution of Laplace’s equation in \(d\)-dimensional Euclidean space \(\mathbb{R}^d\)**

Let \(d = 1, 2, 3, \ldots\) Define \(G^d : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \to \mathbb{R}\), such that

\[
G^d(x, x') = \begin{cases} 
\frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)} \frac{1}{\|x - x'\|^{d-2}} & \text{if } d = 1 \text{ or } d \geq 3, \\
-\frac{1}{2\pi} \log \|x - x'\| & \text{if } d = 2,
\end{cases}
\]

then \(G^d\) is a fundamental solution for \(-\Delta\), where \(\Delta\) is the Laplacian.
Special Functions & Fundamental Solutions

Fundamental solution of the polyharmonic equation in \( \mathbb{R}^d \)

\[ (-\Delta)^k G_k^d(x, x') = \delta(x - x') \]

**Theorem. Fundamental solution of the polyharmonic equation in \( \mathbb{R}^d \)**

Let \( d, k \in \mathbb{N} \). Define \( G_k^d : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \rightarrow \mathbb{R} \), such that

\[
G_k^d(x, x') = \begin{cases} 
\frac{(-1)^{k+d/2+1} \|x - x'\|^{2k-d}}{(k-1)! (k - d/2)! 2^{2k-1} \pi^{d/2}} \left( \log \|x - x'\| - \beta_{k-d/2,d} \right) & \text{if } d \text{ even, } k \geq d/2, \\
\frac{\Gamma(d/2 - k) \|x - x'\|^{2k-d}}{(k-1)! 2^{2k} \pi^{d/2}} & \text{otherwise,}
\end{cases}
\]

where \( \beta_{p,d} \in \mathbb{Q} \) such that \( \beta_{p,d} := \frac{1}{2} \left[ H_p + H_{d/2 + p - 1} - H_{d/2 - 1} \right] \), and \( H_j \in \mathbb{Q} \) is the \( j \)-th harmonic number \( H_0 := 0 \), \( H_j := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j} \), then \( G_k^d \) is a fundamental solution of the polyharmonic equation.
Theory of separation of variables for \((-\Delta)^k f = 0\)

- Symmetry group of polyharmonic equation is conformal group.
- Symmetries are differential operators which map solutions to solutions.
- Conformal symmetries include inversions, reflections, dilatations.
- These extra symmetries imply the existence of both simple and \(R\)-separation of variables.
- \(R\)-separation yields extra coordinate systems:
  - \(R^3\): quadrics – 2nd order surfaces which are two-dimensional generalization of the conic sections:
    \[P_2(x, y, z) = 0\]
  - \(R^3\): cyclides – 4th order surfaces:
    \[c(x^2 + y^2 + z^2)^2 + P_2(x, y, z) = 0, \quad c \in \mathbb{R}\]
    and their \((d - 1)\)-dimensional generalizations.
- We look for rotationally invariant separable coordinate systems.
  - all share trigonometric \(e^{im\phi}\) (poly-)harmonics.
Subgroup-type rotationally-invariant coordinates
Separable rotationally invariant coordinate systems

\( \mathbb{R}^3 : \text{Rotational (quadric)} \)

\( \mathbb{R}^3 : \text{Rotational (cyclidic)} \)
Vilenkin’s polyspherical coordinates

- **The method of trees**: a method for constructing subgroup-type coordinates on the $d$-dimensional hypersphere
- Example: $\mathbb{R}^2$ - putting coordinates on $S^1$

\[
x_1 = r \cos \phi \\
x_2 = r \sin \phi
\]
Vilenkin’s polyspherical coordinates (cont.)

Example: $\mathbb{R}^3$ - putting coordinates on $S^2$

\[
\begin{align*}
x_1 &= r \cos \theta \\
x_2 &= r \sin \theta \cos \phi \\
x_3 &= r \sin \theta \sin \phi
\end{align*}
\]

\[
\begin{align*}
x_1 &= r \cos \theta \cos \phi \\
x_2 &= r \cos \theta \sin \phi \\
x_3 &= r \sin \theta
\end{align*}
\]
Catalan numbers: 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, …

Wedderburn-Etherington numbers: 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, 983, …
Fourier $e^{in\phi}$, Gegenbauer $C_{n}^{\mu}(\cos \theta)$, and Jacobi $P_{n}^{(a,b)}(\cos \theta)$ analysis of a fundamental solution for $(-\Delta)^{k}\Phi(x) = 0$

- We would like to perform Fourier $e^{im\phi}$, Gegenbauer $C_{l}^{\mu}(\cos \theta)$, and Jacobi $P_{l}^{a,b}(\cos \theta)$ analysis of a fundamental solution for the polyharmonic equation.

- Perform an eigenfunction expansion of a fundamental solution of the polyharmonic equation in a basis given by Chebyshev polynomials of the first kind

$$T_{n}(\cos \phi) = \cos(n\phi)$$

and Gegenbauer polynomials.

- Gegenbauer polynomials with argument given in terms of the $\cos \gamma$ are simply hyperspherical harmonics on $\hat{x}, \hat{x}' \in S^{d-1}$.
Separable rotationally invariant coordinate systems

- **Separable curvilinear coordinate systems** are those which transform \((-\Delta)^k \Phi(x) = 0\) into a set of \(d\)-uncoupled ODEs, each separately in terms of \(\xi_i\) such that \(i = 1, 2, \ldots, d\), with \((d - 1)\)-separation constants.

- Consider the following **rotationally invariant** coordinate system:

\[
\begin{align*}
  x_1 &= R(\xi_1, \xi_2, \ldots, \xi_{d-1}) \cos \phi \\
  x_2 &= R(\xi_1, \xi_2, \ldots, \xi_{d-1}) \sin \phi \\
  x_3 &= x_3(\xi_1, \xi_2, \ldots, \xi_{d-1}) \\
  & \vdots \\
  x_d &= x_d(\xi_1, \xi_2, \ldots, \xi_{d-1})
\end{align*}
\]

- **Parametrize** points on the \((d - 1)\)-dimensional half-hyperplane given by \(\phi = \text{const}\) and \(R > 0\) using separable curvilinear coordinates \((\xi_1, \xi_2, \ldots, \xi_{d-1})\).
Fourier expansions for a fundamental solution of $(-\Delta)^k$

- Expand, over the $(d-1)$-separation constants, a fundamental solution for $(-\Delta)^k$ in terms of separable poly-harmonics in each rotationally invariant coordinate system.

- Since all rotationally invariant coordinate systems share $\phi$ as an azimuthal angle, one can always expand a fundamental solution of the polyharmonic equation as a azimuthal Fourier cosine series over $\cos(m(\phi - \phi'))$ with $m \in \mathbb{N}_0$

- In a $(\kappa, d)$ rotationally invariant coordinate system, we can write

$$
||x - x'|| = \sqrt{2RR'} \sqrt{\chi^d_{\kappa}} - \cos(\phi - \phi')
$$

where $\chi^d_{\kappa} > 1$ is defined by

$$
\chi^d_{\kappa} := \frac{R^2 + R'^2 + (x_3 - x'_3)^2 + \ldots + (x_d - x'_d)^2}{2RR'}
$$
Example: \((2, d)\) and \((3, d)\) Euclidean-hyperspherical

Embedded 2 and 3 dimensional hyperspherical coordinates
Example: $(\kappa, d)$ Euclidean-hyperspherical coordinates
Addition theorem for hyperspherical harmonics

\[ C^{d/2-1}_l(\cos \gamma) = \frac{2\pi^{d/2}(d-2)}{(2l+d-2)\Gamma(d/2)} \sum_K Y^K_l(\hat{x}) Y^K_l(\hat{x}'), \]

where \( x, x' \in S^{d-1} \), and \( K \) is a set of quantum numbers which label representations for \( l \) in subgroup-type coordinates which parametrize points on \( S^{d-1} \).
Fourier and Gegenbauer expansions: powers of the distance

\[ \| \mathbf{x} - \mathbf{x}' \|^\nu = \sqrt{\frac{2}{\pi}} \frac{e^{i\pi(v+1)/2}}{\Gamma(-\nu/2)} \left( 2rr' \prod_{i=1}^{d-2} \sin \theta_i \sin \theta'_i \right)^{\nu/2} (\chi^2 - 1)^{(v+1)/4} \]

\[ \times \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} Q_{m-1/2}^{-(v+1)/2}(\chi) \]

\[ \chi := \chi_d = \frac{r^2 + r'^2 - 2rr' \sum_{i=1}^{d-2} \cos \theta_i \cos \theta'_i \prod_{j=1}^{i-1} \sin \theta_j \sin \theta'_j}{2rr' \prod_{i=1}^{d-2} \sin \theta_i \sin \theta'_i} \]

\[ \| \mathbf{x} - \mathbf{x}' \|^\nu = \frac{e^{i\pi(v+d-1)/2} \Gamma \left( \frac{d-2}{2} \right)}{2\sqrt{\pi} \Gamma \left( -\frac{\nu}{2} \right)} \frac{\left( r^2 > r'^2 \right)^{(v+d-1)/2}}{(rr')^{(d-1)/2}} \]

\[ \times \sum_{\lambda=0}^{\infty} (2\lambda + d - 2) Q_{\lambda+(d-3)/2}^{(1-v-d)/2} \left( \frac{r^2 + r'^2}{2rr'} \right) C_{\lambda}^{d/2-1}(\cos \gamma) \]
Example: Spherical coordinates on $\mathbb{R}^3$

$$\|x - x'|^\nu = \sqrt{\frac{2}{\pi}} \frac{e^{i\pi(\nu+1)/2}}{\Gamma(-\nu/2)} \left(2rr' \sin \theta \sin \theta'\right)^{\nu/2} \left(\chi^2 - 1\right)^{(\nu+1)/4}$$

$$\times \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} Q_{m-1/2}^{-(\nu+1)/2}(\chi),$$

$$\chi = \frac{r^2 + r'^2 - 2rr' \cos \theta \cos \theta'}{2rr' \sin \theta \sin \theta'}$$

$$\|x - x'|^\nu = -\frac{e^{i\pi\nu/2}}{2\Gamma(-\nu/2)} \frac{(r_\geq^2 - r_\leq^2)(\nu+2)/2}{rr'} \sum_{l=0}^{\infty} (2l + 1) Q_l^{-(\nu+2)/2} \left(\frac{r^2 + r'^2}{2rr'}\right)$$

$$\times \sum_{m=-l}^{l} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) P_l^m(\cos \theta') e^{im(\phi - \phi')}$$
Sum over configuration space

Alternative double summations in \((l,m)\) space (a) first over \(m\) at fixed \(l\) to form partial sums \(\vartheta_l\) (b) first over \(l\) at fixed \(m\) to form partial sums \(\varphi_m\)
Addition theorems in $\mathbb{R}^3$

- **General addition theorem**

$$Q_{m-1/2}^{-(\nu+1)/2}(\chi) = \frac{i\sqrt{\pi}2^{-(\nu+3)/2}(\sin \theta \sin \theta')^{-\nu/2}}{(\chi^2 - 1)(\nu+1)/4} \left( \frac{r^2_<>}{r r'} \right)^{(\nu+2)/2} \times \sum_{l=|m|}^{\infty} (2l + 1) \frac{(l - m)!}{(l + m)!} Q_{l}^{-(\nu+2)/2} \left( \frac{r^2 + r'^2}{2rr'} \right) P_{l}^{m}(\cos \theta) P_{l}^{m}(\cos \theta')$$

- **Addition theorem for $\nu = -1$**

$$Q_{m-1/2}(\chi) = \pi \sqrt{\sin \theta \sin \theta'} \sum_{l=|m|}^{\infty} \frac{(l - m)!}{(l + m)!} \left( \frac{r_<}{r_>} \right)^{l+1/2} P_{l}^{m}(\cos \theta) P_{l}^{m}(\cos \theta')$$
Generating functions for orthogonal polynomials

- **Chebyshev polynomials of the 1st kind**: \( T_n(\cos \theta) = \cos(n\theta) \)
  \[
  \frac{1 - x\rho}{1 + \rho^2 - 2\rho x} = \sum_{n=0}^{\infty} T_n(x) \rho^n
  \]

- **Chebyshev polynomials of the 2nd kind**: \( U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta} \)
  \[
  \frac{1}{1 + \rho^2 - 2\rho x} = \sum_{n=0}^{\infty} U_n(x) \rho^n
  \]

- **Gegenbauer polynomials**
  \[
  \frac{1}{(1 + \rho^2 - 2\rho x)\mu} = \sum_{n=0}^{\infty} C_{\mu}^{\alpha}(x) \rho^n
  \]

- **Jacobi polynomials**
  \[
  \frac{2^{\alpha+\beta}}{R(1 - \rho + R)^{\alpha}(1 + \rho + R)^{\beta}} = \sum_{n=0}^{\infty} P_{n}^{(\alpha,\beta)}(x) \rho^n
  \]
  where \( R = \sqrt{1 + \rho^2 - 2\rho x} \).  
  Should we study these?
The distance between two points $x, x' \in \mathbb{R}^d$ in a pure hyperspherical coordinate system is given by

$$
\|x - x'\| = \sqrt{r^2 + r'^2 - 2rr' \cos \gamma},
$$

(1)

where $r = \|x\|$, $r' = \|x'\|$, and $\cos \gamma = \frac{(x, x')}{rr'}$. If you define $r_{\leq} := \min_{\max} \{r, r'\}$, then you can rewrite (1) as

$$
\|x - x'\| = r_{>} \sqrt{1 + \left( \frac{r_{\leq}}{r_{>}} \right)^2 - \frac{2r_{\leq}}{r_{>}} \cos \gamma},
$$

or with $\rho := \frac{r_{\leq}}{r_{>}}$, and $x := \cos \gamma$ we have

$$
\|x - x'\| = r_{>} \sqrt{1 + \rho^2 - 2\rho \cos \gamma},
$$

where $\rho \in (0, 1)$. The other option is:

$$
\|x - x'\| = \sqrt{2rr'}\sqrt{z - x}, \quad \text{where} \quad z = \frac{1 + \rho^2}{2\rho} \in (1, \infty)
$$
Generalizations of generating functions

- **The Chebyshev polynomials** can also be defined in terms of Gegenbauer polynomials.
  - **Chebyshev polynomial of the second kind**
    \[ U_n(x) := C_n^{1}(x) \]
  - **Chebyshev polynomial of the first kind**
    \[ T_n(x) := \lim_{\epsilon_n \to 0} \frac{1}{\epsilon_n} \frac{n + \mu}{\mu} C_n^{\mu}(x), \]

- **Generalization of generating function for Gegenbauer polynomials**
  \[ \frac{(z^2 - 1)^{(\nu-\mu)/2-1/4}}{(z - x)^\nu} = \frac{2^{\mu+1/2} \Gamma(\mu) e^{i\pi(\mu-\nu+1/2)}}{\sqrt{\pi} \Gamma(\nu)} \sum_{n=0}^{\infty} (n+\mu) Q_{n+\mu-1/2}(z) C_n^{\mu}(x) \]

- **Generalization of generating function for Chebyshev polynomials**
  \[ \frac{(z^2 - 1)^{\nu/2-1/4}}{(z - x)^\nu} = \sqrt{\frac{2}{\pi}} \frac{e^{-i\pi(\nu-1/2)}}{\Gamma(\nu)} \sum_{n=0}^{\infty} \epsilon_n T_n(x) Q_{n-1/2}(z) \]
Super expansion for $C_n^\mu$ through various limiting procedures

- The generalization of the generating function for Gegenbauer polynomials produces the generalized Heine’s identity in the limit as $\mu \to 0$.

- Produces an analogous generalized identity for Chebyshev polynomials of the second kind.

- Produces Fourier and hyperspherical harmonic expansions for arbitrary powers of the distance between two points between two points in Euclidean space.

- Through limit-derivative technique, produces Fourier and Gegenbauer expansions for logarithmic fundamental solutions of the polyharmonic equation.

- Produces expansions for $(1 - x)^\mu$ and (cf. Szmytkowski (2011)) for $(y - x)^\mu$ where $x, y \in (-1, 1)$ and $y > x$. 
Super expansion for $P_n^{(\alpha,\beta)}(x)$?

- **Super expansion** for **Gegenbauer polynomials** is a consequence of expanding the **generating function** for Gegenbauer polynomials $C_n^{\mu}(x)$ in terms of the complete set $C_n^{\nu}(x)$.

- This expansion is consequence of **connection formulae** for **orthogonal polynomials** which illustrate how one expands an orthogonal polynomial of one index in terms of a finite-sum over the same orthogonal polynomial of a different index.

- One must identify the coefficients of this expansion, which for **Gegenbauer** and **Jacobi polynomials** are given in terms of a $\,_{3}F_{2}$ hypergeometric function of **unit argument**, which can be expressed in terms of a gamma functions using **Watson’s formula** and **Chu’s** (2011) recent extension.
Generating functions for Jacobi polynomials

- The **Gegenbauer polynomial** can be defined in terms of the Jacobi polynomial.

\[ C^{\mu}_{n}(x) := \frac{(2\mu)_n}{(\mu + \frac{1}{2})_n} P^{(\mu-1/2,\mu-1/2)}_{n}(x) \]

- Relevant **generating functions** for generalization

\[
\sum_{n=0}^{\infty} P^{(\alpha,\beta)}_{n}(x) \frac{(\alpha + \beta + 1)_n}{(\beta + 1)_n} \rho^n = \frac{1}{(1 + \rho)^{\alpha+\beta+1}} \times _{2}F_{1}\left(\frac{\alpha + \beta + 1}{2}, \frac{\alpha + \beta + 2}{2}; \beta + 1; \frac{2\rho(1+x)}{(1 + \rho)^2}\right)
\]

and its **companion**

\[
\sum_{n=0}^{\infty} P^{(\alpha,\beta)}_{n}(x) \frac{(\alpha + \beta + 1)_n}{(\alpha + 1)_n} \rho^n = \frac{1}{(1 - \rho)^{\alpha+\beta+1}} \times _{2}F_{1}\left(\frac{\alpha + \beta + 1}{2}, \frac{\alpha + \beta + 2}{2}; \alpha + 1; \frac{-2\rho(1-x)}{(1 - \rho)^2}\right)
\]
Legendre function representations of generating functions

- **Associated Legendre function representation**
  \[
  \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) \frac{(\alpha + \beta + 1)}{(\beta + 1)n} \rho^n = \left( \frac{2}{\rho(1 + x)} \right)^{\beta/2} \\
  \times \frac{\Gamma(\beta + 1)}{(1 + \rho^2 - 2\rho x)^{\alpha + 1/2}} P_{\alpha}^{-\beta} \left( \frac{1 + \rho}{\sqrt{1 + \rho^2 - 2\rho x}} \right)
  \]

- **Ferrers function representation**
  \[
  \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) \frac{(\alpha + \beta + 1)}{(\alpha + 1)n} \rho^n = \left( \frac{2}{\rho(1 - x)} \right)^{\alpha/2} \\
  \times \frac{\Gamma(\alpha + 1)}{(1 + \rho^2 - 2\rho x)^{\beta + 1/2}} P_{\beta}^{-\alpha} \left( \frac{1 - \rho}{\sqrt{1 + \rho^2 - 2\rho x}} \right)
  \]
Extensions

\[ \sum_{n=0}^{\infty} P_{n}^{(\alpha,\beta)}(x)(2n + \alpha + \beta + 1) \frac{\alpha + \beta + 1}{\beta + 1} \frac{n!}{\rho^n} = \frac{(\alpha + \beta + 1)(1 - \rho)}{(1 + \rho)^{\alpha + \beta + 2}} \times {}_2F_1 \left( \begin{array}{c} \frac{\alpha + \beta + 2}{2}, \frac{\alpha + \beta + 3}{2} \\ \beta + 1; \frac{2\rho(1 + x)}{(1 + \rho)^2} \end{array} \right) \]

and its companion

\[ \sum_{n=0}^{\infty} P_{n}^{(\alpha,\beta)}(x)(2n + \alpha + \beta + 1) \frac{\alpha + \beta + 1}{\alpha + 1} \frac{n!}{\rho^n} = \frac{(\alpha + \beta + 1)(1 + \rho)}{(1 - \rho)^{\alpha + \beta + 2}} \times {}_2F_1 \left( \begin{array}{c} \frac{\alpha + \beta + 2}{2}, \frac{\alpha + \beta + 3}{2} \\ \alpha + 1; \frac{-2\rho(1 - x)}{(1 - \rho)^2} \end{array} \right) \]
Legendre function representations of extensions

- **Associated Legendre function representation** of extension

\[
\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n + \alpha + \beta + 1) \frac{(\alpha + \beta + 1)_n}{(\beta + 1)_n} \rho^n = \left( \frac{2}{\rho(1+x)} \right)^{\beta/2} \\
\times \frac{(\alpha + \beta + 1)(1-\rho)\Gamma(\beta + 1)}{(1 + \rho^2 - 2\rho x)^{\alpha/2}} P_{\alpha+1}^{-\beta} \left( \frac{1 + \rho}{\sqrt{1 + \rho^2 - 2\rho x}} \right)
\]

- **Ferrers function representation** of extension

\[
\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n + \alpha + \beta + 1) \frac{(\alpha + \beta + 1)_n}{(\alpha + 1)_n} \rho^n = \left( \frac{2}{\rho(1-x)} \right)^{\alpha/2} \\
\times \frac{(\alpha + \beta + 1)(1+\rho)\Gamma(\alpha + 1)}{(1 + \rho^2 - 2\rho x)^{\beta/2}} P_{\beta+1}^{-\alpha} \left( \frac{1 - \rho}{\sqrt{1 + \rho^2 - 2\rho x}} \right)
\]
Generalization of extension

- **Hypergeometric generalization**

\[
\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)(2n + \alpha + \beta + 1)m \frac{(\alpha + \beta + 1)n}{(\beta + 1)n} \\
\times _2F_1 \left( m, -m + 1; \alpha + \beta + 2n + 2; \frac{-\rho}{1 - \rho} \right) \rho^n \\
= \frac{(\alpha + \beta + 1)m(1 - \rho)^m}{(1 - \rho)^{\alpha + \beta + m + 1}} \\
\times _2F_1 \left( \frac{\alpha + \beta + m + 1}{2}, \frac{\alpha + \beta + m + 2}{2}; \beta + 1; \frac{2\rho(1 + x)}{(1 + \rho)^2} \right)
\]
Generalizations of extensions $+$ Jacobi generating functions

- **Associated Legendre generalization**

\[
\sum_{n=0}^{\infty} P_{n}^{(\alpha,\beta)}(x)(2n + \alpha + \beta + 1)(\alpha + \beta + m + 1)_{2n} \frac{\Gamma(\alpha + \beta + n + 1)}{\Gamma(\beta + n + 1)} \times P_{-m}^{-\alpha-\beta-2n-1} \left( \frac{1 + \rho}{1 - \rho} \right) \\
= \rho^{(\alpha+1)/2}(1 - \rho)^{m} \left( \frac{2}{1 + x} \right)^{\beta/2} \times P_{-\alpha}^{-\beta} \left( \frac{1 + \rho}{\sqrt{1 + \rho^{2} - 2\rho x}} \right)
\]

- And its companions
Generating function for products of Gegenbauer polynomials

\[ \sum_{k=0}^{\infty} \frac{k!}{(2\alpha)_k} C_k^\alpha(x) C_k^\alpha(y) \rho^k = 2 F_1 \left( \frac{\alpha}{2}, \frac{\alpha + 1}{2}; \alpha + \frac{1}{2}; \frac{4(1 - x^2)(1 - y^2)\rho^2}{(1 + \rho^2 - 2xy\rho)^2} \right) \]

\[ = \frac{\Gamma \left( \alpha + \frac{1}{2} \right)}{\sqrt{\pi} \Gamma(\alpha) (\rho \sin \theta \sin \phi)^\alpha} Q_{\alpha - 1} \left( \frac{1 + \rho^2 - 2\rho \cos \theta \cos \phi}{2\rho \sin \theta \sin \phi} \right) \]
Generating functions for Jacobi polynomials

\begin{equation}
\sum_{n=0}^{\infty} \binom{\alpha-2n}{\beta-2n} P_{2n}(x) \rho^n = 2F_1 \left( \frac{-\alpha}{2}, \frac{-\alpha + 1}{2}; \frac{1}{2}; \xi \right) 2F_1 \left( \frac{-\beta}{2}, \frac{-\beta + 1}{2}; \frac{1}{2}; \eta \right)
\end{equation}

\begin{equation}
- \frac{1}{4} \alpha \beta (1 - x^2) \rho \ 2F_1 \left( \frac{-\alpha + 1}{2}, \frac{-\alpha + 2}{2}; \frac{3}{2}; \xi \right) 2F_1 \left( \frac{-\beta + 1}{2}, \frac{-\beta + 2}{2}; \frac{3}{2}; \eta \right)
\end{equation}

\begin{equation}
= \frac{(1 - \xi)^{\alpha/2} (1 - \eta)^{\beta/2}}{4} \left[ \left( \frac{1 + \sqrt{\xi}}{\sqrt{1-\xi}} \right)^\alpha + \left( \frac{1 + \sqrt{\xi}}{\sqrt{1-\xi}} \right)^{-\alpha} \right] \left[ \left( \frac{1 + \sqrt{\eta}}{\sqrt{1-\eta}} \right)^\beta + \left( \frac{1 + \sqrt{\eta}}{\sqrt{1-\eta}} \right)^{-\beta} \right]
\end{equation}

\begin{equation}
- \frac{\rho (1 - x^2)}{4 \sqrt{\xi \eta}} \left[ \left( \frac{1 + \sqrt{\xi}}{\sqrt{1-\xi}} \right)^\alpha - \left( \frac{1 + \sqrt{\xi}}{\sqrt{1-\xi}} \right)^{-\alpha} \right] \left[ \left( \frac{1 + \sqrt{\eta}}{\sqrt{1-\eta}} \right)^\beta - \left( \frac{1 + \sqrt{\eta}}{\sqrt{1-\eta}} \right)^{-\beta} \right]
\end{equation}

where

\begin{equation}
\xi = \frac{(1 + x)^2 \rho}{4}, \text{ and } \eta = \frac{(1 - x)^2 \rho}{4}.
\end{equation}
Generating function for Jacobi polynomials

\[ \sum_{n=0}^{\infty} P_{2n+1}^{(\alpha-2n, \beta-2n)}(x) \rho^n \]

\[ = \frac{(1 - \xi)^{(\alpha+1)/2}(1 - \eta)^{(\beta+1)/2}}{8\sqrt{\xi}} (x - 1) \left[ \left( \frac{1+\sqrt{\xi}}{\sqrt{1-\xi}} \right)^\alpha + \left( \frac{1+\sqrt{\xi}}{\sqrt{1-\xi}} \right)^{-\alpha} \right] \]

\[ \times \left[ \left( \frac{1+\sqrt{\eta}}{\sqrt{1-\eta}} \right)^\beta + \left( \frac{1+\sqrt{\eta}}{\sqrt{1-\eta}} \right)^{-\beta} \right] \]

\[ + \frac{(1 - \xi)^{(\alpha+1)/2}(1 - \eta)^{(\beta+1)/2}}{8\sqrt{\eta}} (x + 1) \left[ \left( \frac{1+\sqrt{\xi}}{\sqrt{1-\xi}} \right)^\alpha - \left( \frac{1+\sqrt{\xi}}{\sqrt{1-\xi}} \right)^{-\alpha} \right] \]

\[ \times \left[ \left( \frac{1+\sqrt{\eta}}{\sqrt{1-\eta}} \right)^\beta - \left( \frac{1+\sqrt{\eta}}{\sqrt{1-\eta}} \right)^{-\beta} \right] \]
A whole host of other Jacobi generating functions can be expressed in terms of associated Legendre functions and Ferrers functions, and in terms of the second kind.

Multi-summation (power-law and logarithmic) addition theorems

Analysis of special functions and fundamental solutions on highly symmetric Riemannian manifolds