Numerical Methods for Partial Differential Equations with Random Data

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Outline

I. Problem statement and discretization

- Example: diffusion equation with random diffusion coefficient
- Discretization by stochastic Galerkin method
- Discretization by stochastic collocation method

II. Solution algorithms

- Multigrid-style methods for various discretizations
- Comparison of solution costs for different discretizations

I. Stochastic Differential Equations with Random Data

Example: diffusion equation

$$-\nabla\cdot(a\nabla u) = f \quad \text{in } \mathcal{D} \subset R^d$$
$$u = g_D \text{ on } \partial\mathcal{D}_D, \quad (a\nabla u)\cdot n = 0 \text{ on } \partial\mathcal{D}_N = \partial\mathcal{D} \setminus \partial\mathcal{D}_D$$

Uncertainty / randomness:

$a = a(\mathrm{x},\omega)$ a *random field*

For each fixed x, $a(x,\omega)$ a random variable

Other possibly uncertain quantities :

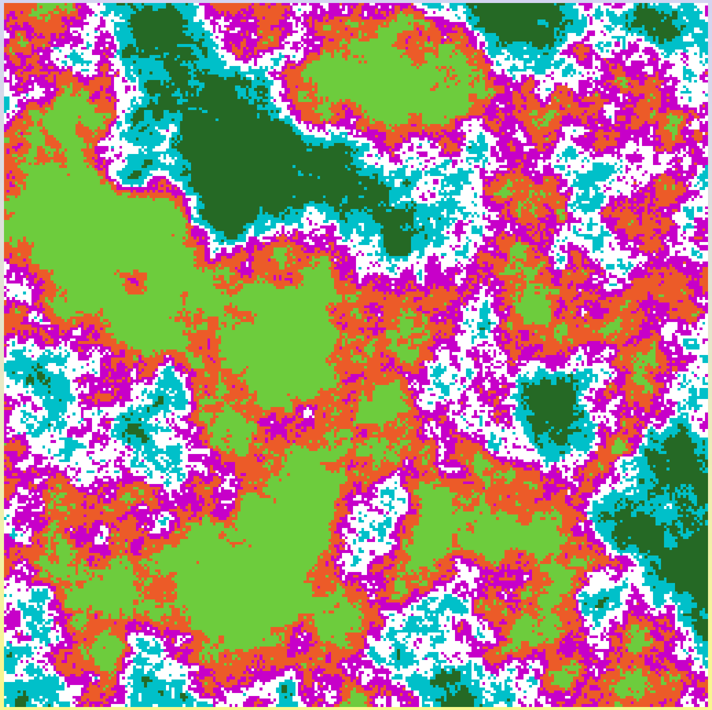
Forcing function f

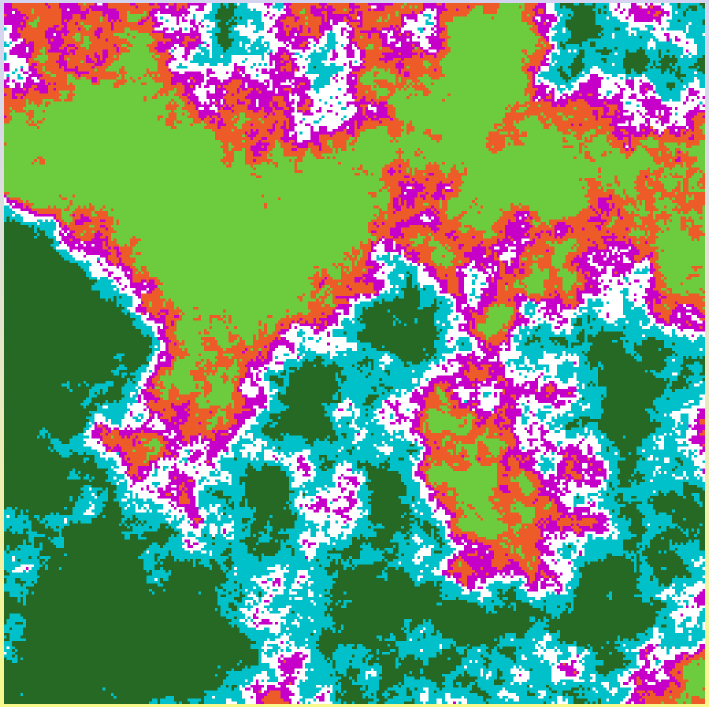
Boundary data g_D

Viscosity ν in Navier-Stokes equations

$$-\nu\,\nabla^2 u + (u\cdot \text{grad})u + \text{grad }p = f$$
$$-\text{div } u = 0$$

Depictions: Random Data on Unit Square





Diffusion Equation with Random Diffusion Coefficient

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } \mathcal{D}$$

Assumptions:

1. Spatial correlation of random field: For $x, y \in \mathcal{D}$:

   Random field $a(x,\omega)$

   Mean $\mu(x) = E(a(x,\cdot))$

   Variance $\sigma(x) = E(a(x,\cdot)^2) - \mu^2$

   Covariance function

   $$c(x,y) = E(\ (a(x,\cdot) - \mu(x))\ (a(y,\cdot) - \mu(y))\ )$$

   is finite

   vs. *white noise*, where c is a δ-function

2. Coercivity $0 < \alpha_1 \le a \le \alpha_2 < \infty$

   $\Longrightarrow$ Problem is well-posed

Monte-Carlo Simulation

Sample $a(x,\omega)$ at all $x \in \mathcal{D}$, solve in usual way

Standard weak formulation: find $u \in H^1_E(\mathcal{D})$ such that

$$a(u,v) = \ell(v)$$

for all $v \in H^1_{E_0}(\mathcal{D})$,

$$a(u,v) = \int_{\mathcal{D}} a \nabla u \cdot \nabla v \, dx, \qquad \ell(v) = \int_{\mathcal{D}} f \, v \, dx$$

Multiple realizations (samples) of $a(x,\cdot)$ $\longrightarrow$
Multiple realizations of u $\longrightarrow$
Statistical properties of u

Problem: convergence is slow, requires many solves

Another Point of View

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } \mathcal{D}$$

Covariance function is finite $\Longrightarrow$
random field (diffusion coefficient) has *Karhunen-Loève* expansion:

$$a(x,\omega) = a_0(x) + \sigma \sum_{r=1}^{\infty} \sqrt{\lambda_r}\, a_r(x) \xi_r(\omega)$$

$$a_0(x) = \mu(x) = E(a(x,\cdot)) \quad \text{mean}$$

$a_r(x), \lambda_r =$ eigenfunctions/eigenvalues of covariance operator

$$(Ca)(x) = \lambda a(x), \qquad (Ca)(x) = \int_{\mathcal{D}} c(x,y) a(y)\, dy$$

$\xi_r(\omega)$ = identically distributed uncorrelated random variables with mean 0 and variance 1

Finite Noise Assumption

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } \mathcal{D}$$

Truncated *Karhunen-Loève* expansion:

$$a(x,\omega) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x) \xi_r(\omega)$$

~ Principal components analysis

Requires: m large enough so that the fluctuation of a is well-represented, i.e. λ_{m+1}/λ_1 is small

More precisely: error from truncation is $\dfrac{|\mathcal{D}|\sigma^2 - \sum_{j=1}^{m} \lambda_j}{|\mathcal{D}|\sigma^2}$

Choose m to make this small

Various Ways to Use This

$$a(x,\omega) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r(\omega)$$

1. **Stochastic Finite Element (Galerkin) Method:**
   Introduce a weak formulation analogous to finite elements in space that handles the "stochastic" component of the problem

2. **Stochastic Collocation Method:**
   Devise a special strategy for sampling $\underline{\xi}$ that converges more quickly than Monte Carlo simulation; derived from interpolation

Ghanem, Spanos, Babuška, Deb, Oden, Matthies, Keese, Karniadakis, Xiu, Hesthaven, Tempone, Nobile, Webster, Schwab, Todor, Ernst, Powell, Furnival, E., Ullmann, Rosseel, Vandewalle

Stochastic Finite Element (Stochastic Galerkin) Method

Probability space $(\Omega, \mathcal{F}, P)$

$$L_P^2(\Omega) \equiv \{\text{square integrable functions wrt } dP(\omega)\}$$

Inner product on $L_P^2(\Omega): \langle v, w \rangle = E(vw) = \int_\Omega v(\omega) w(\omega) dP(\omega)$

Use to concoct weak formulation on product space $H_E^1(\mathcal{D}) \otimes L_P^2(\Omega)$

Find $u \in H_E^1(\mathcal{D}) \otimes L_P^2(\Omega)$ such that

$$\langle a(u,v) \rangle = \langle \ell(v) \rangle \qquad \int_\Omega \int_D a \nabla u \cdot \nabla v \, dx \, dP(\omega)$$

for all $v \in H_{E_0}^1(\mathcal{D}) \otimes L_P^2(\Omega)$

Solution $\boldsymbol{u=u(x,\omega)}$ is itself a random field

For Computation: Return to Finite Noise Assumption

Truncated Karhunen-Loève expansion

$$a(x,\xi(\omega)) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r(\omega)$$

Stochastic weak formulation uses

$$\langle a(u,v) \rangle = \int_{\Omega}\int_{D} a \nabla u \cdot \nabla v \, dx \, dP(\omega) = \int_{\xi(\Omega)}\int_{D} a(x,\underline{\xi}) \nabla u \cdot \nabla v \, dx \, \rho(\underline{\xi}) d\underline{\xi}$$

Bilinear form entails integral over *image* of random variables $\underline{\xi}$

Require joint density function associated with $\underline{\xi}$

$\underline{\xi}$ plays the role of a Cartesian coordinate

Statement of Problem Becomes

Find $u \in H_E^1(\mathcal{D}) \otimes L_P^2(\Gamma)$ such that

$$\int_\Gamma \int_{\mathcal{D}} a(x,\underline{\xi}) \nabla u \cdot \nabla v \, dx \, \rho(\underline{\xi}) d\underline{\xi} = \int_\Gamma \int_{\mathcal{D}} f v \, dx \, \rho(\underline{\xi}) d\underline{\xi}$$

for all $v \in H_{E_0}^1(\mathcal{D}) \otimes L_P^2(\Gamma) \quad (\Gamma = \underline{\xi}(\Omega))$

Like an ordinary Galerkin (or Petrov-Galerkin) problem on a (d+m)-dimensional "continuous" space

d = dimension of spatial domain
m = dimension of stochastic space

Discretization

$$\int_{\Gamma}\int_{\mathcal{D}} a(x,\underline{\xi})\nabla u\cdot\nabla v\,dx\,\rho(\underline{\xi})d\underline{\xi} = \int_{\Gamma}\int_{\mathcal{D}} f\,v\,dx\,\rho(\underline{\xi})d\underline{\xi}$$

Finite dimensional spaces:

- spatial discretization: $S_h \subset H_0^1(\mathcal{D})$, spanned by $\{\varphi_j\}_{j=1}^{N_x}$

  for example: piecewise linear on triangles

- stochastic discretization: $T_p \subset L^2(\Gamma)$, spanned by $\{\psi_l\}_{l=1}^{N_\xi}$

  for example: polynomial chaos = m-variate Hermite polynomials (orthogonal wrt Gaussian measure)

Discrete weak formulation:

$$a(u_{hp}, v_{hp}) = \ell(v_{hp}) \quad \text{for all } v_{hp} \in S_h \otimes T_p$$

$$u_{hp} = \sum_{j=1}^{N_x}\sum_{l=1}^{N_\xi} u_{jl}\varphi_j(x)\psi_l(\xi)$$

Basis Functions for Stochastic Space

Underlying space: $L^2(\Gamma) = \left\{ \mathrm{v}(\underline{\xi}) \,\middle|\, \int_\Gamma \mathrm{v}(\underline{\xi})^2 \rho(\underline{\xi})\, d\underline{\xi} < \infty \right\}$

$$\rho(\underline{\xi}) = \rho_1(\xi_1)\rho_2(\xi_2)\cdots\rho_M(\xi_M)$$

Let $q_j^{(k)}(\xi_k) =$ polynomial of degree *j* orthogonal wrt ρ_k

Examples: if ρ_k ~ *Gaussian measure* ⟶ Hermite polynomials

ρ_k ~ *uniform distribution* ⟶ Legendre polynomials

Any ρ_k can be handled computationally (Gautschi)

⟶ Rys polynomials

$T_p \subset L^2(\Gamma)$ spanned by $\{q_{j_1}^{(1)}(\xi_1) q_{j_2}^{(2)}(\xi_2) \cdots q_{j_m}^{(m)}(\xi_m)\}$

Orthogonality of basis functions ⟶ sparsity of coefficient matrix

Matrix Equation *Au=f*

$$a(x,\xi(\omega)) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r(\omega)$$

$$A = G_0 \otimes A_0 + \sum_{r=1}^{m} G_r \otimes A_r$$

$$[A_0]_{jk} = \int_{\mathcal{D}} a_0(x) \nabla \varphi_j(x) \cdot \nabla \varphi_k(x)\, dx$$

$$[A_r]_{jk} = \sqrt{\lambda_r} \int_{\mathcal{D}} \sigma(x) a_r(x) \nabla \varphi_j(x) \cdot \nabla \varphi_k(x)\, dx$$

$$[G_0]_{lq} = \langle \psi_l, \psi_q \rangle, \quad [G_r]_{lq} = \langle \xi_r \psi_l, \psi_q \rangle$$

$$[f]_{kq} = \int_{\Gamma} \int_{\mathcal{D}} f(x,\xi) \varphi_k(x) \psi_q(\xi)\, dx\, \rho(\xi) d\xi$$

Properties of *A*:

- order = N_x *x* N_ξ = (size of spatial basis) *x* (size of stochastic basis)
- sparsity: inherited from that of $\{G_r\}$ and $\{A_r\}$

Dimensions of Discrete Stochastic Space

$$T_p \subset L^2(\Gamma) \text{ spanned by } \{q_{j_1}^{(1)}(\xi_1)q_{j_2}^{(2)}(\xi_2)\cdots q_{j_m}^{(m)}(\xi_m)\}$$

Full tensor product basis: $0 \le j_i \le p, \quad i = 1,\ldots,m$

Dimension: $(p+1)^m$ Too large

"Complete" polynomial basis: $j_1 + j_2 + \cdots + j_m \le p$

Dimension: $\binom{m+p}{p} = \dfrac{(m+p)!}{m!\,p!}$ More manageable

Order these in a systematic way $\longrightarrow$

$$\psi_1(\underline{\xi}), \psi_2(\underline{\xi}), \ldots, \psi_{N_\xi}(\underline{\xi})$$

Example

$T_p \subset L^2(\Gamma)$ spanned by $\{q_{j_1}^{(1)}(\xi_1) q_{j_2}^{(2)}(\xi_2) \cdots q_{j_m}^{(m)}(\xi_m)\}$

"Complete" polynomial basis: $j_1 + j_2 + \cdots + j_m \leq p$

$$m=2,\ p=3 \longrightarrow \binom{m+p}{p} = \binom{5}{2} = 10$$

Orthogonal (Hermite) polynomials in 1D:

$$H_0(\xi) = 1,\ H_1(\xi) = \xi,\ H_2(\xi) = \xi^2 - 1,\ H_3(\xi) = \xi^3 - 3\xi$$

Gives basis set:

$$\psi_1(\underline{\xi}) = 1 \qquad \psi_6(\underline{\xi}) = \xi_1 \xi_2$$

$$\psi_2(\underline{\xi}) = \xi_1 \qquad \psi_7(\underline{\xi}) = (\xi_1^2 - 1)\xi_2$$

$$\psi_3(\underline{\xi}) = \xi_1^2 - 1 \qquad \psi_8(\underline{\xi}) = (\xi_2^2 - 1)$$

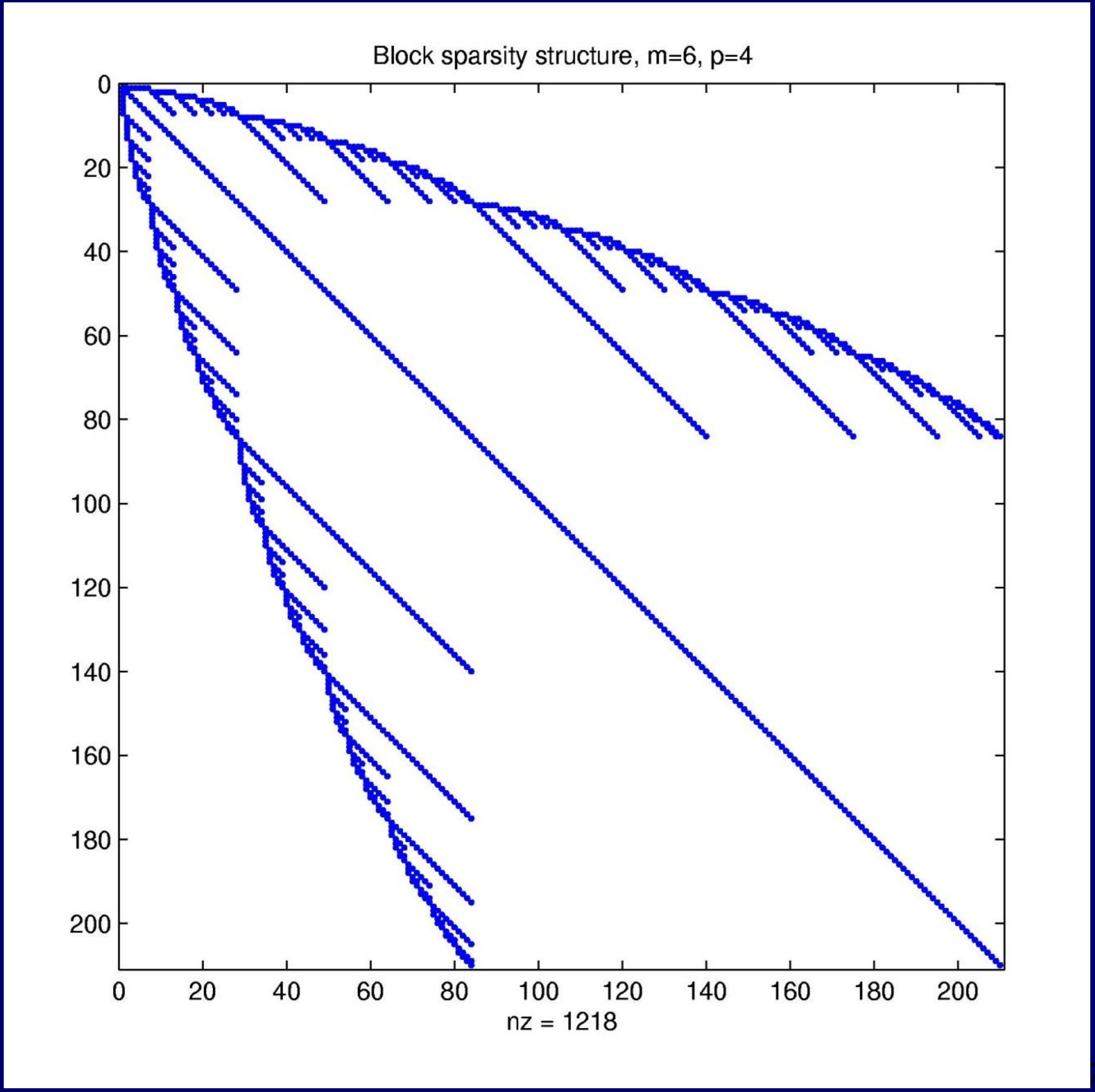
$$\psi_4(\underline{\xi}) = \xi_1^3 - 3\xi_1 \qquad \psi_9(\underline{\xi}) = (\xi_2^2 - 1)\xi_1$$

$$\psi_5(\underline{\xi}) = \xi_2 \qquad \psi_{10}(\underline{\xi}) = \xi_2^3 - 3\xi_2$$

Example of Sparsity Pattern

For m-variate polynomials of total degree p:

$$N_\xi = \frac{(m+p)!}{m!p!} = \frac{10!}{6!4!} = 210$$



Uses of the Computed Solution:

$$u_{hp} = \sum_{l=1}^{N_\xi} \underbrace{\sum_{j=1}^{N_x} u_{jl}\varphi_j(x)}_{u_l(x)} \psi_l(\underline{\xi}) = \sum_{l=1}^{N_\xi} u_l(x)\psi_l(\underline{\xi})$$

1. **Moments:** First moment of *u* (expected value):

$$E(u_{hp}) = \sum_{l=1}^{M} u_l(x) \int_\Gamma \psi_l(\underline{\xi})\rho(\underline{\xi})\, d\underline{\xi}$$

Free!

$$= u_1(x) = \sum_{j=1}^{N} u_{j1}\varphi_j(x)$$

using orthogonality of stochastic basis functions

Similarly for second moment / covariance

Uses of the Computed Solution:

$$u_{hp} = \underbrace{\sum_{l=1}^{N_\xi} \sum_{j=1}^{N_x} u_{jl} \varphi_j(x)}_{u_l(x)} \psi_l(\underline{\xi}) = \sum_{l=1}^{N_\xi} u_l(x) \psi_l(\underline{\xi})$$

2. **Cumulative distribution functions**

E.g.: $P(u_{hp}(x,\xi) > \alpha)$ at some point x

Sample $\underline{\xi}$

Evaluate $u_{hp}(x,\underline{\xi}) = \sum_{l=1}^{N_\xi} u_l(x)\, \psi_l(\underline{\xi})$

$u_l(x)$ → Precomputed

Repeat

Not free, but **<u>no solves required</u>**

Stochastic Collocation Method

Given $a(x,\xi) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r$ as above

Let $\underline{\xi}$ be a specified realization (~ Monte Carlo) $\longrightarrow$

Weak formulation:

$$\int_{\mathcal{D}} \left(a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r\right) \nabla u \cdot \nabla v \, dx = \int_{\mathcal{D}} f\, v\, dx$$

Discretize in space in usual way.

Stochastic collocation: choose special set $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(N_\xi)}$
from considerations of interpolation

Advantage: Spatial systems are decoupled

Multi-Dimensional Interpolation

Given $\underline{\xi}^{(1)}, \underline{\xi}^{(2)}, \ldots, \underline{\xi}^{(N_\xi)}$, and $v(\underline{\xi})$, consider an interpolant

$$(Iv)(\underline{\xi}) \equiv \sum_{k=1}^{N_\xi} v(\underline{\xi}^{(k)}) L_k(\underline{\xi}) \approx v(\underline{\xi}),$$

where $L_k(\underline{\xi}^{(j)}) = \delta_{jk}$, Lagrange interpolating polynomial

If $u_h^{(k)}$ solves the discrete (in space) version of

$$\int_{\mathcal{D}} (a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x) \xi_r) \nabla u \cdot \nabla v \, dx = \int_{\mathcal{D}} f\, v\, dx$$

with $\underline{\xi} = \underline{\xi}^{(k)}$, then the ***collocated*** solution is

$$u_{hp}(x, \underline{\xi}) = \sum_{k=1}^{N_\xi} u_h^{(k)}(x) L_k(\underline{\xi})$$

To Compute Statistical Quantities

Solution $$u_{hp}(x,\underline{\xi}) = \sum_{k=1}^{N_\xi} u_h^{(k)}(x) L_k(\underline{\xi})$$

1. Moments

$$E(u_{hp})(x) = \sum_{k=1}^{N_\xi} u_h^{(k)}(x) \underbrace{\int_\Gamma L_k(\underline{\xi})\rho(\underline{\xi})d\underline{\xi}}$$

Not free but can be precomputed

2. Distribution functions

Obtained by sampling, cheap

Strategy for Interpolation

$$(Iv)(\underline{\xi}) \equiv \sum_{k=1}^{N_\xi} v(\underline{\xi}^{(k)}) L_k(\underline{\xi}) \approx v(\underline{\xi}),$$

One choice of $\{L_k\}$: $\quad L_k(\underline{\xi}) = \ell_{k_1}(\xi_1)\ell_{k_2}(\xi_2)\cdots\ell_{k_m}(\xi_m)$

$\ell_{k_j} =$ 1D interpolating polynomial

$0 \le k_j \le p$

Advantage: easy to construct

Disadvantage: "curse of dimensionality,"
dimension = $(p+1)^m$

Detour: Sparse Grids

Given: 1D interpolation rule $(U^{(k)}v)(y^{(k)}) = \sum_{j=1}^{m_k} v(y_j^{(k)})\ell_j(y^{(k)})$

Derived from (1D) grid $Y^{(k)} = \{y_1^{(k)}, \ldots, y_{m_k}^{(k)}\}$

Multidimensional rule above is induced by ***fully populated*** multidimensional grid $Y^{(1)} \times Y^{(2)} \times \cdots \times Y^{(m)}$.

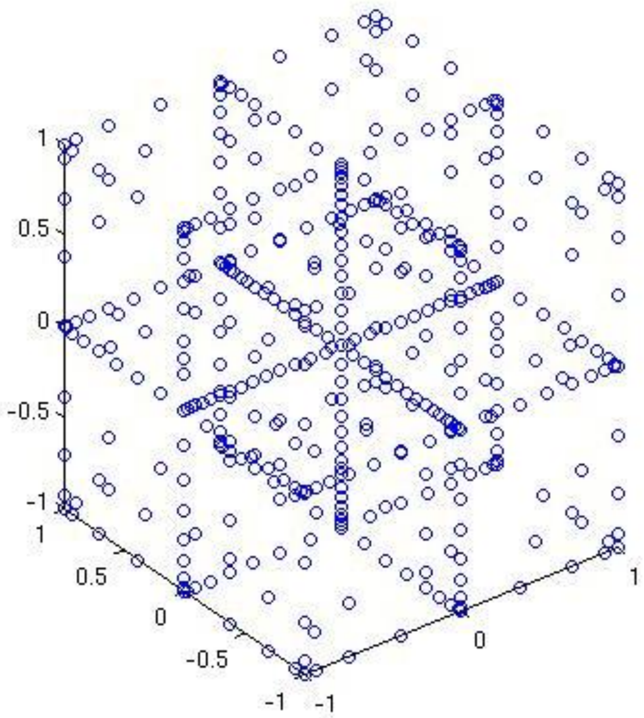
$$|Y^{(k)}| = m_k = p + 1$$

Alternative: multidimensional ***sparse grid*** (Smolyak)

$$\mathcal{H}(m+p, m) \equiv \bigcup_{p-m+1 \le i_1 + \cdots + i_m \le p} (Y^{(i_1)} \times Y^{(i_2)} \times \cdots \times Y^{(i_m)})$$

Sparse Grid Interpolation

Example of sparse grid for m=3, p=16



For v of the form $v(\underline{\xi}) = v_1(\xi_1)v_2(\xi_2)\cdots v_m(\xi_m)$, interpolating function takes the form

$$(\mathcal{I}v)(\underline{\xi}) = \sum_{i_1+\cdots+i_m \le p} (U^{(i_1)} - U^{(i_1-1)})v_1(\xi_1) \otimes (U^{(i_2)} - U^{(i_2-1)})v_2(\xi_2) \otimes \cdots \otimes (U^{(i_m)} - U^{(i_m-1)})v_m(\xi_m)$$

Sparse Grid Interpolation

Theorem (Novak, Ritter, Wasilkowski, Wozniakowski)

For $\underline{\xi} \in$ sparse grid and $v(\underline{\xi})$ a tensor product polynomial of total degree at most *p*,

$$v(\underline{\xi}) = q^{(1)}_{j_1}(\xi_1) q^{(2)}_{j_2}(\xi_2) \cdots q^{(m)}_{j_m}(\xi_m), \qquad j_1 + j_2 + \cdots + j_m \le p$$

$$(\mathcal{I}v)(\underline{\xi}) = v(\underline{\xi}).$$

That is: sparse grid interpolation evaluates the set of complete *m*-variate polynomials exactly

Overhead: number of sparse grid points to achieve this (= # stochastic dof) is larger than for Galerkin

$$\approx 2^{p} \binom{m+p}{p} \quad \text{vs.} \quad \binom{m+p}{p}$$

Analysis (Babuška, Tempone, Zouraris, Nobile, Webster)

Monte-Carlo:

$$E(u) - E_s(u_h) = \underbrace{(E(u) - E(u_h))}_{\le c_1 h E(|u|_2)} + \underbrace{(E(u_h) - E_s(u_h))}_{\sim 1/\sqrt{s}}$$

Convergence is slow wrt number of samples but independent of number of random variables m

Stochastic Galerkin and Collocation:

$$E(u) - E(u_{hp}) = \underbrace{(E(u) - E(u_h))}_{\le c_1 h E(|u|_2)} + \underbrace{(E(u_h) - E(u_{hp}))}_{\le c_2 r^p,\ r<1}$$

Exponential in polynomial degree p

Constants (c_2 , r) depend on m

Rule of thumb: the same p gives the same error (for all versions of SG and collocation)

More dof for collocation than SG

Recapitulating

Monte-Carlo methods:

Many samples needed for statistical quantities

Many systems to solve

Systems are independent

Statistical quantities are free (once data is accumulated)

With ***s*** realizations: $E_s(u_h) = \frac{1}{s}\sum_{r=1}^{s} u_h^{(r)}(x)$

Convergence is slow but independent of m

Stochastic Galerkin methods:

One large system to solve

Statistical quantities are free or (relatively) cheap

Stochastic collocation methods:

Systems are independent

Fewer systems than Monte Carlo

More degrees of freedom than Galerkin

Statistical quantities are (relatively) cheap

Similar convergence behavior

Faster than MC

Depends on m

II. Computing with the Stochastic Galerkin and Collocation Methods

For both: compute a discrete solution, a random field $u_{hp}(x,\underline{\xi})$

Stochastic Galerkin:

$$u_{hp}(x,\underline{\xi}) = \sum_{l=1}^{N_\xi} \sum_{j=1}^{N_x} u_{jl} \varphi_j(x) \psi_l(\underline{\xi}) = \sum_{l=1}^{N_\xi} u_l(x) \psi_l(\underline{\xi})$$

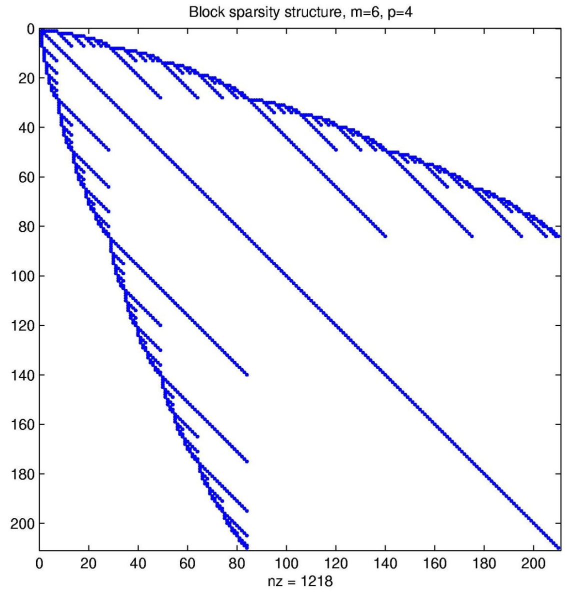
Stochastic Collocation:

$$u_{hp}(x,\underline{\xi}) = \sum_{l=1}^{N_\xi} \sum_{j=1}^{N_x} u_{jl} \varphi_j(x) L_l(\underline{\xi}) = \sum_{l=1}^{N_\xi} u_l(x) L_l(\underline{\xi})$$

Postprocess to get statistics

Computational Issues

Stochastic Galerkin: Solve one large system of order $N_x \times N_\xi$



$$N_\xi = \binom{m+p}{p}$$

Frequently cited as a problem for this methodology

Stochastic Collocation: Solve N_ξ "ordinary" algebraic systems (of order N_x), one for each sparse grid point

Here: $N_\xi^{(collocation)} \sim 2^p N_\xi^{(Galerkin)}$

Some savings possible

Multigrid Solution of Matrix Equation I (E. & Furnival)

Solving **$Au=f$**

$$A = G_0 \otimes A_0 + \sum_{r=1}^{m} G_r \otimes A_r$$

$$[A_r]_{jk} = \sqrt{\lambda_r}\,\sigma \int_D a_r(x)\nabla\varphi_j(x)\cdot\nabla\varphi_k(x)\,dx,$$

$$[G_r]_{lq} = \int_\Omega \psi_l(\xi)\psi_q(\xi)\xi_r\rho(\xi)\,d\xi$$

$A_r = A_r^{(h)}, \quad A = A^{(h)},$ spatial discretization parameter h

$A_r = A_r^{(2h)}, \quad A = A^{(2h)},$ spatial discretization parameter $2h$

Develop MG algorithm for spatial component of the problem

Multigrid Algorithm (Two-grid)

Let $A^{(h)} = Q - N,$ $Q =$ smoothing operator

for *i=0,1,…*

for *j=1:k* k smoothing steps

$$u^{(h)} \leftarrow (I - Q^{-1}A^{(h)})u^{(h)} + Q^{-1}f^{(h)}$$

end

$r^{(2h)} = \mathcal{R}(f^{(h)} - A^{(h)}u^{(h)})$ Restriction

Solve $A^{(2h)}c^{(2h)} = r^{(2h)}$ Coarse grid correction

$u^{(h)} \leftarrow u^{(h)} + \mathcal{P}c^{(2h)}$ Prolongation

end

Prolongation and restriction:

$\mathcal{P} = I \otimes P,$ induced by natural inclusion in spatial domain

$\mathcal{R} = \mathcal{P}^T = I \otimes R, \quad R = P^T$

Convergence Analysis: Use "Standard" Approach

Error propagation matrix:

$$e^{(i+1)} = [(A^{(h)})^{-1} - \mathcal{P}(A^{(2h)})^{-1}\mathcal{R})]\,[A^{(h)}(I - Q^{-1}A^{(h)})^{k}\,]\,e^{(i)}$$

Establish ***approximation property***

$$\left\| [(A^{(h)})^{-1} - \mathcal{P}(A^{(2h)})^{-1}\mathcal{R}]\,y \right\|_{A^{(h)}} \le C\|y\|_2 \quad \forall y$$

and ***smoothing property***

$$\left\| [A^{(h)}(I - Q^{-1}A^{(h)})^{k}\,]\,y \right\|_2 \le \eta(k)\|y\|_{A^{(h)}} \quad \forall y, \quad \eta(k) \xrightarrow{k\ \text{increases}} 0$$

Analysis is:

$$\begin{aligned}\| e^{(i+1)} \|_{A^{(h)}} &\le \| (A^{(h)})^{-1} - \mathcal{P}(A^{(2h)})^{-1}\mathcal{R})][A^{(h)}(I - Q^{-1}A^{(h)})^{k}\,]\,e^{(i)} \|_{A^{(h)}} \\ &\le C \| [A^{(h)}(I - Q^{-1}A^{(h)})^{k}\,]\,e^{(i)} \|_2 \\ &\le C\eta(k) \| e^{(i)} \|_{A^{(h)}}\end{aligned}$$

Approximation Property

"Standard" MG analysis for deterministic problem:

$$\left\| [(A^{(h)})^{-1} - \mathcal{P}(A^{(2h)})^{-1}\mathcal{R}]\, y \right\|_{A^{(h)}} = \left\| u^{(h)} - u^{(2h)} \right\|_{A^{(h)}}$$

$$= \left\| u_h - u_{2h} \right\|_a \quad (= a(u_h - u_{2h}, u_h - u_{2h})^{1/2})$$

$$\leq \left\| u_h - u \right\|_a + \left\| u - u_{2h} \right\|_a$$

Approximability

$$\leq \sqrt{\alpha_2}\left(Ch \left\| D^2 u \right\|_{L^2(\mathcal{D})} + C2h \left\| D^2 u \right\|_{L^2(\mathcal{D})} \right)$$

Regularity

$$\leq Ch \left\| f \right\|_{L^2(\mathcal{D})}$$

Property of mass matrix

$$\leq C \left\| y \right\|_2$$

For Approximation Property in Stochastic Case

Introduce *semi-discrete* space $H_0^1(\mathcal{D}) \otimes \underbrace{T_p}$ → Discrete stochastic space

Weak formulation

$$a(u_p, v_p) = \ell(v_p) \quad \text{for all } v_p \in H_0^1(\mathcal{D}) \otimes T_p$$

Solution u_p

$$\left\| [(A^{(h)})^{-1} - \mathcal{P}(A^{(2h)})^{-1}\mathcal{R}]\, y \right\|_{A^{(h)}} = \left\| u_{hp} - u_{2h,p} \right\|_a$$

$$\leq \left\| u_h - u_p \right\|_a + \left\| u_p - u_{2h} \right\|_a$$

Approximation (in 2D):

$$\left\| u_p - u_{hp} \right\|_a \leq Ch \left\| D^2 u_p \right\|_{L^2(\mathcal{D}) \otimes L^2(\Gamma)}$$

Established using best approximation property of u_{hp}
and interpolant $\tilde{u}_p(x_j, \xi) = u_p(x_j, \xi) \;\; \forall \xi$

Similarly for other steps used for deterministic analysis

Comments

- Establishes convergence of multigrid with rate independent of spatial discretization size *h*
- No dependence on stochastic parameters *m*, *p*
- Applies to any basis of stochastic space
- Coarse grid operator: $G = a_0 G_0 + \sigma \sum_{r=1}^{m} a_r \sqrt{\lambda_r} G_r,$ size O(N_ξ)

  G_r derives from basis of multivariate polynomials of total degree *p*, orthogonal wrt probability measure $\rho(\xi)d\xi$

  Maximum eigenvalue η = max root of orthogonal polynomial, bounded for bounded measure

  $$\Rightarrow 0 < a_0^{1x1} - \sigma\eta\left(\sum_{r=1}^{m} a_r^{1x1}\sqrt{\lambda_r}\right) \le \lambda(G) \le a_0^{1x1} + \sigma\eta\left(\sum_{r=1}^{m} a_r^{1x1}\sqrt{\lambda_r}\right),$$

  CG iteration is an option

Iteration Counts / Normal Distribution

terms (m) in KL-expansion

h=1/16

Polynomial degree

	m=1	m=2	m=3	m=4
p=1	8	8	8	8
p=2	8	8	8	8
p=3	9	9	9	9
p=4	9	10	10	10

h=1/32

Polynomial degree

	m=1	m=2	m=3	m=4
p=1	7	7	8	8
p=2	8	8	8	8
p=3	8	8	9	9
p=4	9	9	9	9

Multigrid Solution of Matrix Equation II

Solving ***Au=f***

$$A = G_0 \otimes A_0 + \sum_{r=1}^{m} G_r \otimes A_r$$

$$[A_r]_{jk} = \sqrt{\lambda_r}\, \sigma \int_{\mathcal{D}} a_r(x) \nabla \varphi_j(x) \cdot \nabla \varphi_k(x)\, dx,$$

$$[G_r]_{lq} = \int_{\Omega} \psi_l(\xi) \psi_q(\xi) \xi_r \rho(\xi)\, d\xi$$

Preconditioner for use with CG: $Q = G_0 \otimes A_0$ (Kruger, Pellissetti, Ghanem)

$$A_0 \sim \int_{\mathcal{D}} a_0(x) \nabla \varphi_j(x) \cdot \nabla \varphi_k(x)\, dx$$

Deterministic diffusion, from mean

$$G_0 = I$$

Analysis (Powell & E.)

Recall $a(x,\omega) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r(\omega)$

$$\longrightarrow \quad A = G_0 \otimes A_0 + \sum_{r=1}^{\mathrm{m}} G_r \otimes A_r$$

$$Q = G_0 \otimes A_0$$

Theorem : For μ constant, the Rayleigh quotient satisfies

$$1 - \tau \le \frac{(w, Aw)}{(w, Qw)} \le 1 + \tau$$

$$\tau = (\sigma/\mu)\, c(p) \sum_{r=1}^{m} \sqrt{\lambda_r}\, \| a_r \|_\infty$$

Consequence: $\kappa \le \frac{1+\tau}{1-\tau}$ dictates convergence of PCG

Sketch of Proof

$$\tau = (\sigma/\mu)\, c(p) \sum_{r=1}^{m} \sqrt{\lambda_r}\, \| a_r \|_\infty$$

$$A = G_0 \otimes A_0 + \sum_{\mathrm{r=1}}^{\mathrm{m}} G_r \otimes A_r$$

In spatial domain:

$$(\varphi, A_r \varphi) \sim \sigma \sqrt{\lambda_r} \int_D a_r(x) \nabla \varphi(x) \cdot \nabla \varphi(x)\, dx$$

$$\leq \sigma \sqrt{\lambda_r}\, \| a_r \|_\infty \int_D \nabla \varphi(x) \cdot \nabla \varphi(x)\, dx$$

$$= (\sigma / \mu) \sqrt{\lambda_r}\, \| a_r \|_\infty\, (\varphi, A_0 \varphi)$$

From stochastic component: as above

$c(p)$ bounded by largest root of scalar orthogonal polynomial

Multigrid Variant of this Idea

Replace action of A_0^{-1} with multigrid ⟶ preconditioner

$$Q_{MG} = G_0 \otimes A_{0,MG}$$

(Le Maitre, et al.)

Analysis:
$$\frac{(w, Aw)}{(w, Q_{MG}w)} = \frac{(w, Aw)}{(w, Qw)} \underbrace{\frac{(w, Qw)}{(w, Q_{MG}w)}}_{\in [\beta_1, \beta_2]}$$

Spectral equivalence of MG approximation to diffusion operator

$$\Longrightarrow \quad \kappa \le \frac{(1+\tau)}{(1-\tau)} \frac{\beta_2}{\beta_1}$$

Experiment

Starting with a with specified covariance and small σ (=*.01)*:

Compare Monte-Carlo simulation with SFEM, for

$$-\nabla \cdot (a \nabla u) = f$$

N.B.: No negative samples of diffusion obtained in MC

		# Samples ***s***			
Max	SFEM	100	1000	10,000	40,000
Mean	.06311	.06361	.06330	.06313	.06313
Variance	2.360(-5)	2.161(-5)	2.407(-5)	2.258(-5)	2.316(-5)

Solve one system of order 210x225

Solve ***s*** systems of size 225

Comparison of Galerkin and Collocation

Recall, for **stochastic collocation**

Discrete solution $u_{hp}(x,\underline{\xi}) = \sum_{k=1}^{N_\xi} u_h^{(k)}(x) L_k(\underline{\xi})$

Obtained by solving

$$\int_{\mathcal{D}} \left(a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x) \xi_r\right) \nabla u \cdot \nabla v \, dx = \int_{\mathcal{D}} f\, v\, dx$$

For set of samples $\{\underline{\xi}^{(k)}\}$ situated in a sparse grid

Advantage of this approach: simpler (decoupled) systems

Disadvantage: larger stochastic space for comparable accuracy

larger by factor approximately 2^p

Dimensions of Stochastic Space

m (#KL)	p	Galerkin	Collocation Sparse	Collocation Tensor
4	1	5	9	16
	2	15	41	81
	3	35	137	256
	4	70	401	625
10	1	11	21	1024
	2	66	221	59,049
	3	286	1582	1,048,576
	4	1001	8,801	9,765,625
30	1	31	61	1.07(9)
	2	496	1861	2.06(14)
	3	5456	37,941	1.15(18)
	4	46,376	582,801	9.31(20)

~ size of coarse grid space for MG / Version 1

systems for collocation MG / Version II

Experiment

(E., Miller, Phipps, Tuminaro)

- Solve the stochastic diffusion equation by both methods
- Compare the accuracy achieved for different parameter sets[1]
- For parameter choices giving comparable accuracy, compare solution costs
- Spatial discretization fixed (32x32 finite difference grid)

Solution algorithm for both discretizations:

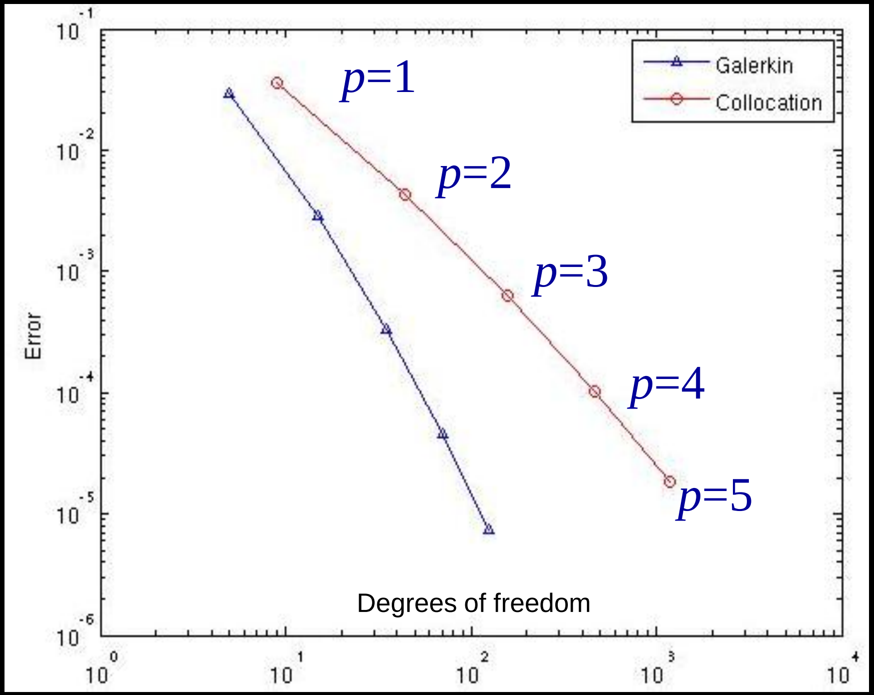
Preconditioned conjugate gradient with mean-based preconditioning, using AMG for the approximate diffusion solve

[1]Estimated using a high-degree (p=10) Galerkin solution.

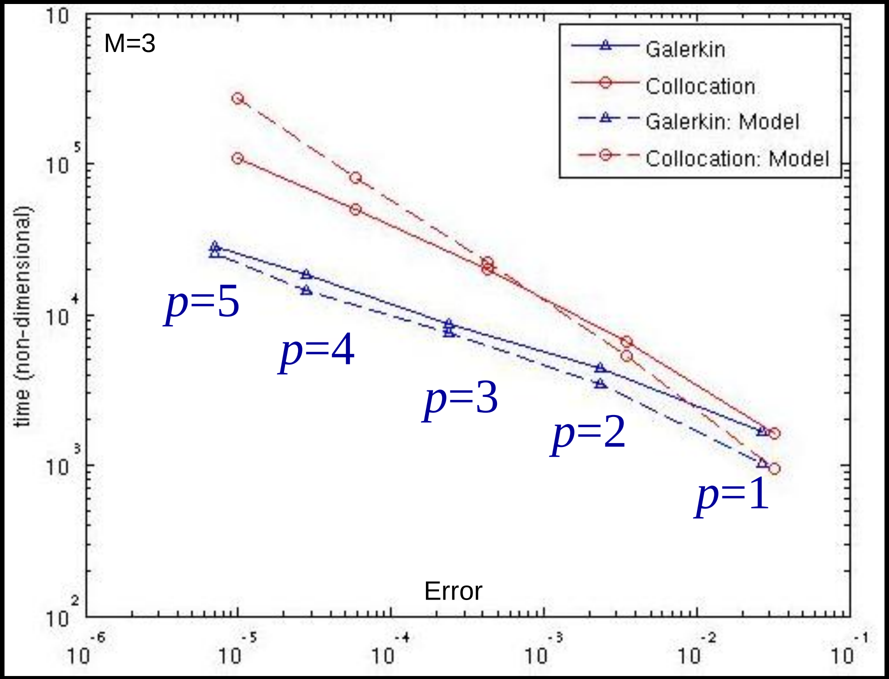
Experimental Results

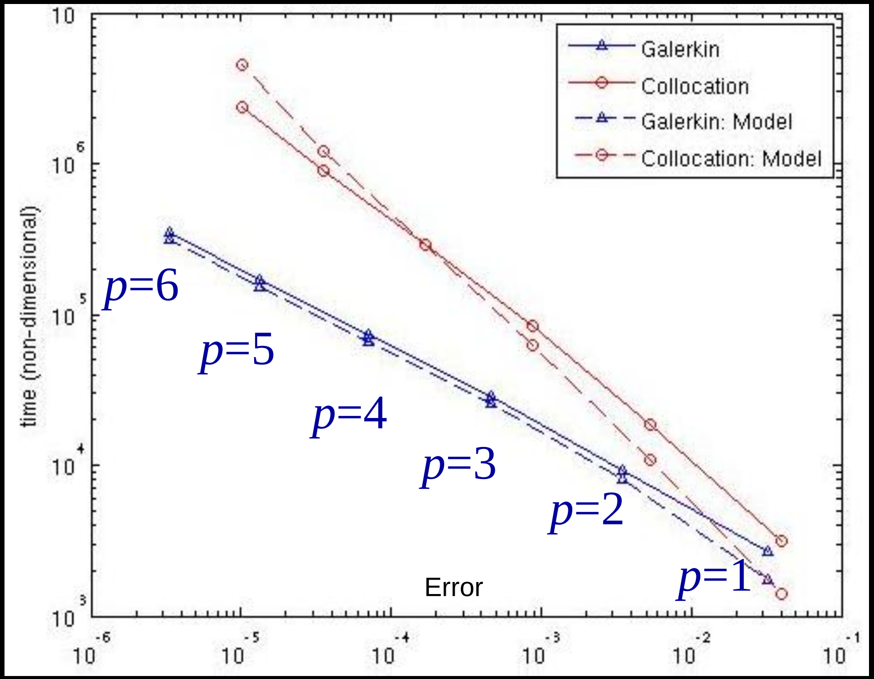
Accuracy:

for fixed m=4: similar p= {polynomial degree for SG / "level" for collocation} produces comparable errors



Performance:





Experimental Results: Performance

Performed on a serial machine with C code and CG/AMG code from Trilinos

Observation: Galerkin faster, more so as number of stochastic variables (KL terms) grows

CPU times for larger *m* = #KL terms:

	Galerkin			Collocation		
p	m=5	m=10	m=15	m=5	m=10	m=15
1	.058	.147	.32-	.069	.163	.286
2	.269	1.20	3.80	.532	2.13	5.08
3	1.20	13.14	51.45	2.41	16.99	57.98
4	3.50	53.79	168.11	8.31	102.60	493.04
5	6.51	117.73		24.56	515.75	

More General Problems

For the problem discussed, based on a KL expansion, has a *linear* dependence on the stochastic variable $\underline{\xi}$

Other models have *nonlinear* dependence. For example

$$a(x,\xi) = a_{\min} + \underbrace{e^{c(x,\xi)}}_{\text{Nonlinear}}$$

$$c(x,\xi) = a_0(x) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\, a_r(x)\xi_r$$

For Gaussian *c*, called a *log-normal* distribution

In particular: coercivity is guaranteed with this choice

More General Problems

For stochastic Galerkin, need a finite term *expansion* for *a*

$$a(x,\underline{\xi}) = a_0(x) + \sigma \sum_{r=1}^{M} \sqrt{\lambda_r}\, a_r(x)\psi_r(\underline{\xi})$$

Note: not ξ_r

⟶ matrix

$$A = G_0 \otimes A_0 + \sum_{r=1}^{M} G_r \otimes A_r$$

$$[G_r]_{ij} = \langle \psi_r \psi_i \psi_j \rangle$$

Less sparse

More importantly: # terms M will be larger
perhaps as large as $2N_\xi$

⟹ mvp will be more expensive

In Contrast

Collocation is less dependent on this expansion

$A^{(k)}$ comes from $\int_{\mathcal{D}} a(x, \underline{\xi}^{(k)}) \nabla u \cdot \nabla v \, dx$ for each sparse grid point $\underline{\xi}^{(k)}$

Many matrices to assemble, but mvp is not a difficulty

Concluding Remarks

- Exciting new developments models of PDEs with uncertain coefficients
- Replace pure simulation (Monte Carlo) with finite-dimensional models that simulate sampling at potentially lower cost
- Two techniques, the *stochastic Galerkin* method and the *stochastic collocation* method, were presented, each with some advantages
- Solution algorithms are available for both methods, and work continues in this direction