

The Scalable Parallel Random Number Generators (SPRNG) Library and Modern Computer Architectures

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Outline of the Talk

Types of random numbers and Monte Carlo Methods

Pseudorandom number generation

Types of pseudorandom numbers

Properties of these pseudorandom numbers

Parallelization of pseudorandom number generators

New directions for SPRNG

Quasirandom number generation

The Koksma-Hlawka inequality

Discrepancy

The van der Corput sequence

Methods of quasirandom number generation

Randomization

Conclusions and Future Work



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Monte Carlo Methods: Numerical Experimental that Use Random Numbers

- ▶ A Monte Carlo method is any process that consumes random numbers
- 1. Each calculation is a numerical experiment
 - ▶ Subject to known and unknown sources of error
 - ▶ Should be reproducible by peers
 - ▶ Should be easy to run anew with results that can be combined to reduce the variance
- 2. Sources of errors must be controllable/isolatable
 - ▶ Programming/science errors under your control
 - ▶ Make possible RNG errors approachable
- 3. Reproducibility
 - ▶ Must be able to rerun a calculation with the same numbers
 - ▶ Across different machines (modulo arithmetic issues)
 - ▶ Parallel and distributed computers?



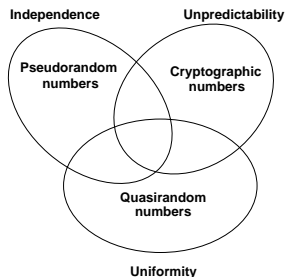
What are Random Numbers Used For?

1. Random numbers are used extensively in simulation, statistics, and in *Monte Carlo* computations
 - ▶ Simulation: use random numbers to “randomly pick” event outcomes based on statistical or experiential data
 - ▶ Statistics: use random numbers to generate data with a particular distribution to calculate statistical properties (when analytic techniques fail)
2. There are many Monte Carlo applications of great interest
 - ▶ Numerical quadrature “all Monte Carlo is integration”
 - ▶ Quantum mechanics: Solving Schrödinger’s equation with Green’s function Monte Carlo via random walks
 - ▶ Mathematics: Using the Feynman-Kac/path integral methods to solve partial differential equations with random walks
 - ▶ Defense: neutronics, nuclear weapons design
 - ▶ Finance: options, mortgage-backed securities



What are Random Numbers Used For?

3. There are many types of random numbers
 - ▶ “Real” random numbers: uses a ‘physical source’ of randomness
 - ▶ Pseudorandom numbers: deterministic sequence that passes tests of randomness
 - ▶ Quasirandom numbers: well distributed (low discrepancy) points



Pseudorandom Numbers

- ▶ Pseudorandom numbers mimic the properties of ‘real’ random numbers
- A.** Pass statistical tests
- B.** Reduce error is $O(N^{-\frac{1}{2}})$ in Monte Carlo
 - ▶ Some common pseudorandom number generators (RNG):
 1. Linear congruential: $x_n = ax_{n-1} + c \pmod{m}$
 2. Implicit inversive congruential: $x_n = a\overline{x_{n-1}} + c \pmod{p}$
 3. Explicit inversive congruential: $x_n = a\bar{n} + c \pmod{p}$
 4. Shift register: $y_n = y_{n-s} + y_{n-r} \pmod{2}$, $r > s$
 5. Additive lagged-Fibonacci: $z_n = z_{n-s} + z_{n-r} \pmod{2^k}$, $r > s$
 6. Combined: $w_n = y_n + z_n \pmod{p}$
 7. Multiplicative lagged-Fibonacci: $x_n = x_{n-s} \times x_{n-r} \pmod{2^k}$, $r > s$



Pseudorandom Numbers

- ▶ Some properties of pseudorandom number generators, integers: $\{x_n\}$ from modulo m recursion, and $U[0, 1], z_n = \frac{x_n}{m}$
- A.** Should be a purely periodic sequence (e.g.: DES and IDEA are not provably periodic)
- B.** Period length: $\text{Per}(x_n)$ should be large
- C.** Cost per bit should be moderate (not cryptography)
- D.** Should be based on theoretically solid and empirically tested recursions
- E.** Should be a totally reproducible sequence

Splitting RNGs for Use In Parallel

- ▶ We consider splitting a single PRNG:
 - ▶ Assume $\{x_n\}$ has $\text{Per}(x_n)$
 - ▶ Has the fast-leap ahead property: leaping L ahead costs no more than generating $O(\log_2(L))$ numbers
- ▶ Then we associate a single block of length L to each parallel subsequence:

1. Blocking:

- ▶ First block: $\{x_0, x_1, \dots, x_{L-1}\}$
- ▶ Second : $\{x_L, x_{L+1}, \dots, x_{2L-1}\}$
- ▶ i th block: $\{x_{(i-1)L}, x_{(i-1)L+1}, \dots, x_{iL-1}\}$

2. The Leap Frog Technique: define the leap ahead of

$$\ell = \left\lfloor \frac{\text{Per}(x_i)}{L} \right\rfloor :$$

- ▶ First block: $\{x_0, x_\ell, x_{2\ell}, \dots, x_{(L-1)\ell}\}$
- ▶ Second block: $\{x_1, x_{1+\ell}, x_{1+2\ell}, \dots, x_{1+(L-1)\ell}\}$
- ▶ i th block: $\{x_i, x_{i+\ell}, x_{i+2\ell}, \dots, x_{i+(L-1)\ell}\}$



Splitting RNGs for Use In Parallel

3. The Lehmer Tree, designed for splitting LCGs:

- ▶ Define a right and left generator: $R(x)$ and $L(x)$
- ▶ The right generator is used within a process
- ▶ The left generator is used to spawn a new PRNG stream
- ▶ Note: $L(x) = R^W(x)$ for some W for **all** x for an LCG
- ▶ Thus, spawning is just jumping a fixed, W , amount in the sequence

4. Recursive Halving Leap-Ahead, use fixed points or fixed leap aheads:

- ▶ First split leap ahead: $\left\lfloor \frac{\text{Per}(x_i)}{2} \right\rfloor$
- ▶ i th split leap ahead: $\left\lfloor \frac{\text{Per}(x_i)}{2^{i+1}} \right\rfloor$
- ▶ This permits effective user of all remaining numbers in $\{x_n\}$ without the need for *a priori* bounds on the stream length L



Generic Problems Parallelizing via Splitting

1. Splitting for parallelization is not scalable:

- ▶ It usually costs $O(\log_2(\text{Per}(x_i)))$ bit operations to generate a random number
- ▶ For parallel use, a given computation that requires L random numbers per process with P processes must have $\text{Per}(x_i) = O((LP)^e)$
- ▶ Rule of thumb: never use more than $\sqrt{\text{Per}(x_i)}$ of a sequence $\rightarrow e = 2$
- ▶ Thus cost per random number is not constant with number of processors!!

Generic Problems Parallelizing via Splitting

2. Correlations within sequences are generic!!

- ▶ Certain offsets within any modular recursion will lead to extremely high correlations
- ▶ Splitting in any way converts auto-correlations to cross-correlations between sequences
- ▶ Therefore, splitting generically leads to interprocessor correlations in PRNGs

New Results in Parallel RNGs: Using Distinct Parameterized Streams in Parallel

1. Default generator: additive lagged-Fibonacci,
$$x_n = x_{n-s} + x_{n-r} \pmod{2^k}, r > s$$
 - ▶ Very efficient: 1 add & pointer update/number
 - ▶ Good empirical quality
 - ▶ Very easy to produce distinct parallel streams
2. Alternative generator #1: prime modulus LCG,
$$x_n = ax_{n-1} + c \pmod{m}$$
 - ▶ Choice: Prime modulus (quality considerations)
 - ▶ Parameterize the multiplier
 - ▶ Less efficient than lagged-Fibonacci
 - ▶ Provably good quality
 - ▶ Multiprecise arithmetic in initialization



New Results in Parallel RNGs: Using Distinct Parameterized Streams in Parallel

3. Alternative generator #2: power-of-two modulus LCG,

$$x_n = ax_{n-1} + c \pmod{2^k}$$

- ▶ Choice: Power-of-two modulus (efficiency considerations)
- ▶ Parameterize the prime additive constant
- ▶ Less efficient than lagged-Fibonacci
- ▶ Provably good quality
- ▶ Must compute as many primes as streams

Parameterization Based on Seeding

- ▶ Consider the Lagged-Fibonacci generator:

$$x_n = x_{n-5} + x_{n-17} \pmod{2^{32}} \text{ or in general:}$$

$$x_n = x_{n-s} + x_{n-r} \pmod{2^k}, r > s$$

- ▶ The seed is 17 32-bit integers; 544 bits, longest possible period for this linear generator is $2^{17 \times 32} - 1 = 2^{544} - 1$
- ▶ Maximal period is $\text{Per}(x_n) = (2^{17} - 1) \times 2^{31}$
- ▶ Period is maximal \iff at least one of the 17 32-bit integers is odd
- ▶ This seeding failure results in only even “random numbers”
- ▶ Are $(2^{17} - 1) \times 2^{31 \times 17}$ seeds with full period
- ▶ Thus there are the following number of full-period equivalence classes (ECs):

$$E = \frac{(2^{17} - 1) \times 2^{31 \times 17}}{(2^{17} - 1) \times 2^{31}} = 2^{31 \times 16} = 2^{496}$$



The Equivalence Class Structure

With the “standard” l.s.b., b_0 :

m.s.b.				l.s.b.	
b_{k-1}	b_{k-2}	...	b_1	b_0	
□	□	...	0	0	x_{r-1}
0	□	...	□	0	x_{r-2}
⋮	⋮	⋮	⋮	⋮	
□	0	...	□	0	x_1
□	□	...	□	1	x_0

or a special b_0 (adjoining 1's):

m.s.b.				l.s.b.	
b_{k-1}	b_{k-2}	...	b_1	b_0	
□	□	...	□	b_{0n-1}	x_{r-1}
□	□	...	□	b_{0n-2}	x_{r-2}
⋮	⋮	⋮	⋮	⋮	
□	□	...	□	b_{01}	x_1
0	0	...	0	b_{00}	x_0

Parameterization of Prime Modulus LCGs

- ▶ Consider only $x_n = ax_{n-1} \pmod{m}$, with m prime has maximal period when a is a primitive root modulo m
- ▶ If α and a are primitive roots modulo m then $\exists l$ s.t. $\gcd(l, m-1) = 1$ and $\alpha \equiv a^l \pmod{m}$
- ▶ If $m = 2^{2^n} + 1$ (Fermat prime) then all odd powers of α are primitive elements also
- ▶ If $m = 2q + 1$ with q also prime (Sophie-Germain prime) then all odd powers (save the q th) of α are primitive elements

Parameterization of Power-of-Two Modulus LCGs

- ▶ $x_n = ax_{n-1} + c_i \pmod{2^k}$, here the c_i 's are distinct primes
- ▶ Can prove (Percus and Kalos) that streams have good spectral test properties among themselves
- ▶ Best to choose $c_i \approx \sqrt{2^k} = 2^{k/2}$
- ▶ Must enumerate the primes, uniquely, not necessarily exhaustively to get a unique parameterization
- ▶ Note: in $0 \leq i < m$ there are $\approx \frac{m}{\log_2 m}$ primes via the prime number theorem, thus if $m \approx 2^k$ streams are required, then must exhaust all the primes modulo $\approx 2^{k+\log_2 k} = 2^k k = m \log_2 m$
- ▶ Must compute distinct primes on the fly either with table or something like Meissel-Lehmer algorithm



Parameterization of MLFGs

1. Recall the MLFG recurrence:

$$x_n = x_{n-s} \times x_{n-r} \pmod{2^k}, \quad r > s$$

2. One of the r seed elements is even \rightarrow eventually all become even
3. Restrict to only odd numbers in the MLFG seeds
4. Allows the following parameterization for odd integers modulo a power-of-two $x_n = (-1)^{y_n} 3^{z_n} \pmod{2^k}$, where $y_n \in \{0, 1\}$ and where
 - ▶ $y_n = y_{n-s} + y_{n-r} \pmod{2}$
 - ▶ $z_n = z_{n-s} + z_{n-r} \pmod{2^{k-2}}$
5. Last recurrence means we can use ALFG parameterization, z_n , and map to MLFGs via modular exponentiation

Parallel Neutronics: A Difficult Example

1. The structure of parallel neutronics
 - ▶ Use a parallel queue to hold unfinished work
 - ▶ Each processor follows a distinct neutron
 - ▶ Fission event places a new neutron(s) in queue with initial conditions
2. Problems and solutions
 - ▶ Reproducibility: each neutron is queued with a new generator (and with the next generator)
 - ▶ Using the binary tree mapping prevents generator reuse, even with extensive branching
 - ▶ A global seed reorders the generators to obtain a statistically significant new but reproducible result

Many Parameterized Streams Facilitate Implementation/Use

1. Advantages of using parameterized generators

- ▶ Each unique parameter value gives an “independent” stream
- ▶ Each stream is uniquely numbered
- ▶ Numbering allows for absolute reproducibility, even with MIMD queuing
- ▶ Effective serial implementation + enumeration yield a portable scalable implementation
- ▶ Provides theoretical testing basis

Many Parameterized Streams Facilitate Implementation/Use

2. Implementation details

- ▶ Generators mapped canonically to a binary tree
- ▶ Extended seed data structure contains current seed and next generator
- ▶ Spawning uses new next generator as starting point: assures no reuse of generators

3. All these ideas in the **Scalable Parallel Random Number Generators (SPRNG) library**: <http://sprng.org>

Many Different Generators and A Unified Interface

1. Advantages of having more than one generator
 - ▶ An application exists that stumbles on a given generator
 - ▶ Generators based on different recursions allow comparison to rule out spurious results
 - ▶ Makes the generators real experimental tools
2. Two interfaces to the SPRNG library: simple and default
 - ▶ Initialization returns a pointer to the generator state:
`init_SPRNG()`
 - ▶ Single call for new random number: `SPRNG()`
 - ▶ Generator type chosen with parameters in `init_SPRNG()`
 - ▶ Makes changing generator very easy
 - ▶ Can use more than one generator *type* in code
 - ▶ Parallel structure is extensible to new generators through dummy routines

New Directions for SPRNG

- ▶ SPRNG was originally designed for distributed-memory multiprocessors
- ▶ HPC architectures are increasingly based on commodity chips with architectural variations
 1. Microprocessors with more than one processor core (multicore)
 2. The IBM cell processor (not very successful even though it was in the Sony Playstation)
 3. Microprocessors with accelerators, most popular being GPGPUs (video games)
- ▶ We will consider only two of these:
 1. Multicore support using OpenMP
 2. GPU support using CUDA (Nvidia) and/or OpenCL (standard)



SPRNG Update Overview

- ▶ SPRNG uses independent full-period cycles for each processor
 1. Organizes the independent use of generators without communication
 2. Permits reproducibility
 3. Initialization of new full-period generators is slow for some generators
- ▶ A possible solution
 1. Keep the independent full-period cycles for “top-level” generators
 2. Within these (multicore processor/GPU) use cycle splitting to service threads

Experience with Multicore

- ▶ We have implemented an OpenMP version of SPRNG for multicore using these ideas
- ▶ OpenMP is now built into the main compilers, so it is easy to access
- ▶ Our experience has been
 1. It works as expected giving one access to Monte Carlo on all the cores
 2. Permits reproducibility but with some work: must know the number of threads
 3. Near perfect parallelization is expected and seen
 4. Comparison with independent spawning vs. cycle splitting is not as dramatic as expected
- ▶ Backward reproducibility is something that we can provide, but forward reproducibility is trickier
- ▶ This version is a prototype, but will be used for the eventual creation of the multicore version of SPRNG



Expectations for SPRNG on GPGPUs

- ▶ SPRNG for the GPU will be simple in principal, but harder for users
 1. The same technique that was used for multicore will work for GPUs with many of the same issues
 2. The concept of reproducibility will have to be modified as well
 3. Successful exploitation of GPU threads will require that SPRNG calls be made to insure that the data and the execution are on the GPU
- ▶ The software development may not be the hardest aspect of this work
 1. Clear documentation with descriptions of common coding errors will be essential for success
 2. An extensive library of examples will be necessary to provide most users with code close to their own to help use the GPU efficiently for Monte Carlo
- ▶ We are currently working on putting many Monte Carlo codes on GPUs in anticipation of this



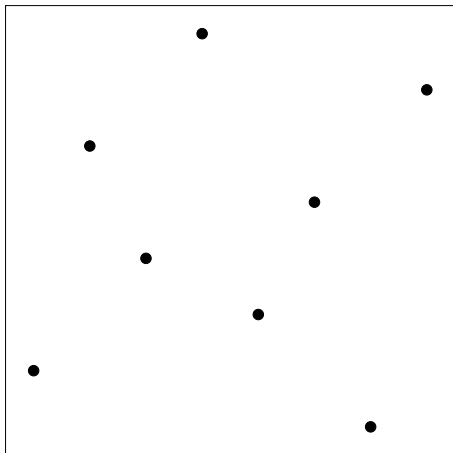
Quasirandom Numbers

- ▶ Many problems require uniformity, not randomness: “quasirandom” numbers are highly uniform deterministic sequences with small *star discrepancy*
- ▶ **Definition:** The *star discrepancy* D_N^* of x_1, \dots, x_N :

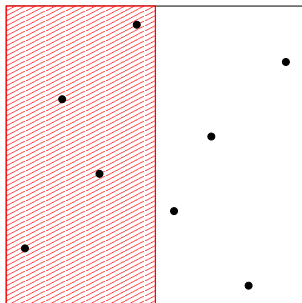
$$\begin{aligned} D_N^* &= D_N^*(x_1, \dots, x_N) \\ &= \sup_{0 \leq u \leq 1} \left| \frac{1}{N} \sum_{n=1}^N \chi_{[0,u)}(x_n) - u \right|, \end{aligned}$$

where χ is the characteristic function

Star Discrepancy in 2D



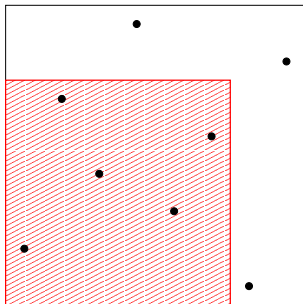
Star Discrepancy in 2D



▶ $\left| \frac{1}{2} - \frac{4}{8} \right| = 0$



Star Discrepancy in 2D



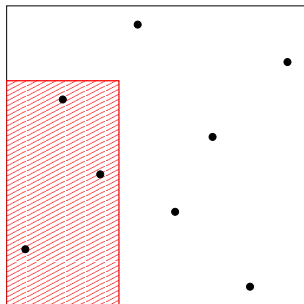
▶ $\left| \frac{1}{2} - \frac{4}{8} \right| = 0$

▶ $\left| \frac{9}{16} - \frac{5}{8} \right| = \frac{1}{16} = 0.0625$

▶

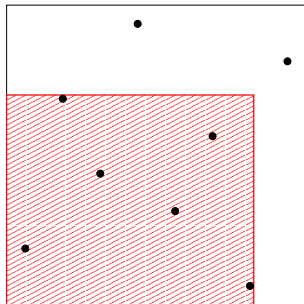
▶

Star Discrepancy in 2D



- ▶ $\left| \frac{1}{2} - \frac{4}{8} \right| = 0$
- ▶ $\left| \frac{9}{16} - \frac{5}{8} \right| = \frac{1}{16} = 0.0625$
- ▶ $\left| \frac{9}{32} - \frac{3}{8} \right| = \frac{3}{32} = 0.09375$
- ▶

Star Discrepancy in 2D



- ▶ $\left| \frac{1}{2} - \frac{4}{8} \right| = 0$
- ▶ $\left| \frac{9}{16} - \frac{5}{8} \right| = \frac{1}{16} = 0.0625$
- ▶ $\left| \frac{9}{32} - \frac{3}{8} \right| = \frac{3}{32} = 0.09375$
- ▶ $\left| \frac{143}{256} - \frac{6}{8} \right| = \frac{49}{256} \approx 0.19140625$

Quasirandom Numbers

- ▶ **Theorem** (Koksma, 1942): if $f(x)$ has bounded variation $V(f)$ on $[0, 1]$ and $x_1, \dots, x_N \in [0, 1]$ with star discrepancy D_N^* , then:

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x) dx \right| \leq V(f) D_N^*,$$

this is the Koksma-Hlawka inequality

- ▶ Note: Many different types of discrepancies are definable



Discrepancy Facts

- ▶ Real random numbers have (the law of the iterated logarithm):

$$D_N^* = O(N^{-1/2}(\log \log N)^{-1/2})$$

- ▶ Klaus F. Roth (Fields medalist in 1958) proved the following lower bound in 1954 for the star discrepancy of N points in s dimensions:

$$D_N^* \geq O(N^{-1}(\log N)^{\frac{s-1}{2}})$$

- ▶ Sequences (indefinite length) and point sets have different "best discrepancies" at present
 - ▶ Sequence: $D_N^* \leq O(N^{-1}(\log N)^{s-1})$
 - ▶ Point set: $D_N^* \leq O(N^{-1}(\log N)^{s-2})$



Some Types of Quasirandom Numbers

- ▶ Must choose point sets (finite #) or sequences (infinite #) with small D_N^*
- ▶ Often used is the *van der Corput sequence* in base b :
 $x_n = \Phi_b(n - 1), n = 1, 2, \dots$, where for $b \in \mathbb{Z}, b \geq 2$:

$$\Phi_b \left(\sum_{j=0}^{\infty} a_j b^j \right) = \sum_{j=0}^{\infty} a_j b^{-j-1} \quad \text{with}$$
$$a_j \in \{0, 1, \dots, b - 1\}$$



Some Types of Quasirandom Numbers

- ▶ For the van der Corput sequence

$$ND_N^* \leq \frac{\log N}{3 \log 2} + O(1)$$

- ▶ With $b = 2$, we get $\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8} \dots\}$
- ▶ With $b = 3$, we get $\{\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9} \dots\}$

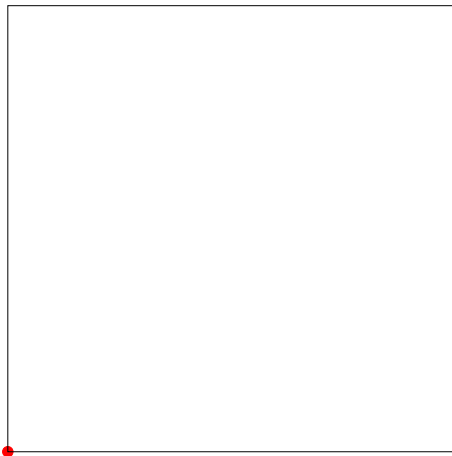
Some Types of Quasirandom Numbers

► Other small D_N^* points sets and sequences:

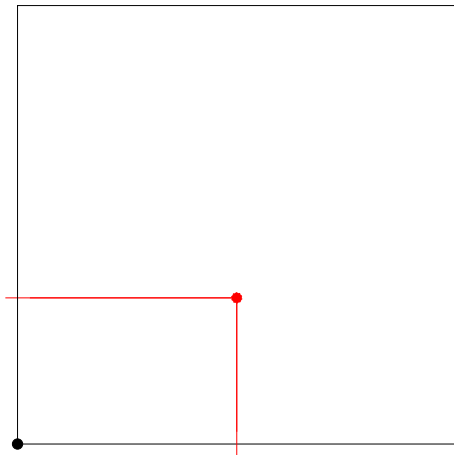
1. Halton sequence: $\mathbf{x}_n = (\Phi_{b_1}(n-1), \dots, \Phi_{b_s}(n-1))$,
 $n = 1, 2, \dots$, $D_N^* = O(N^{-1}(\log N)^s)$ if b_1, \dots, b_s pairwise relatively prime
2. Hammersley point set:
 $\mathbf{x}_n = (\frac{n-1}{N}, \Phi_{b_1}(n-1), \dots, \Phi_{b_{s-1}}(n-1))$, $n = 1, 2, \dots, N$,
 $D_N^* = O(N^{-1}(\log N)^{s-1})$ if b_1, \dots, b_{s-1} are pairwise relatively prime



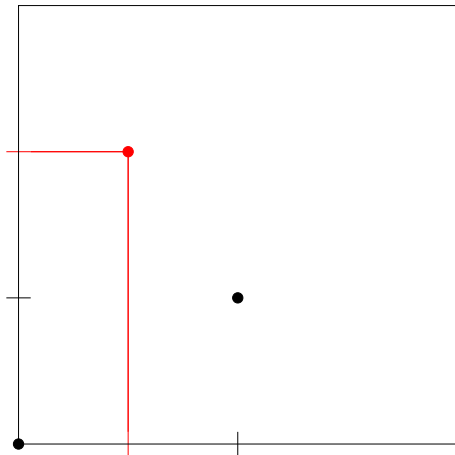
Halton sequence: example



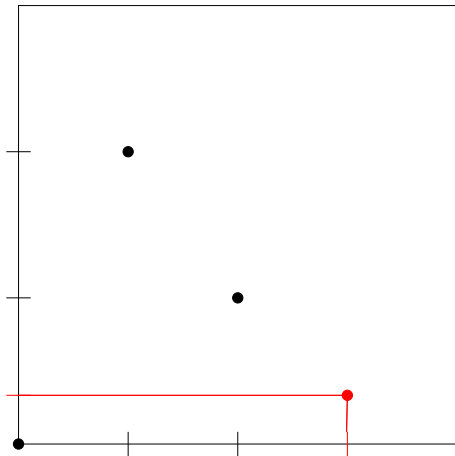
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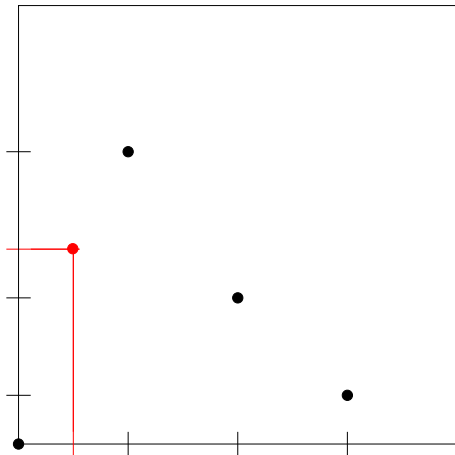
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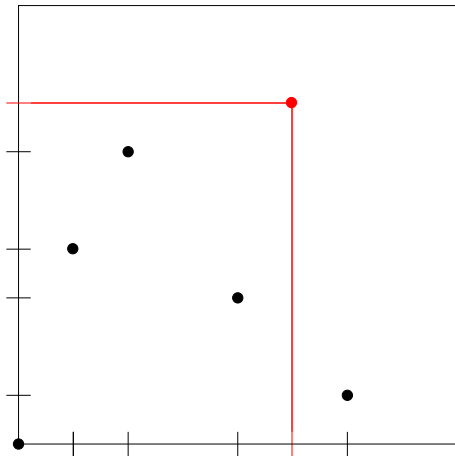
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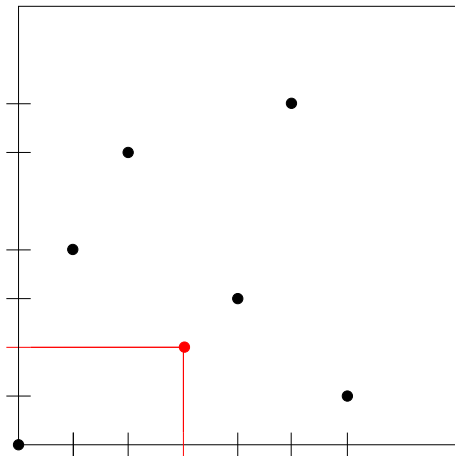
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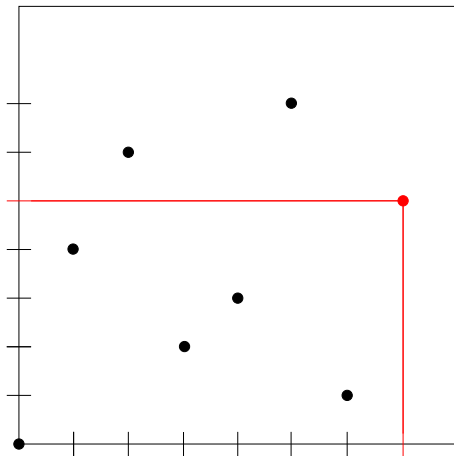
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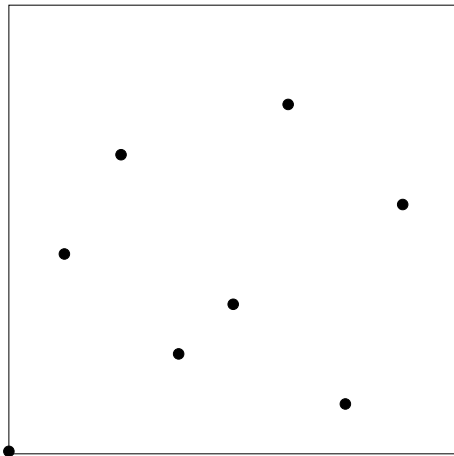
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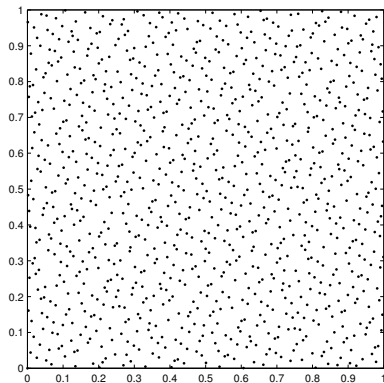
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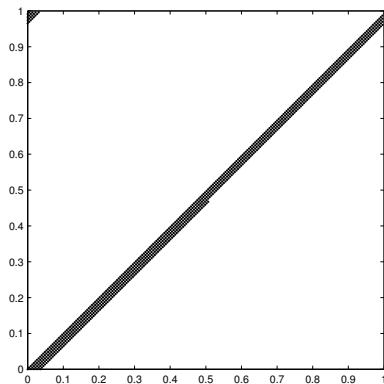
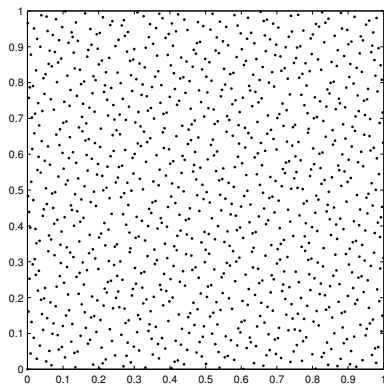
Halton sequence: example



Good Halton points vs poor Halton points



Good Halton points vs poor Halton points



Some Types of Quasirandom Numbers

3. Ergodic dynamics: $\mathbf{x}_n = \{n\alpha\}$, where $\alpha = (\alpha_1, \dots, \alpha_s)$ is irrational and $\alpha_1, \dots, \alpha_s$ are linearly independent over the rationals then for almost all $\alpha \in \mathbb{R}^s$,
 $D_N^* = O(N^{-1}(\log N)^{s+1+\epsilon})$ for all $\epsilon > 0$
4. Other methods of generation
 - ▶ Method of good lattice points (Sloan and Joe)
 - ▶ Sobol' sequences
 - ▶ Faure sequences (more later)
 - ▶ Niederreiter sequences

Continued-Fractions and Irrationals

Infinite continued-fraction expansion for choosing good irrationals:

$$r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

$a_i \leq K \rightarrow$ sequence is a low-discrepancy sequence

Choose all $a_i = 1$. Then

$$r = 1 + \frac{1}{1 + \frac{1}{1 + \dots}}$$

is the golden ratio.

0.618, 0.236, 0.854, 0.472, 0.090, ...

Irrational sequence in more dimensions is not a low-discrepancy sequence.



Lattice

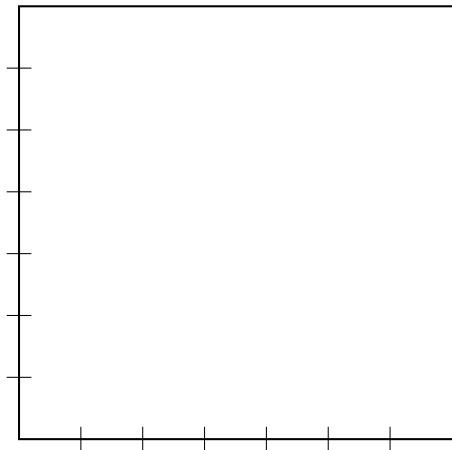
- ▶ Fixed N
- ▶ Generator vector $\vec{g} = (g_1, \dots, g_d) \in \mathbb{Z}^d$.

We define a rank-1 lattice as

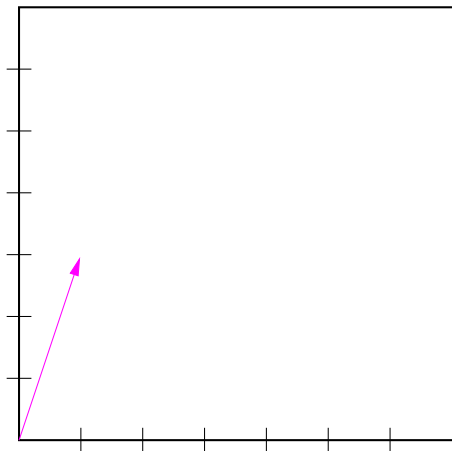
$$P_{\text{lattice}} := \left\{ \vec{x}_i = \frac{i\vec{g}}{N} \bmod 1 \right\}.$$



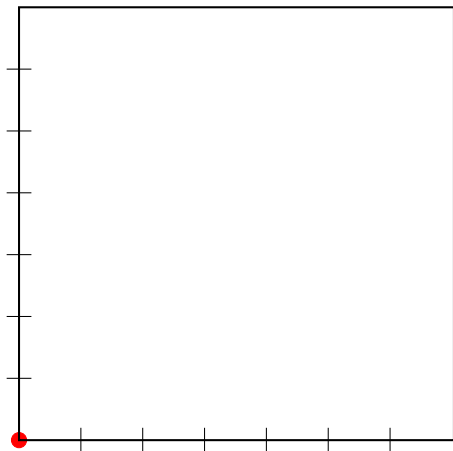
An example lattice



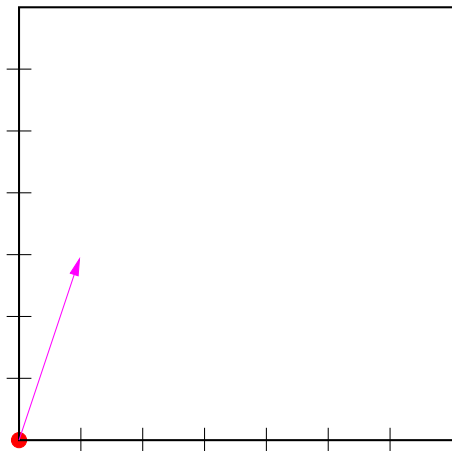
An example lattice



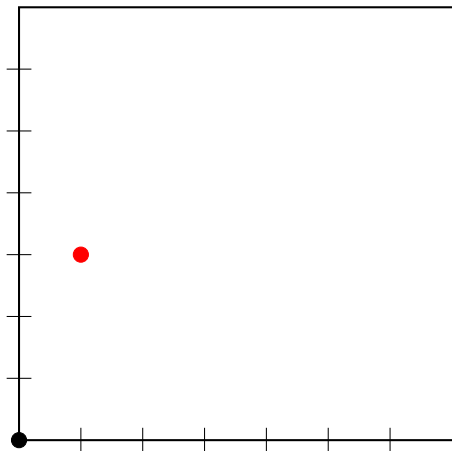
An example lattice



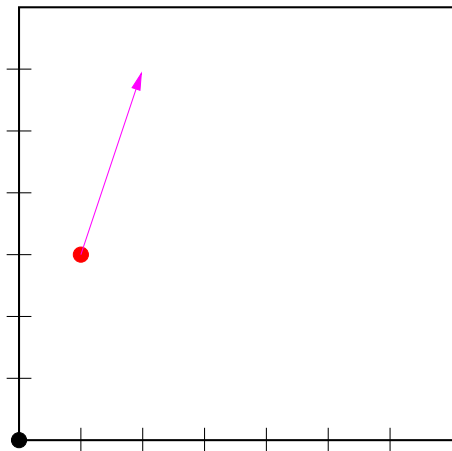
An example lattice



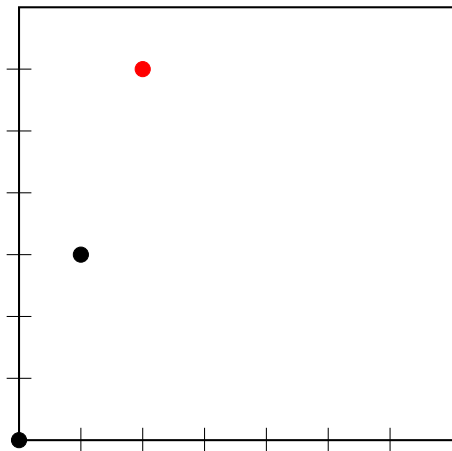
An example lattice



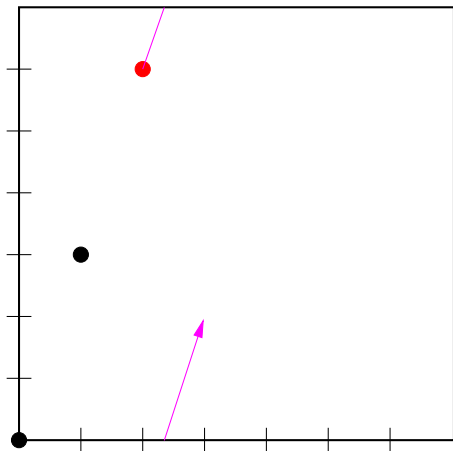
An example lattice



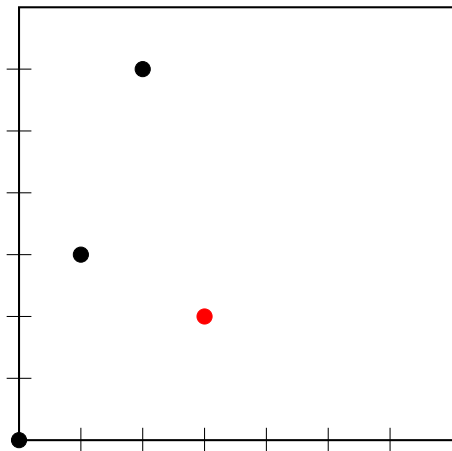
An example lattice



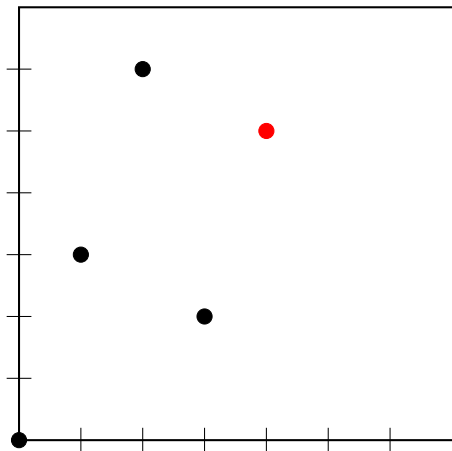
An example lattice



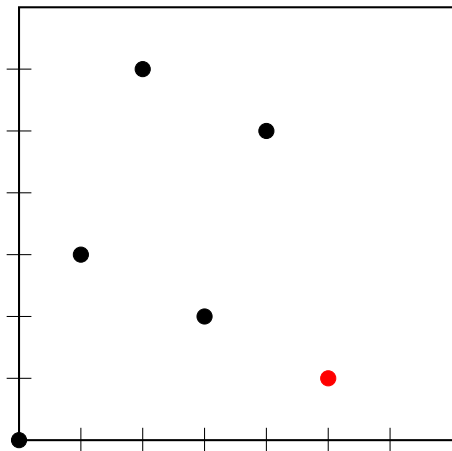
An example lattice



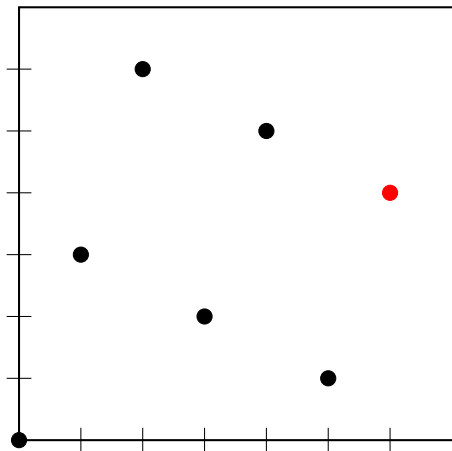
An example lattice



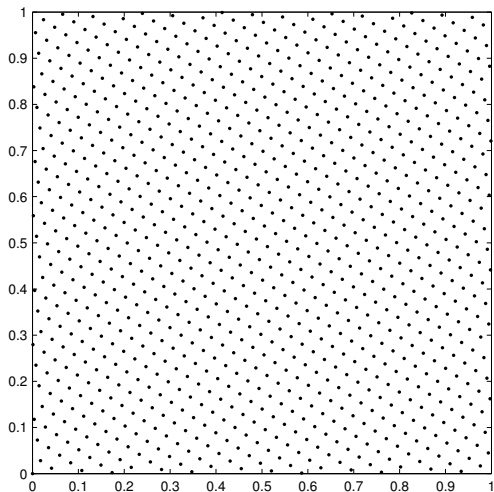
An example lattice



An example lattice



Lattice with 1031 points

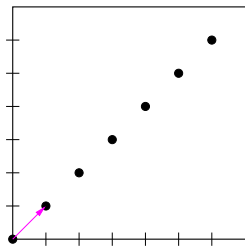


Lattice

- ▶ After N points the sequence repeats itself,
- ▶ Projection on each axe gives the set $\{\frac{0}{N}, \frac{1}{N}, \dots, \frac{N-1}{N}\}$.

Not every generator gives a good point set.

E.g. $g_1 = g_2 = \dots = g_d = 1$, gives $\{(\frac{i}{N}, \dots, \frac{i}{N})\}$.



Some Types of Quasirandom Numbers

1. Another interpretation of the v.d. Corput sequence:

- ▶ Define the i th ℓ -bit “direction number” as: $v_i = 2^i$ (think of this as a bit vector)
- ▶ Represent $n - 1$ via its base-2 representation
 $n - 1 = b_{\ell-1}b_{\ell-2} \dots b_1b_0$
- ▶ Thus we have

$$\Phi_2(n - 1) = 2^{-\ell} \bigoplus_{i=0, b_i=1}^{i=\ell-1} v_i$$

2. The Sobol' sequence works the same!!

- ▶ Use recursions with a primitive binary polynomial define the (dense) v_i
- ▶ The Sobol' sequence is defined as:

$$s_n = 2^{-\ell} \bigoplus_{i=0, b_i=1}^{i=\ell-1} v_i$$

- ▶ Use Gray-code ordering for speed

Some Types of Quasirandom Numbers

- ▶ (t, m, s) -nets and (t, s) -sequences and generalized Niederreiter sequences
1. Let $b \geq 2$, $s > 1$ and $0 \leq t \leq m \in \mathbb{Z}$ then a b -ary box, $J \subset [0, 1)^s$, is given by

$$J = \prod_{i=1}^s \left[\frac{a_i}{b^{d_i}}, \frac{a_i + 1}{b^{d_i}} \right)$$

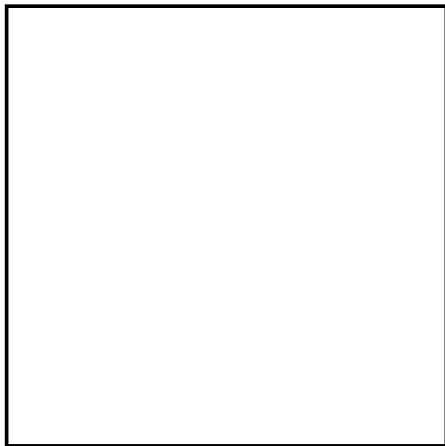
where $d_i \geq 0$ and the a_i are b -ary digits, note that $|J| = b^{-\sum_{i=1}^s d_i}$

Some Types of Quasirandom Numbers

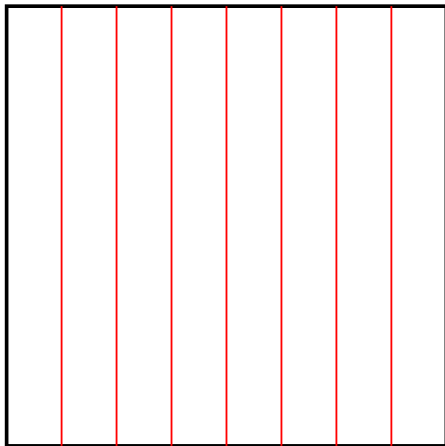
2. A set of b^m points is a (t, m, s) -net if each b -ary box of volume b^{t-m} has exactly b^t points in it
3. Such (t, m, s) -nets can be obtained via Generalized Niederreiter sequences, in dimension j of s :

$$y_i^{(j)}(n) = C^{(j)} a_i(n), \text{ where } n \text{ has the } b\text{-ary representation}$$
$$n = \sum_{k=0}^{\infty} a_k(n) b^k \text{ and } x_i^{(j)}(n) = \sum_{k=1}^m y_k^{(j)}(n) q^{-k}$$

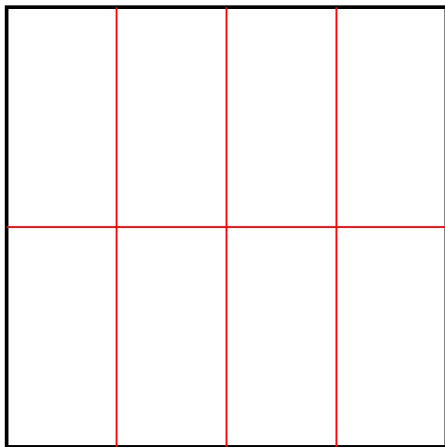
Nets: Example



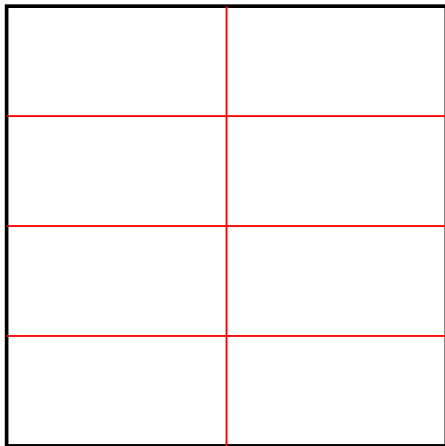
Nets: Example



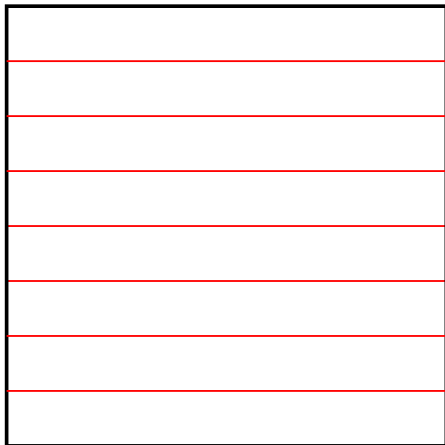
Nets: Example



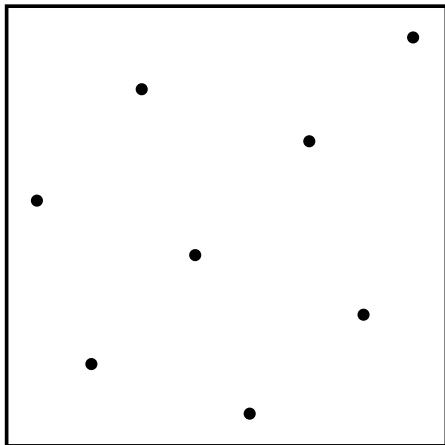
Nets: Example



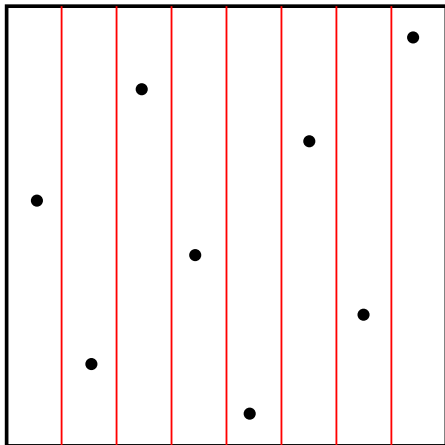
Nets: Example



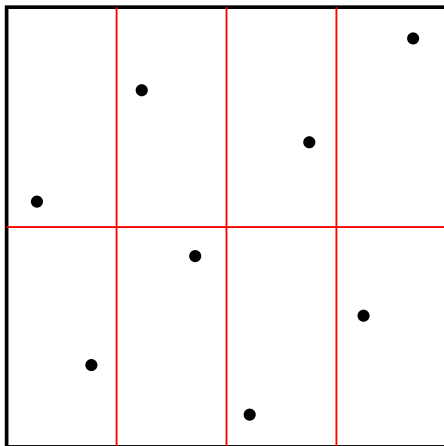
Nets: Example



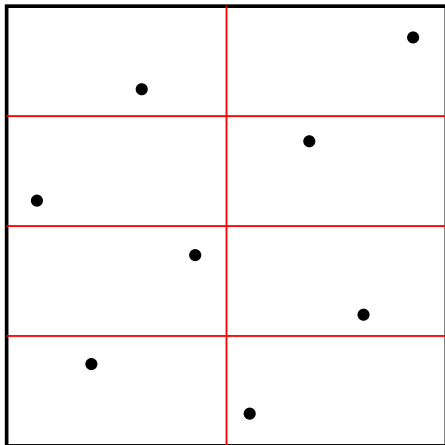
Nets: Example



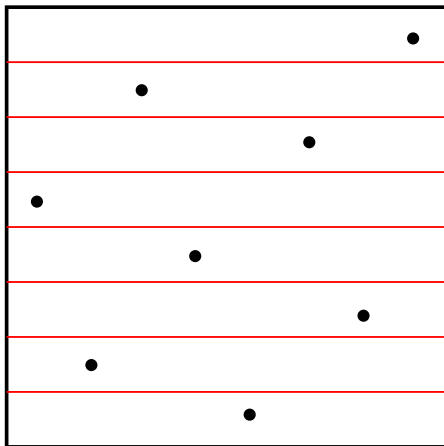
Nets: Example



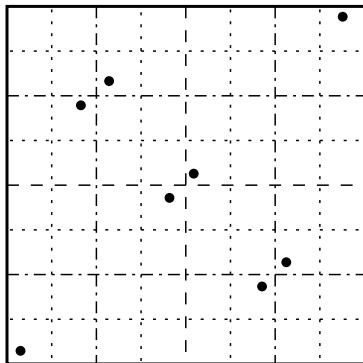
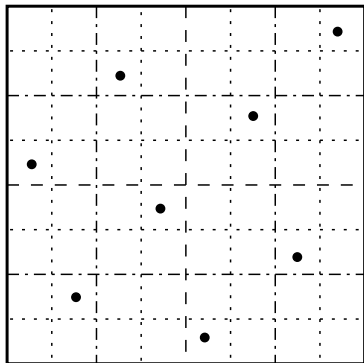
Nets: Example



Nets: Example



Good vs poor net



Randomization of the Faure Sequence

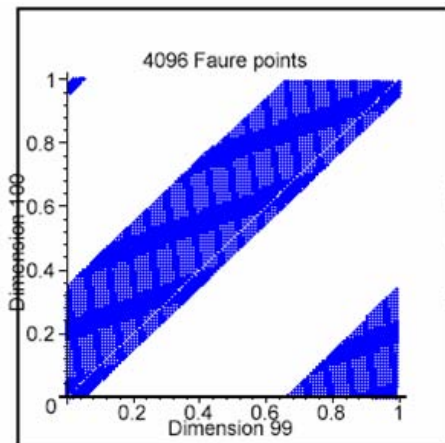
1. A problem with all QRNs is that the Koksma-Hlawka inequality provides no practical error estimate
2. A solution is to randomize the QRNs and then consider each randomized sequence as providing an independent sample for constructing confidence intervals
3. Consider the s -dimensional Faure series is:

$$(\phi_p(C^{(0)}(n)), \phi_p(C^{(1)}(n)), \dots, \phi_p(P^{s-1}(n)))$$
 - ▶ $p > s$ is prime
 - ▶ $C^{(j-1)}$ is the generator matrix for dimension $1 \leq j \leq s$
 - ▶ For Faure $C^{(j)} = P^{j-1}$ is the Pascal matrix:

$$P_{r,k}^{j-1} = \binom{r-1}{k-1} (j-1)^{(r-k)} \pmod{p}$$

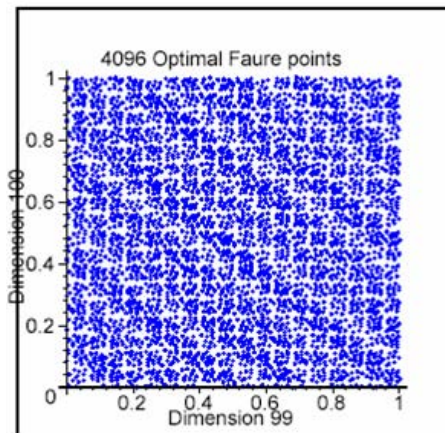
Another Reason for Randomization

QRNs have inherently bad low-dimensional projections



Another Reason for Randomization

Randomization (scrambling) helps



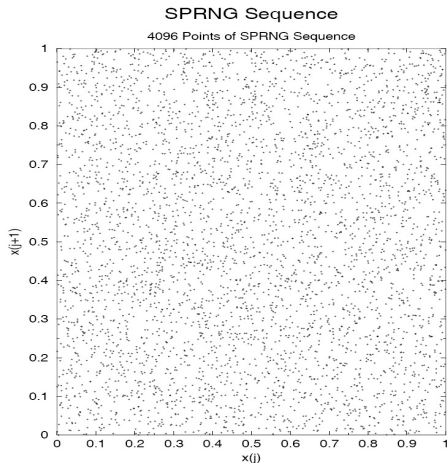
General Randomization Techniques

1. Random shifting: $z_n = x_n + r \pmod{1}$
 - ▶ $x_n \in [0, 1]^s$ is the original QRN
 - ▶ $r \in [0, 1]^s$ is a random point
 - ▶ $z_n \in [0, 1]^s$ scrambled point
2. Digit permutation
 - ▶ Nested scrambling (Owen)
 - ▶ Single digit scrambling like linear scrambling
3. Randomization of the generator matrices, i.e. Tezuka's GFaure , $C^{(j)} = A^{(j)} P^{j-1}$ where A^j is a random nonsingular lower-triangular matrix modulo p

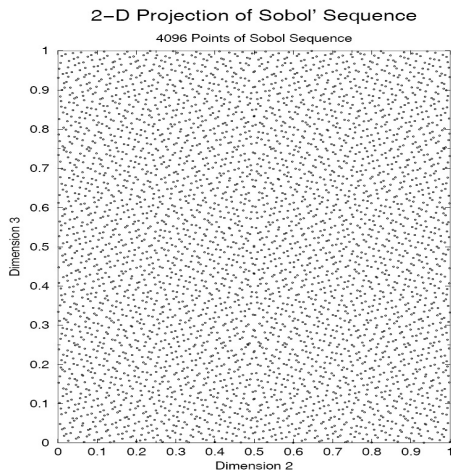
Derandomization and Applications

1. Given that a randomization leads to a family of QRNs, is there a best?
 - ▶ Must make the family small enough to exhaust over, so one uses a small family of permutations like the linear scramblings
 - ▶ There must be a quality criterion that is indicative and cheap to evaluate
2. Applications of randomization: tractable error bounds, parallel QRNs
3. Applications of derandomization: finding more rapidly converging families of QRNs

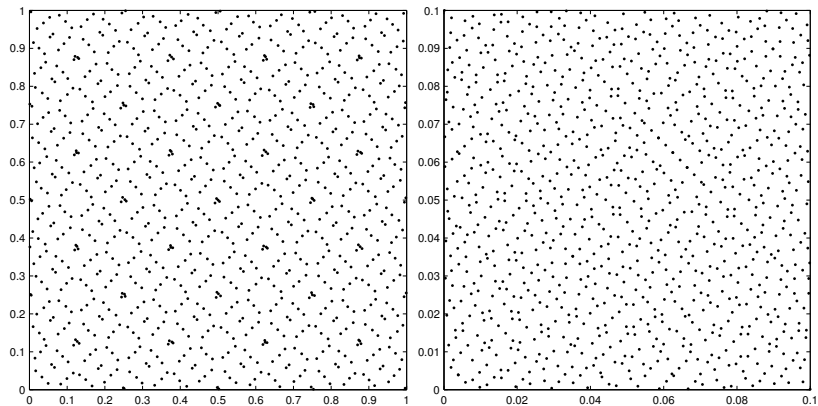
A Picture is Worth a 1000 Words: 4K Pseudorandom Pairs



A Picture is Worth a 1000 Words: 4K Quasirandom Pairs






Sobol' sequence





Conclusions and Future Work

1. SPRNG is at version 4 requires: some cleaning at the moment
2. Splitting SPRNG up
 - ▶ Base SPRNG in C++
 - ▶ Module for MPI
 - ▶ Module for Fortran users
 - ▶ Testing module
3. Other SPRNG updates
 - ▶ Spawn-intensive/small-memory footprint generators: MLFGs
 - ▶ “QPRNG”
 - ▶ Update test suite with TestU01

For Further Reading I

-  [Y. Li and M. Mascagni (2005)]
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Monte Carlo Methods and Applications, **11**: 39–55.
-  [H. Chi, M. Mascagni and T. Warnock (2005)]
On the Optimal Halton Sequence,
Mathematics and Computers in Simulation, **70(1)**: 9–21.
-  [M. Mascagni and H. Chi (2004)]
Parallel Linear Congruential Generators with
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Parallel Computing, **30**: 1217–1231.

For Further Reading II

-  [M. Mascagni and A. Srinivasan (2004)]
Parameterizing Parallel Multiplicative Lagged-Fibonacci
Generators,
Parallel Computing, **30**: 899–916.
-  [M. Mascagni and A. Srinivasan (2000)]
Algorithm 806: SPRNG: A Scalable Library for
Pseudorandom Number Generation,
ACM Transactions on Mathematical Software, **26**: 436–461.

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