Combinatorial Trigonometry (and a method to DIE for)

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Combinatorics

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \text{number of size } k \text{ subsets of } \{1, 2, \ldots, n\}$$

$$= \text{row } n \text{ column } k \text{ entry of Pascal's triangle.}$$

Example: $n = 4$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

1 4 6 4 1

counts subsets of $\{1, 2, 3, 4\}$

size	0	1	2	3	4
	$\varnothing$	1	12	123	1234
		2	13	124	
		3	14	134	
		4	23	234	
			24		
			34		

Pascal's Triangle

Row 0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Pascal's Triangle

Row 0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$

Patterns in Pascal's Triangle

								Row Sum
Row 0	1							1
1	1	1						2
2	1	2	1					4
3	1	3	3	1				8
4	1	4	6	4	1			16
5	1	5	10	10	5	1		32
6	1	6	15	20	15	6	1	64

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}=64=2^6$$

$$\sum_{k=0}^{6}\binom{6}{k}=2^6.$$

Combinatorial Identities

Identity: For $n \geq 0$, $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Trigonometry

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Pythagorean Theorem

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$$\cos^2 x + \sin^2 x = 1$$

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Trigonometric Identities

$$\cos(2x) = 2\cos^2 x - 1$$

$$\sin(2x) = 2\sin x \cos x$$

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Combinatorial Proof:

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$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

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$$\text{Thus, } \cos^2 x + \sin^2 x =$$

$$(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

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$$= 1 + 0x + 0x^2 + 0x^3 + 0x^4 + \cdots = 1.$$

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Why?

$$\cos^2 x + \sin^2 x =$$

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has constant term 1

has no odd terms

How about the even terms?

Coefficient of x^2 : $\quad -\frac{1}{2} + -\frac{1}{2} + 1 = 0.$

$$\cos^2 x + \sin^2 x =$$

$$(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

$$\text{Coefficient of } x^4 \quad = \frac{1}{4!} + \frac{1}{2!2!} + \frac{1}{4!} - \frac{1}{3!} - \frac{1}{3!}$$

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$$\text{Coefficient of } x^4 \quad = \frac{1}{4!} - \frac{1}{3!} + \frac{1}{2!2!} - \frac{1}{3!} + \frac{1}{4!}$$

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$$\text{Coefficient of } x^4 = \frac{1}{0!4!} - \frac{1}{1!3!} + \frac{1}{2!2!} - \frac{1}{3!1!} + \frac{1}{4!0!}$$

$$= \frac{1}{4!}\left[\frac{4!}{0!4!} - \frac{4!}{1!3!} + \frac{4!}{2!2!} - \frac{4!}{3!1!} + \frac{4!}{4!0!}\right]$$

$$= \frac{1}{4!}\left[\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}\right]$$

$$= \frac{1}{4!}\,[1 - 4 + 6 - 4 + 1] = 0$$

$$\cos^2 x + \sin^2 x =$$

$$(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

Coefficient of x^6

$$= -\frac{1}{6!} - \frac{1}{2!4!} - \frac{1}{4!2!} - \frac{1}{6!} + \frac{1}{5!} + \frac{1}{3!3!} + \frac{1}{5!}$$

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$$= -\frac{1}{6!}\left[\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6}\right]$$

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$$= -\frac{1}{6!}\left[1 - 6 + 15 - 20 + 15 - 6 + 1\right] = 0$$

What is the coefficient of x^n?

When n=0, coefficient is 1.

When n is odd, coefficient is 0.

What is the coefficient of x^n?

When n > 0 is even, coefficient is

$$\frac{(-1)^{n/2}}{n!}\left[\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\cdots+\binom{n}{n}\right]$$

$$=\frac{(-1)^{n/2}}{n!}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^k\right]$$

Goal: Prove for all even n > 0,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$$

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Goal: Prove for all even n > 0,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$$

(And it's even true for odd n too!)

Identity: For $n > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0.$$

Algebraic Proof:

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Binomial Theorem: $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$

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Thus,
$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^k &= (1-1)^n \\ &= 0^n \\ &= 0 \end{aligned}$$

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$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

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$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Prove: For the set {1,2,...,n} where n > 0,

# of subsets of even size = # of subsets of odd size

$$\text{Identity: For } n > 0, \ \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0.$$

Example: n=4

Even subsets	Odd subsets
∅	1
12	2
13	3
14	4
23	123
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Toggle the number 1.

Even subsets	f	Odd subsets
$\varnothing$	⟷	1
12	⟷	2
13	⟷	3
14	⟷	4
23	⟷	123
24	⟷	124
34	⟷	134
1234	⟷	234

Toggle the number 1.

$$f(X) = X \oplus 1$$

In general, every subset X of {1,2,...,n}

holds hands with a subset of opposite parity.

$$X \longleftrightarrow X \oplus 1$$

The number of even subsets of {1,2,...,n}
= the number of odd subsets of {1,2,...,n}

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

☺

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X and $f(X)$ have opposite sign in the sum.

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X and $f(X)$ have opposite sign in the sum.

Here, $|X|$ and $|X \oplus 1|$ have opposite parity.

Alternating sums arise in combinatorial problems when using the Principle of Inclusion-Exclusion.

P.I.E.

But we will use a different method.

How about the partial sum?

For $0 \le m \le n$, $\sum_{k=0}^{m} \binom{n}{k} (-1)^k = \ ???$

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$\sum_{k=0}^{m} \binom{n}{k}$ counts

subsets of $\{1, 2, \ldots, n\}$ with *at most* m elements.

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Note: The positive sum has NO CLOSED FORM.

$$\text{For } 0 \leq m \leq n, \;\; \sum_{k=0}^{m} \binom{n}{k} (-1)^k = \;\; ???$$

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Unmatched

For $0 \le m \le n$, $\sum_{k=0}^{m} \binom{n}{k} (-1)^k = \ ???$

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except for

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The involution $f(X) = X \oplus 1$ is well-defined except for size m subsets of {1,2,...,n} that don't contain 1.

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All exceptions have the same sign: $(-1)^m$

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Compromise: We adopt a peaceful interpretation with a violent acronym.

P.I.E.

D.I.E.

D.I.E.

Description.

D.I.E.

Description.

Involution.

D.I.E.

Description.

Involution.

Exception.

D.I.E.

Description.

Describe a set of objects that is being counted when we ignore the sign term.

Involution.

Exception.

D.I.E.

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Describe a set of objects that is being counted when we ignore the sign term.

Involution.

Find an involution between positive objects and negative objects.

Exception.

D.I.E.

Description.

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Involution.

Find an involution between positive objects and negative objects.

Exception.

Describe the exceptions, where the involution is undefined. Count these exceptions, and note their sign.

The Fibonacci Numbers

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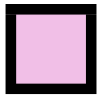
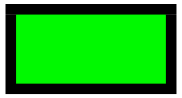
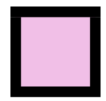
Q: How many ways to tile a $1 \times n$ board with squares  and dominoes  ?

What do Fibonacci numbers count?

Q: How many ways to tile a *1* × *n* board with squares  and dominoes  ?

$n = 1$

$n = 2$

$n = 3$

$n = 4$

$n = 5$

What do Fibonacci numbers count?

Q: How many ways to tile a *1* × *n* board with squares and dominoes ?

$n = 1$ — 1 way

$n = 2$

$n = 3$

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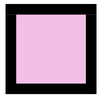
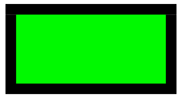
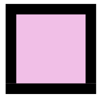
Q: How many ways to tile a *1* × *n* board with squares  and dominoes  ?

$n = 1$  1 way

$n = 2$  2 ways

$n = 3$

$n = 4$

$n = 5$

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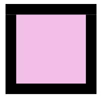
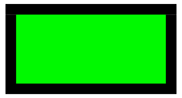
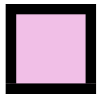
Q: How many ways to tile a *1* × *n* board with squares  and dominoes  ?

$n = 1$  1 way

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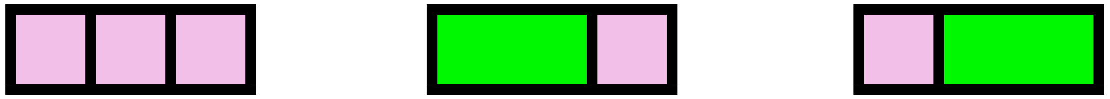
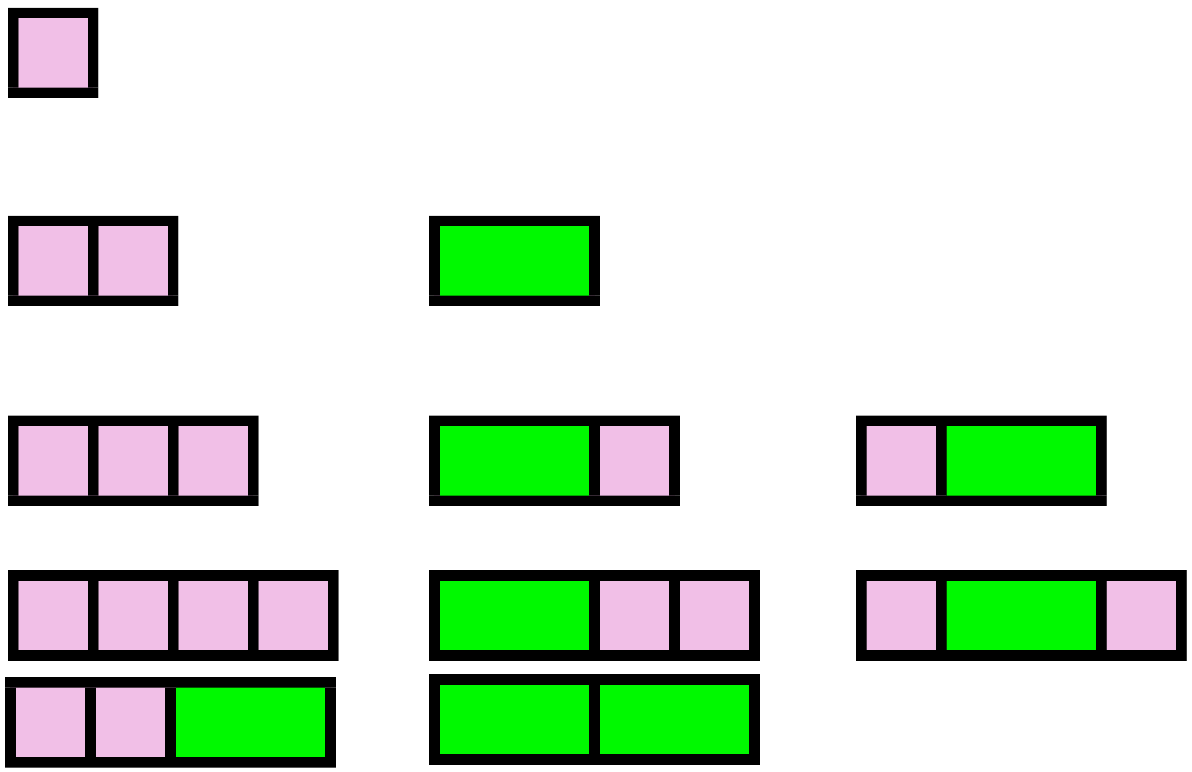
$n = 3$  3 ways

$n = 4$

$n = 5$

What do Fibonacci numbers count?

Q: How many ways to tile a $1 \times n$ board with squares and dominoes ?



$n = 1$ — 1 way

$n = 2$ — 2 ways

$n = 3$ — 3 ways

$n = 4$ — 5 ways

$n = 5$

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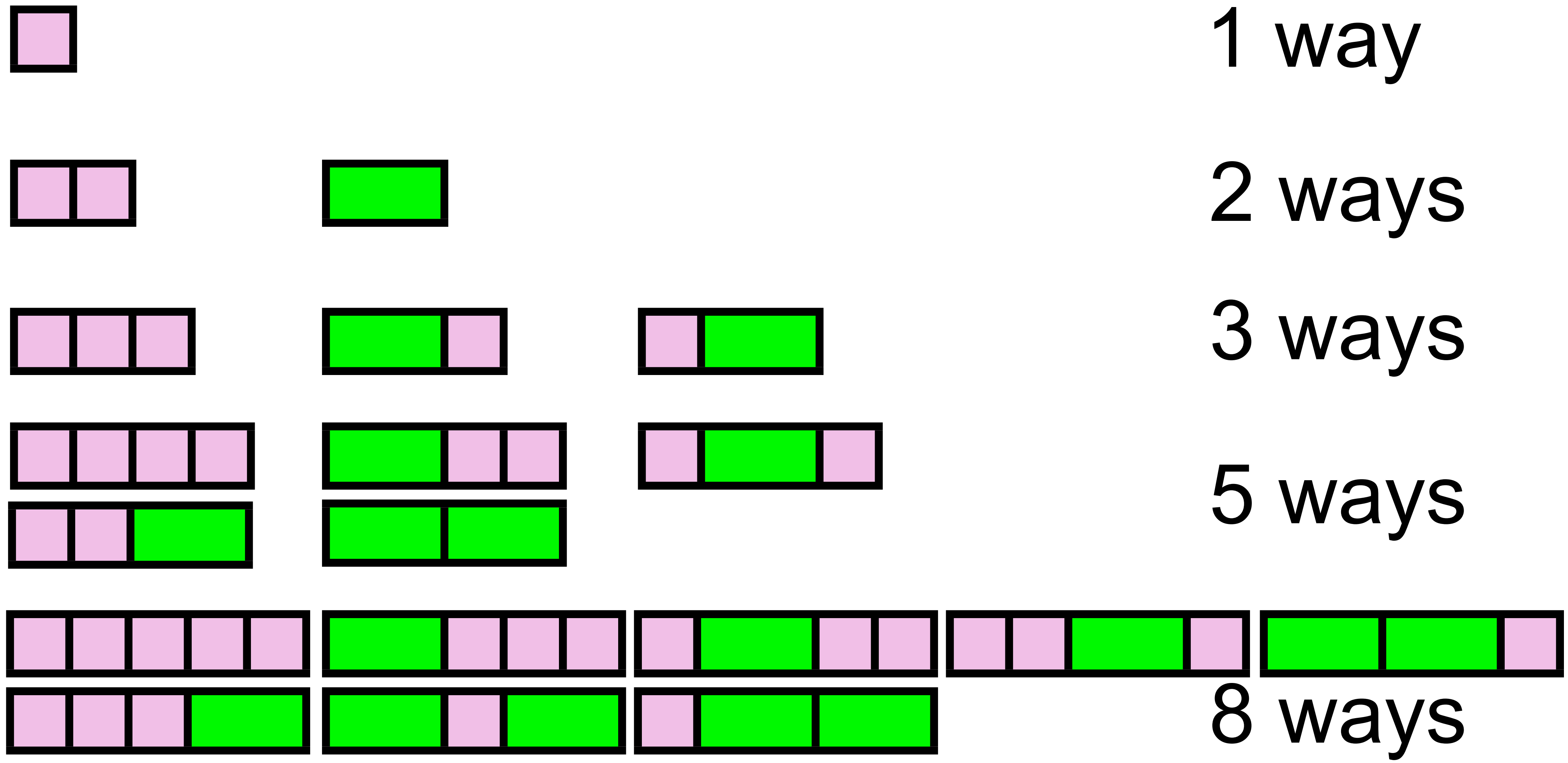

n = 1 — 1 way

n = 2 — 2 ways

n = 3 — 3 ways

n = 4 — 5 ways

n = 5 — 8 ways

What do Fibonacci numbers count?

Q: How many ways to tile a *1* × *n* board with squares and dominoes ?

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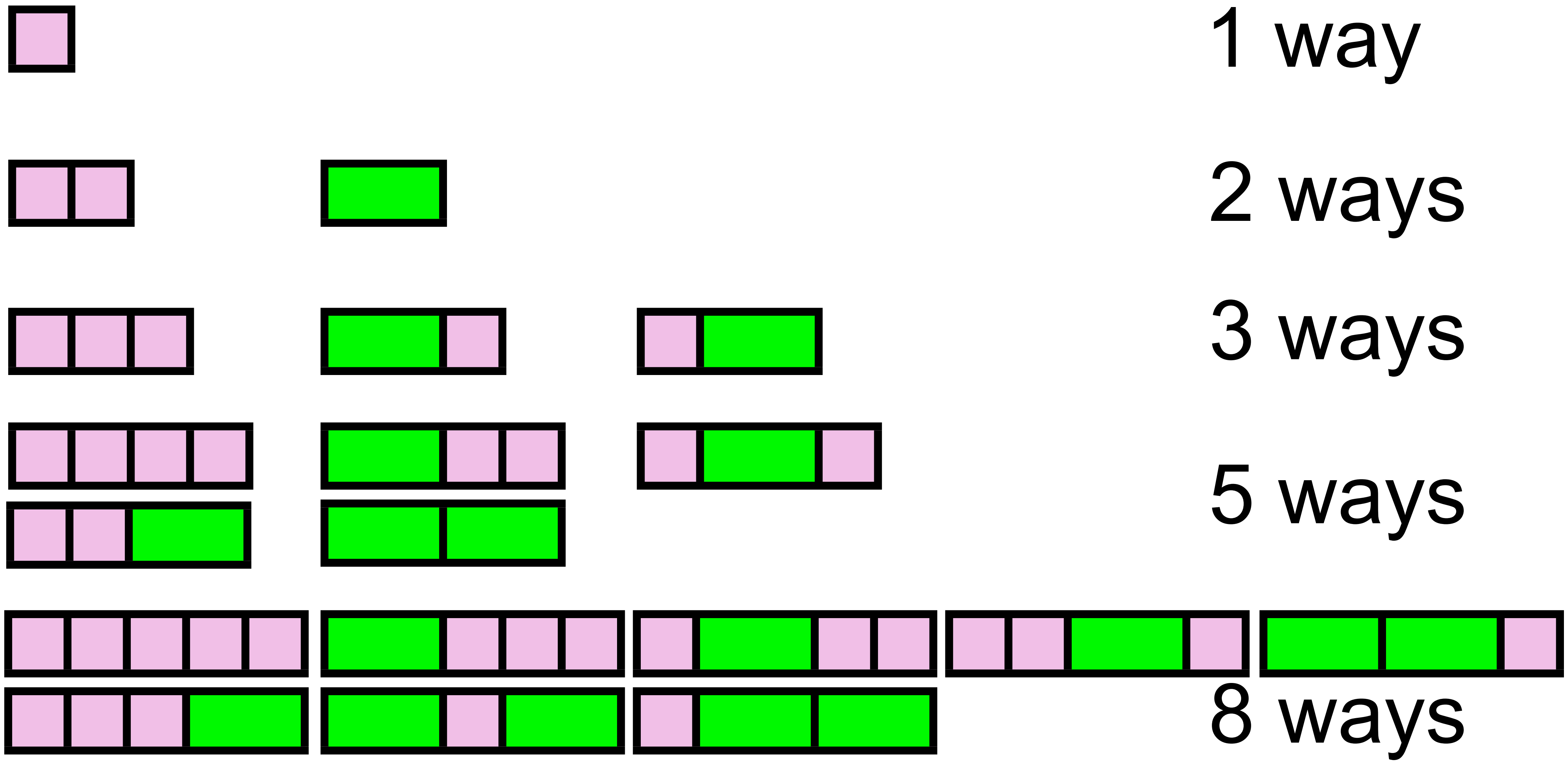

A: The *n*-th Fibonacci number!

Alternating sum of Fibonacci numbers

$$-f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - \cdots \pm f_n$$

$$-1 + 2 - 3 + 5 - 8 + 13 - \cdots \pm f_n$$

Alternating sum of Fibonacci numbers

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$$-1 + 2 - 3 + 5 - 8 + 13 - \cdots \pm f_n$$

-1

Alternating sum of Fibonacci numbers

$$-f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - \cdots \pm f_n$$

$$-1 + 2 - 3 + 5 - 8 + 13 - \cdots \pm f_n$$

-1 1

Alternating sum of Fibonacci numbers

$$-f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - \cdots \pm f_n$$

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$$-1 + 2 - 3 + 5 - 8 + 13 - \cdots \pm f_n$$

-1 1 -2 3

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$$-1 + 2 - 3 + 5 - 8 + 13 - \cdots \pm f_n$$

-1 1 -2 3 -5 8

$$-f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - \cdots \pm f_n = (-1)^n f_{n-1}.$$

Identity: For $n \geq 1$, $\sum_{k=1}^{n} f_k (-1)^k = (-1)^n f_{n-1}$

Description. $\sum_{k=1}^{n} f_k$ counts

All tilings with (positive) length at most n.

Hence, $\sum_{k=1}^{n} f_k (-1)^k$ is the number of even length tilings minus the odd length tilings (up to length n).

Identity: For $n \geq 1$, $\sum_{k=1}^{n} f_k(-1)^k = (-1)^n f_{n-1}$

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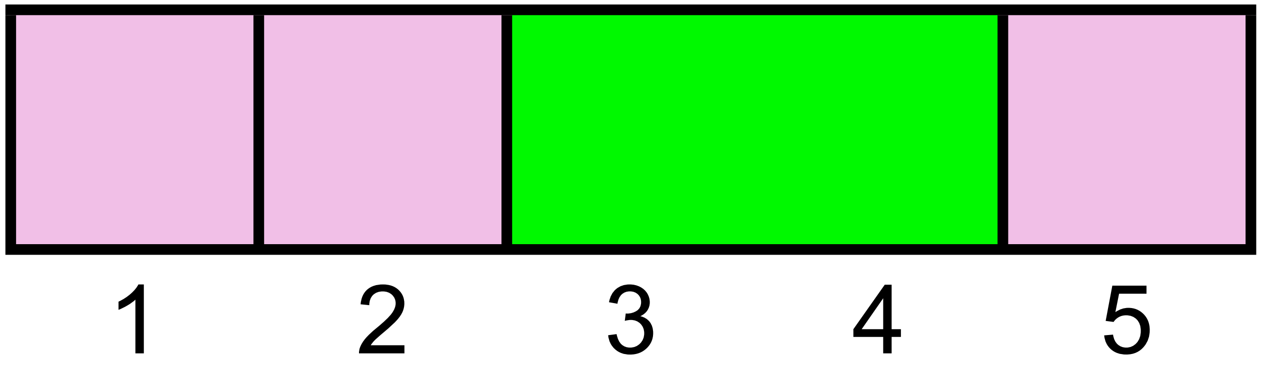
All tilings with (positive) length at most n.

Involution. What is the *second easiest* way to change the parity of the length of a tiling?

X =



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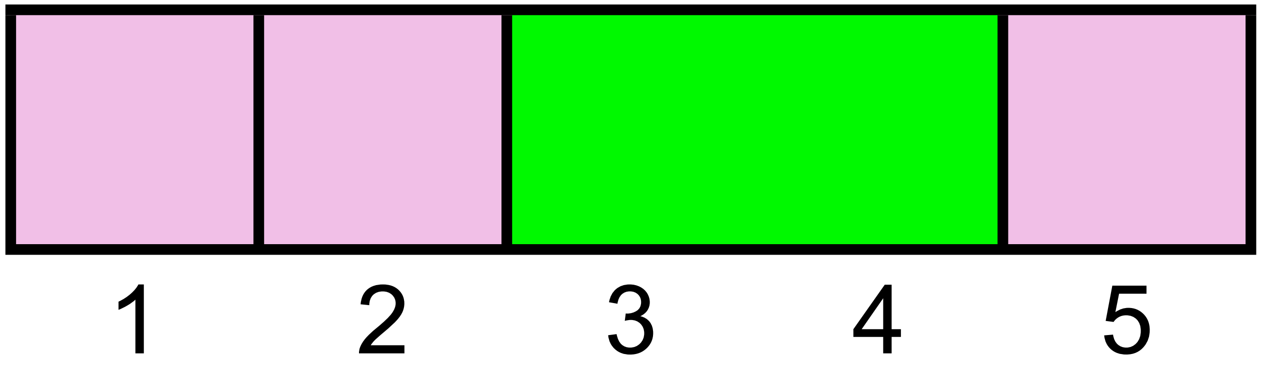
X = 

Add a square?

Identity: For $n \geq 1$, $\sum_{k=1}^{n} f_k(-1)^k = (-1)^n f_{n-1}$

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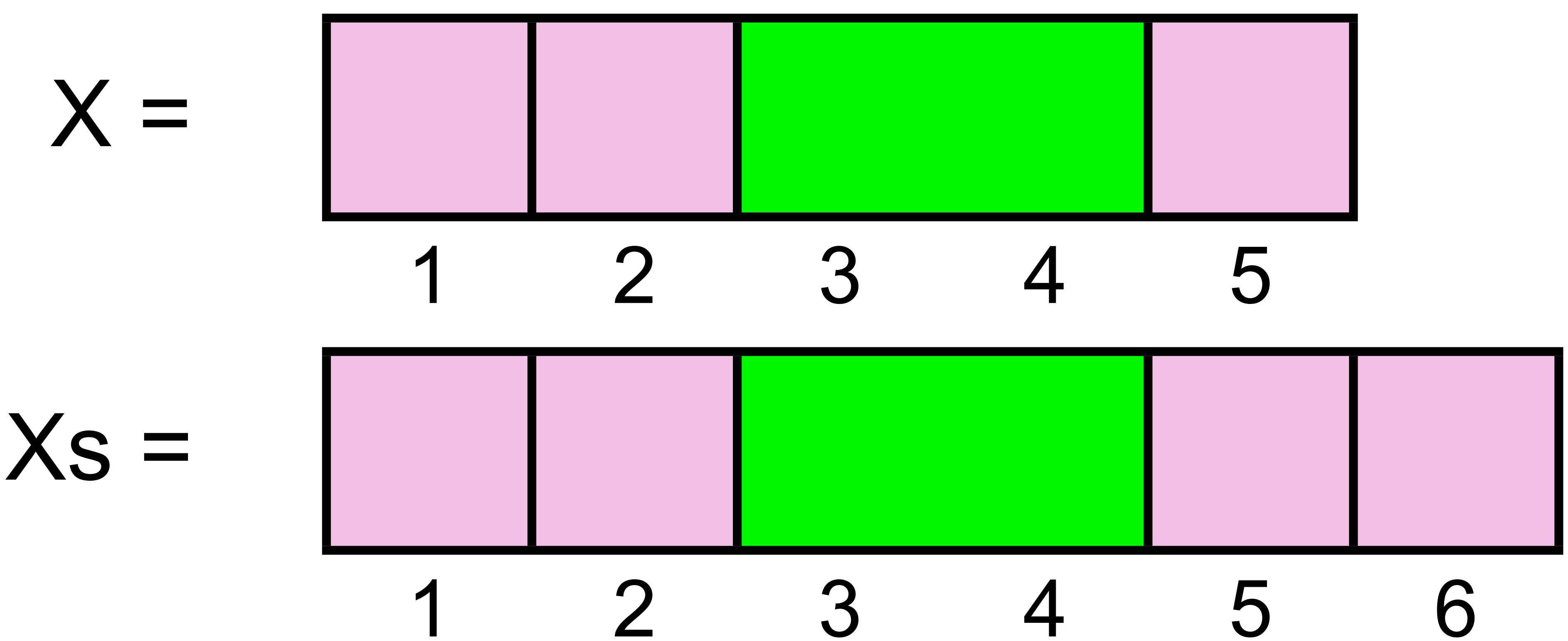

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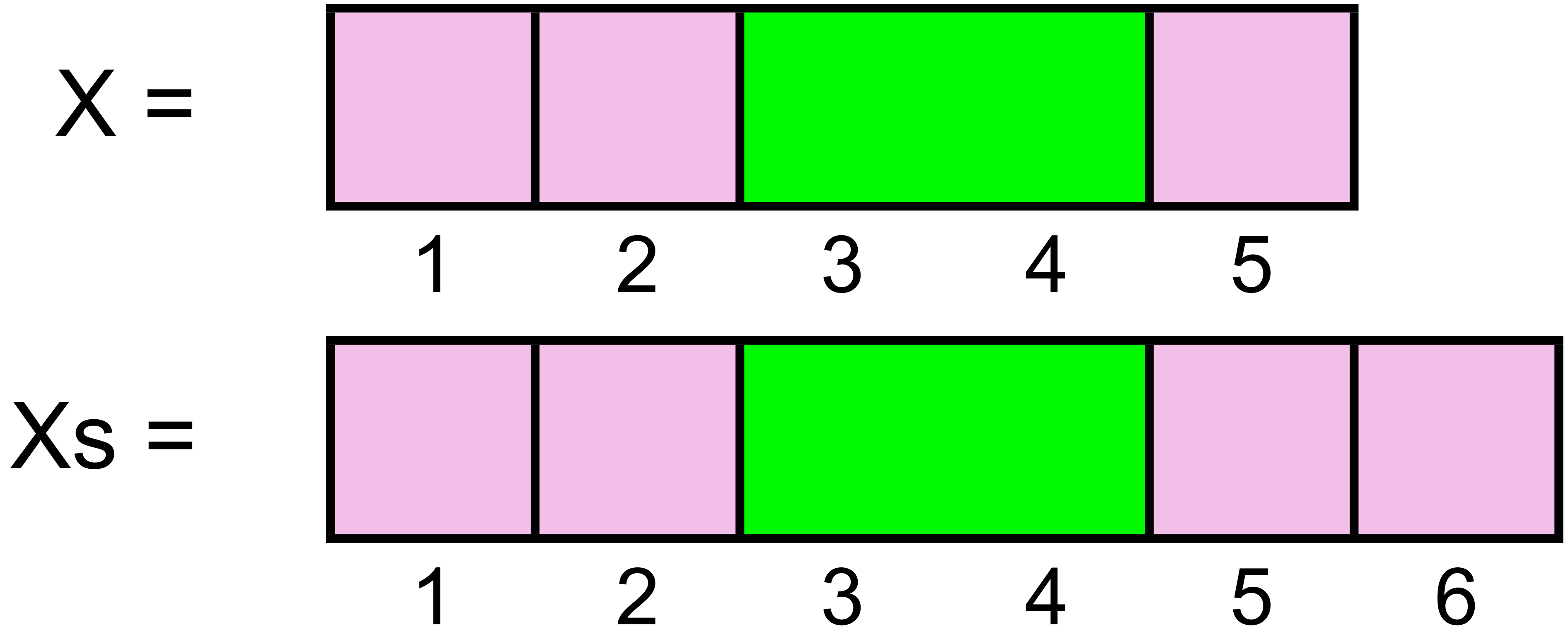

Not an involution!

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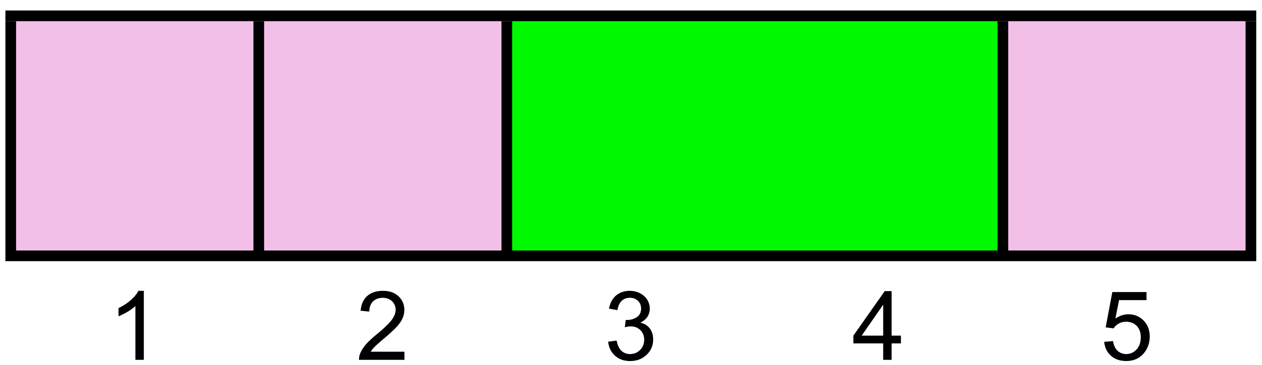
X = 

Toggle the last tile!

Identity: For $n \geq 1$, $\sum_{k=1}^{n} f_k(-1)^k = (-1)^n f_{n-1}$

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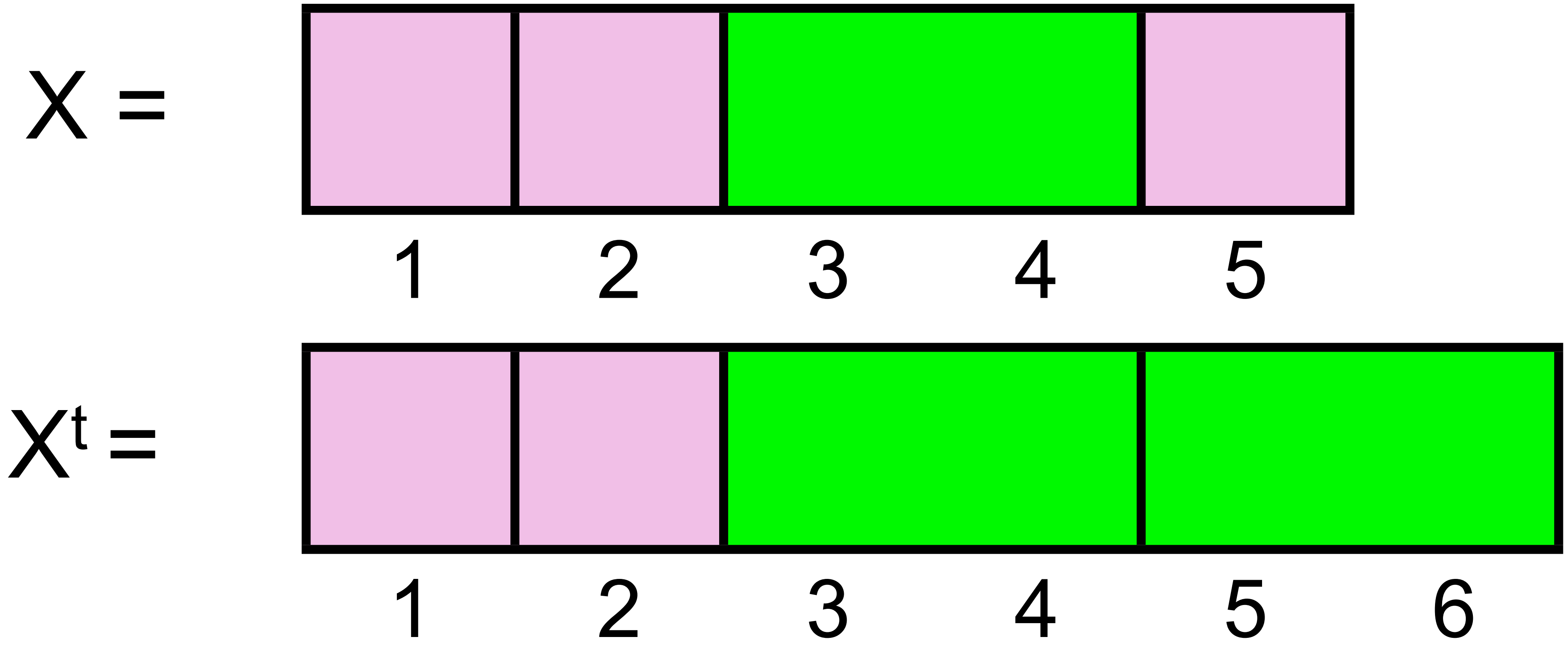

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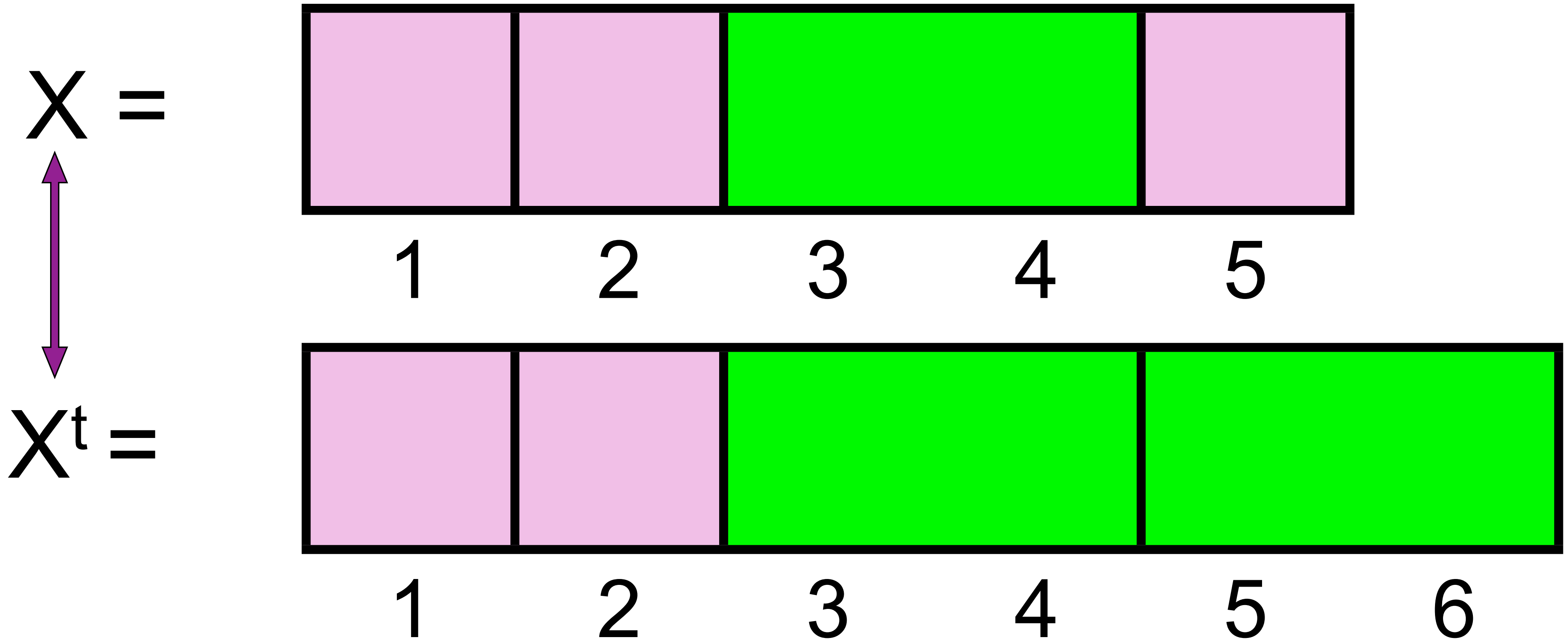

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All tilings with (positive) length at most n.

Involution. Toggle the last tile. $f(X) = X^t$

Exception.

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All tilings with (positive) length at most n.

Involution. Toggle the last tile. $f(X) = X^t$

Exception. Involution is undefined when
X has length n and ends with a square.

(since f(X) would have length n+1 -- too big)

Identity: For $n \geq 1$, $\sum_{k=1}^{n} f_k (-1)^k = (-1)^n f_{n-1}$

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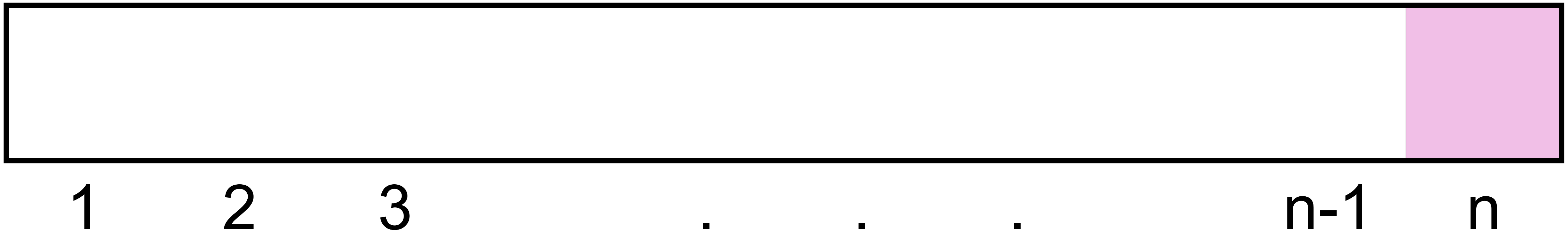

How many exceptions?

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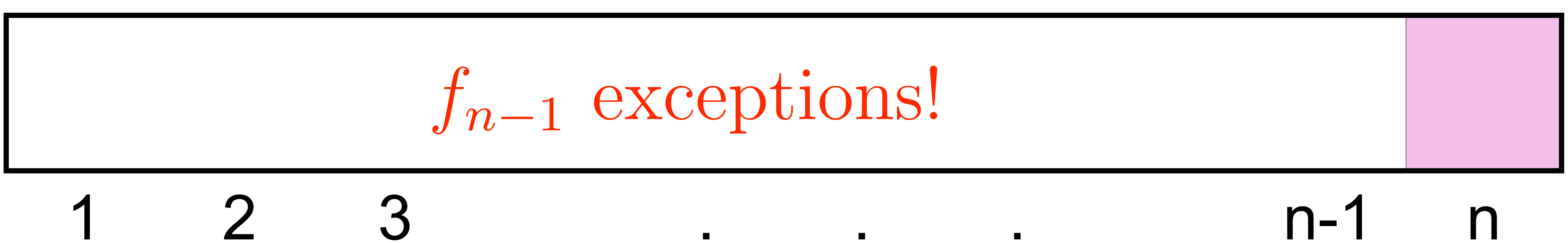

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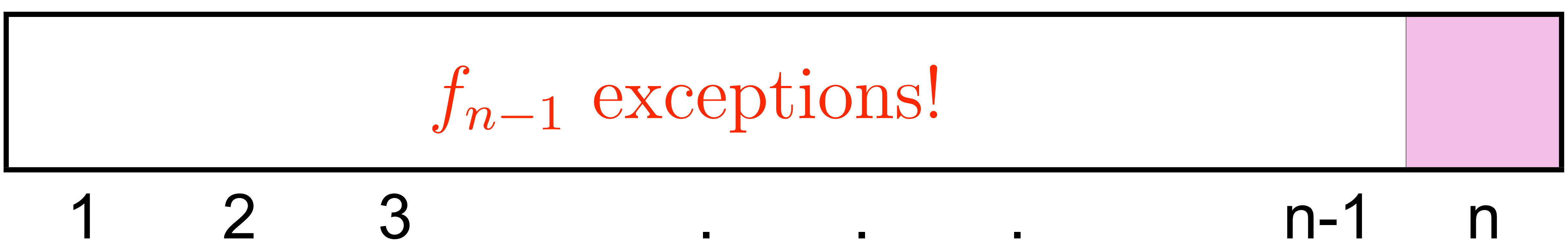

Sign of exceptions?

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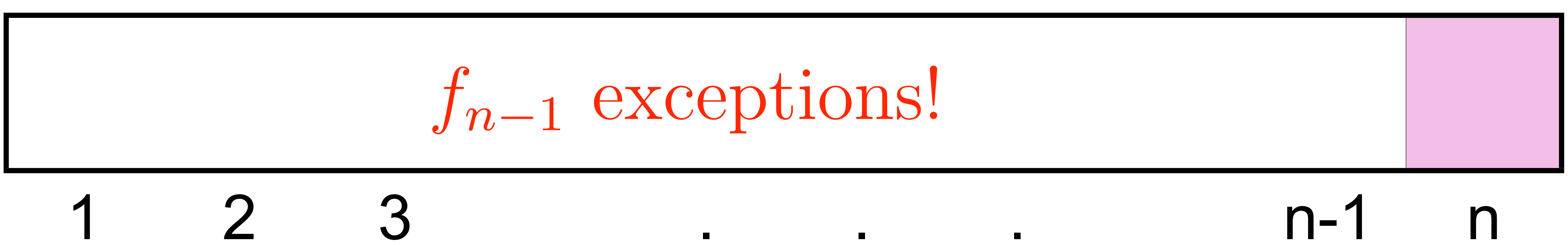

Sign of exceptions? $(-1)^n$ (since all exceptions have length n)

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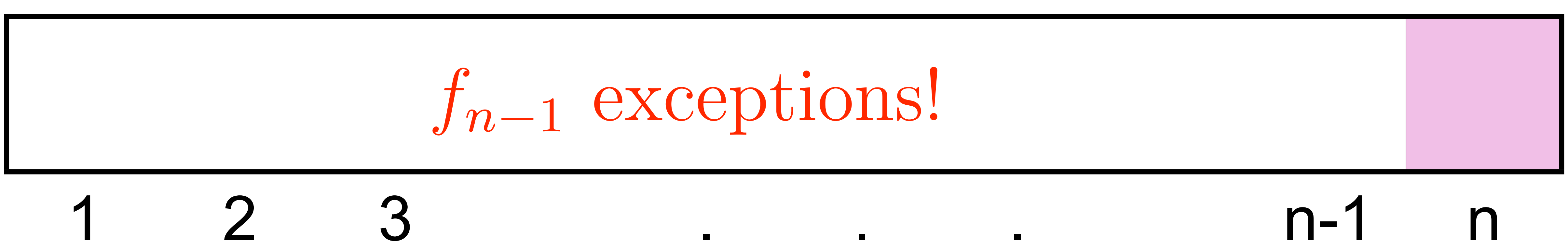

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Pascal's Triangle: Diagonal Sums

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Pascal's Triangle: Diagonal Sums

1	1						
	1	1					
	1	2	1				
	1	3	3	1			
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Pascal's Triangle: Diagonal Sums

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1	1	1					
2	1	2	1				
	1	3	3	1			
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$$\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8 = f_5$$

Pascal's Triangle: Diagonal Sums

1	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
5	1	4	6	4	1		
8	1	5	10	10	5	1	
13	1	6	15	20	15	6	1

$$\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13 = f_6$$

Pascal's Triangle: Diagonal Sums

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1	1	1					
2	1	2	1				
3	1	3	3	1			
5	1	4	6	4	1		
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$$\sum_{k \geq 0} \binom{n-k}{k} = f_n.$$

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$$\sum_{k \geq 0} \binom{n-k}{k} = f_n.$$

Note: $\binom{n-k}{k}$ is nonzero when $k \leq n-k$

(when $k \leq n/2$)

Identity: For all $n \geq 0$, $\displaystyle\sum_{k\geq 0} \binom{n-k}{k} = f_n.$

Combinat**o**ria**l** Proof:

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Q: How many tilings of length n?

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Answer 2:

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Answer 1: f_n

Answer 2: What does k represent?

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Q: How many tilings of length n?

Answer 1: f_n

Answer 2: What does k represent?

The number of dominoes!

Identity: For all $n \geq 0$, $\sum_{k \geq 0} \binom{n-k}{k} = f_n.$

Combinat**o**ria**l** Proof:

Answer 1: f_n

Answer 2: Consider the number of dominoes.

Identity: For all $n \geq 0$, $\sum_{k\geq 0} \binom{n-k}{k} = f_n.$

Combinat**o**rial Proof:

Answer 1: f_n

Answer 2: Consider the number of dominoes.

A tiling of length n with exactly k dominoes has

n - 2k squares

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Can choose k of the tiles to be dominoes in

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☺

Identity: For all $n \geq 0$, $$\sum_{k \geq 0} \binom{n-k}{k} = f_n.$$

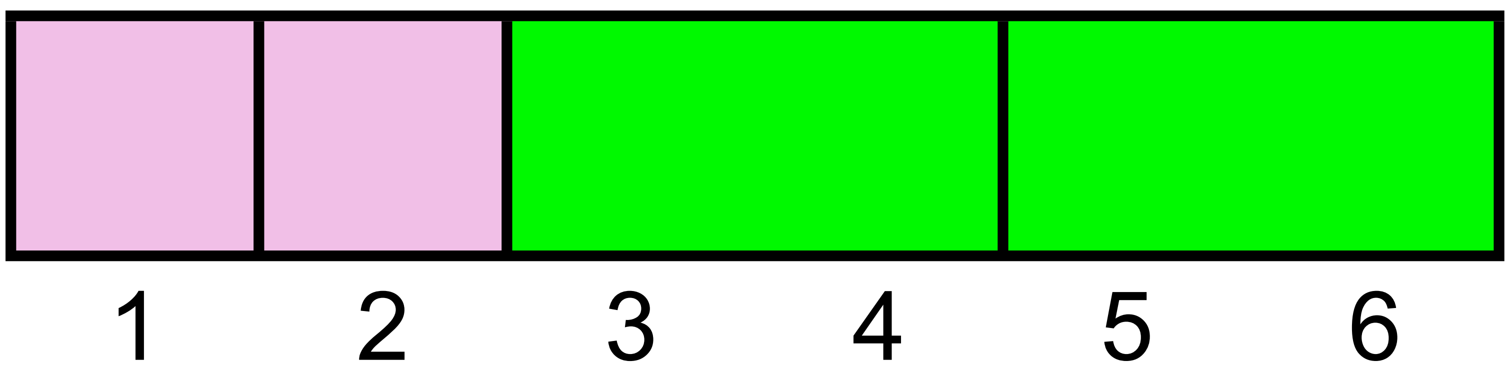
Answer 2: Consider the number of dominoes.

A tiling of length n with exactly k dominoes has

n - 2k squares

Total: n - k tiles.

Example: n = 6, k = 2 dominoes



has 4 tiles

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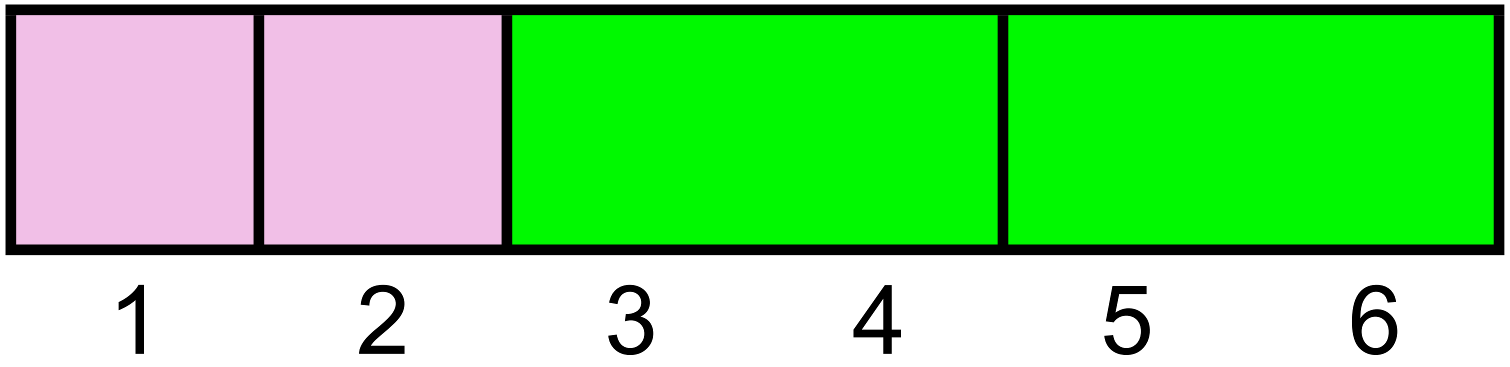

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Here, we have chosen dominoes to be tiles 3 and 4.

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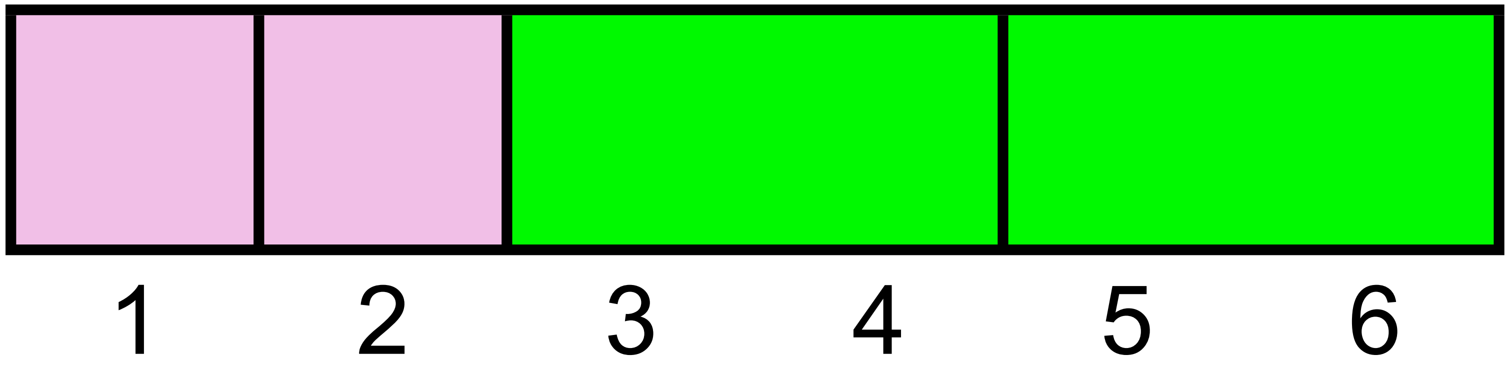

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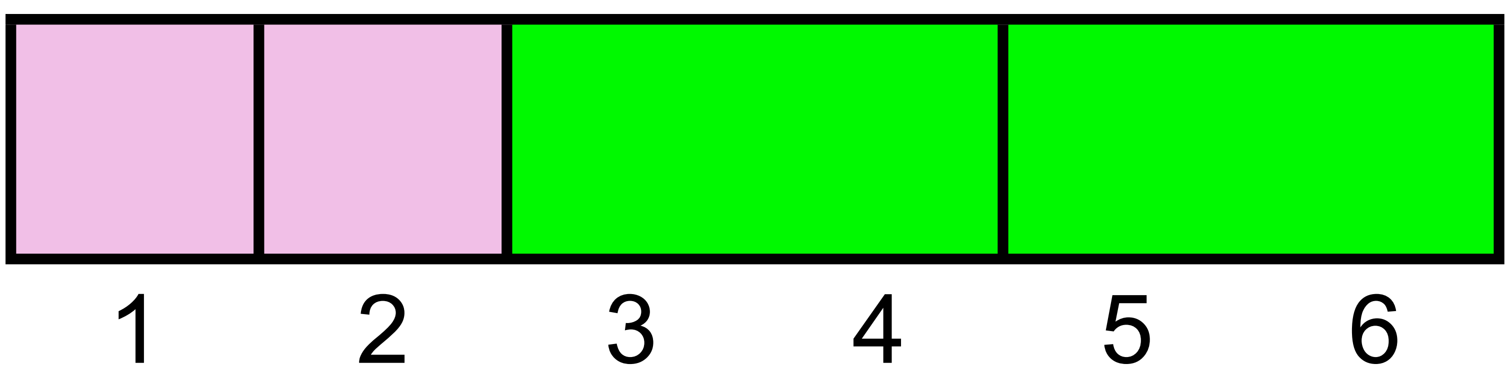

has 4 tiles

Here, we have chosen dominoes to be tiles 3 and 4.

# of length 6 tilings with 2 dominoes is $\binom{4}{2}$

Pascal's Triangle: Alternate Diagonal Sums

1								
1	-1							
1	-2	1						
1	-3	3	-1					
1	-4	6	-4	1				
1	-5	10	-10	5	-1			
1	-6	15	-20	15	-6	1		
1	-7	21	-35	35	-21	7	-1	
1	-8	28	-56	70	-56	28	-8	1

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	1	-5	10	-10	5	-1			
	1	-6	15	-20	15	-6	1		
1	1	-7	21	-35	35	-21	7	-1	
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Pascal's Triangle: Alternate Diagonal Sums

1	1								
1	1	-1							
0	1	-2	1						
-1	1	-3	3	-1					
-1	1	-4	6	-4	1				
0	1	-5	10	-10	5	-1			
1	1	-6	15	-20	15	-6	1		
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Pattern: 1,1,0,-1,-1,0, 1,1,0,-1,-1,0, ...

Identity: For $n \geq 0$,

$$\sum_{k \geq 0} \binom{n-k}{k} (-1)^k = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 6 \\ 0 & \text{if } n \equiv 2 \text{ or } 5 \pmod 6 \\ -1 & \text{if } n \equiv 3 \text{ or } 4 \pmod 6 \end{cases}$$

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$$= \cos \frac{\pi}{3} n + \frac{1}{\sqrt{3}} \sin \frac{\pi}{3} n$$

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Goal. There are almost as many length n tilings with an even number of dominoes as tilings with an odd number of dominoes.

Identity: For $n \geq 0$,

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Nope. That changes the length.

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Description. Tilings of length n.

Involution. Toggle the last tile?

Nope. That changes the length.

We must change the parity of the number of dominoes without changing the length of the tiling.

Description. Tilings of length n.

Involution. If the tiling starts with a domino

then replace the domino with 2 squares (and vice versa)

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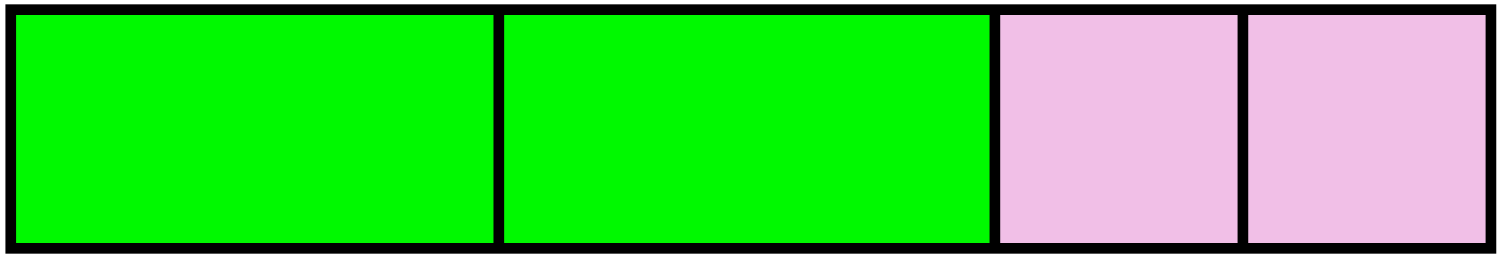
Example:

X = 

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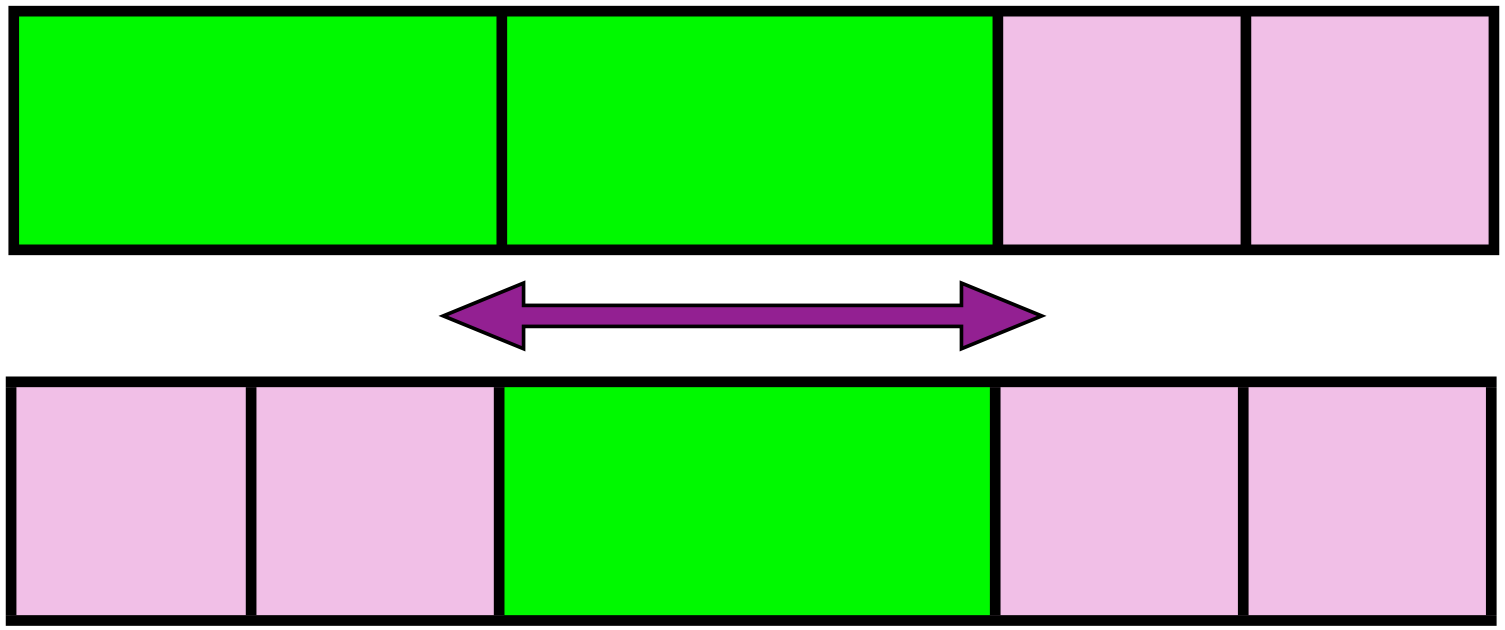

$X^t =$

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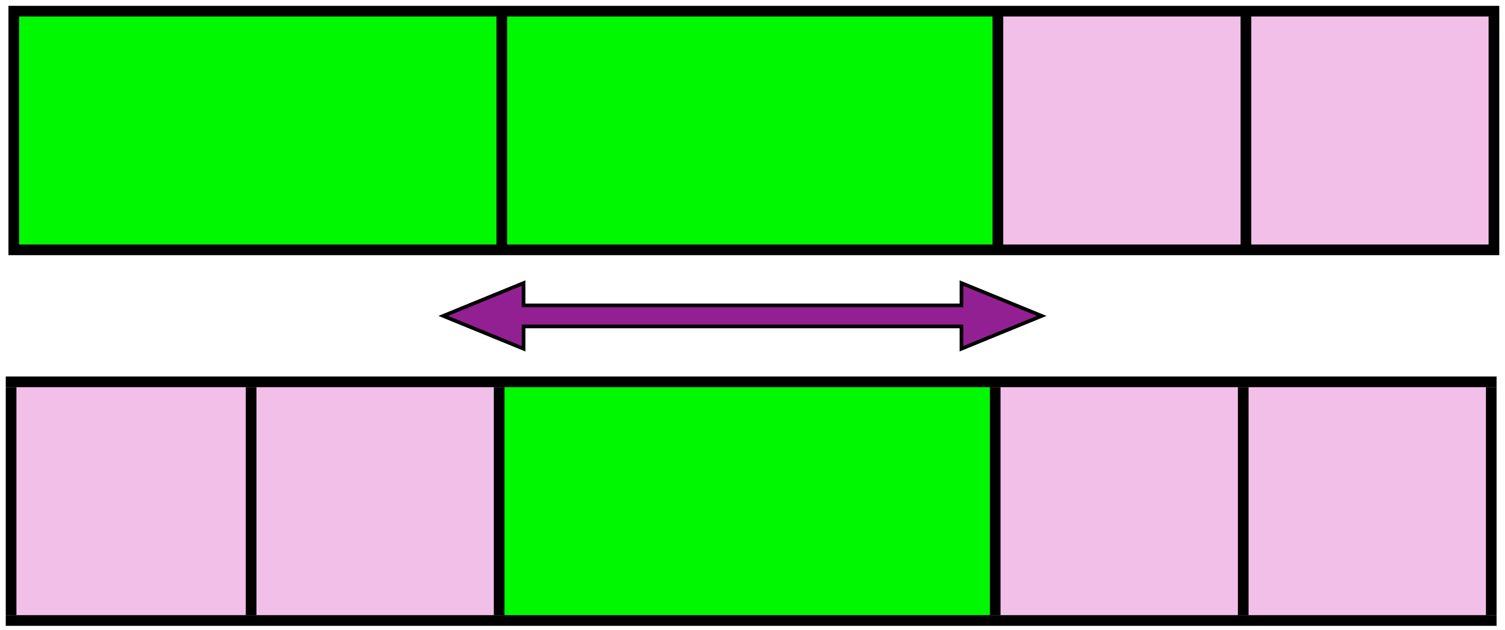

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Convenient notation: $dY \longleftrightarrow ssY$

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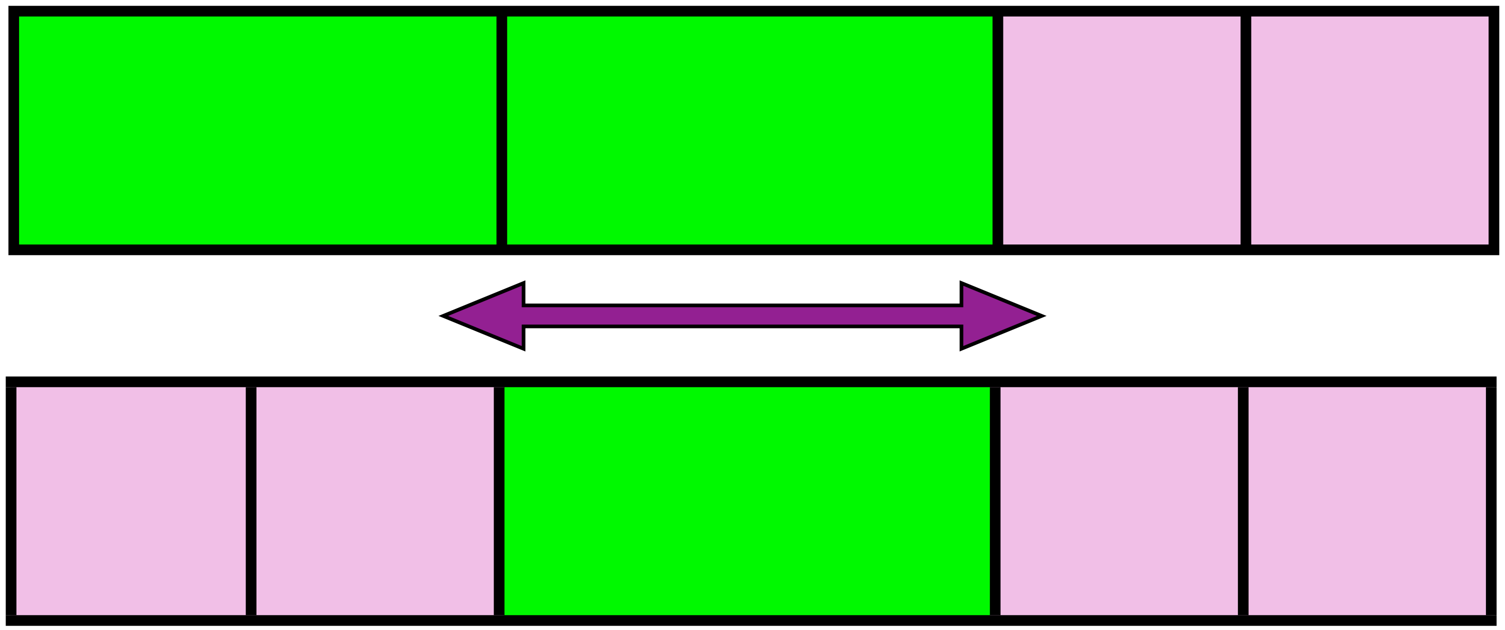


Convenient notation: $dY \longleftrightarrow ssY$

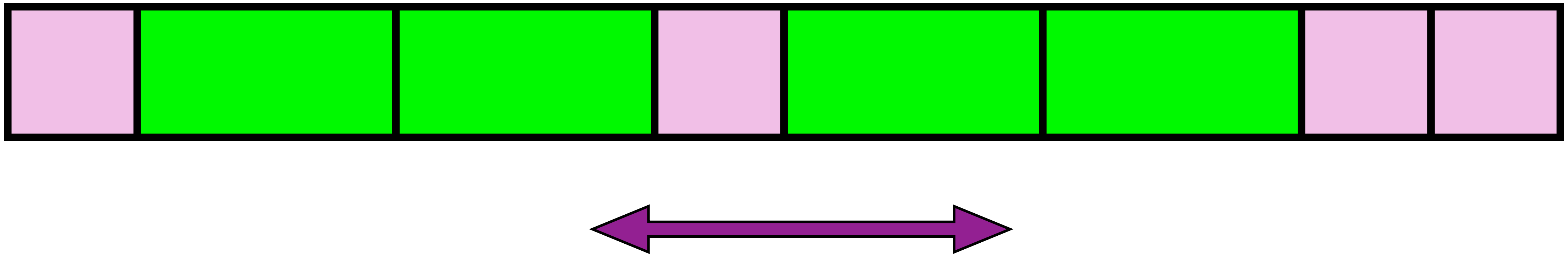
The number of dominoes changes by ± 1.

Description. Tilings of length n.

Involution. $dY \longleftrightarrow ssY$

But what if X begins square-domino, say X = sdY?

Then ignore the sd, and try to toggle what comes next.



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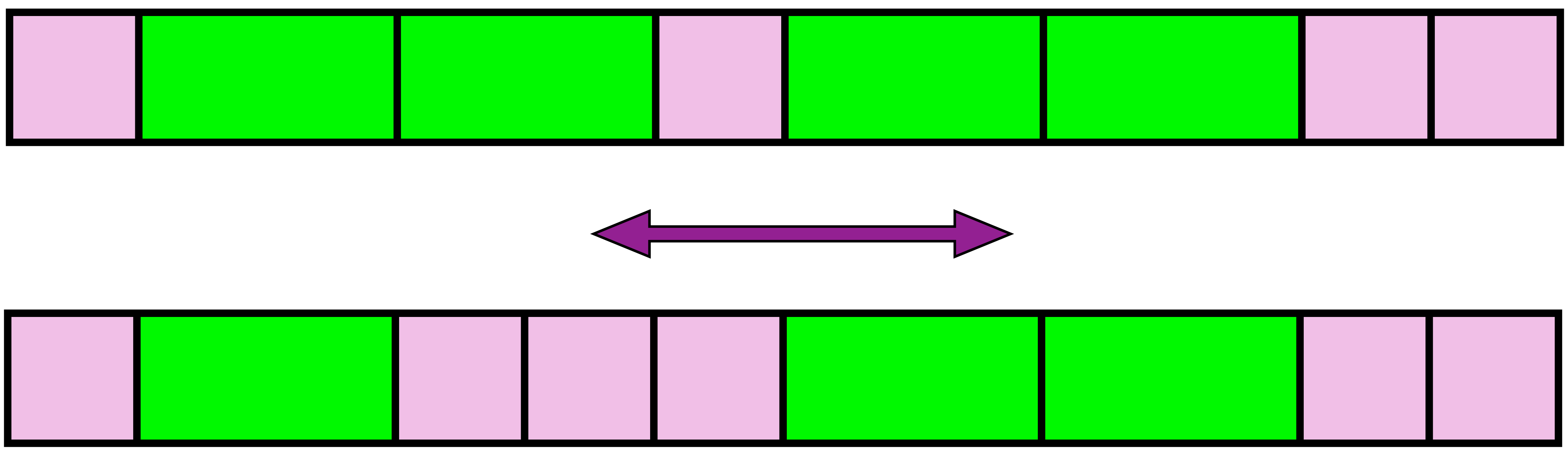
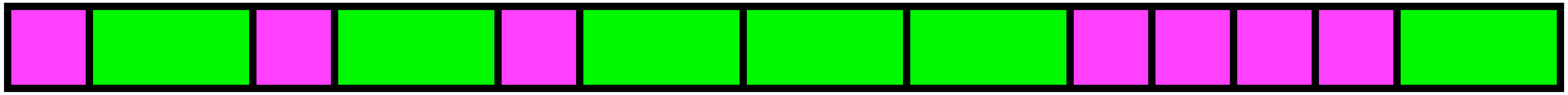


$$sddY \longleftrightarrow sdssY$$

What about ...



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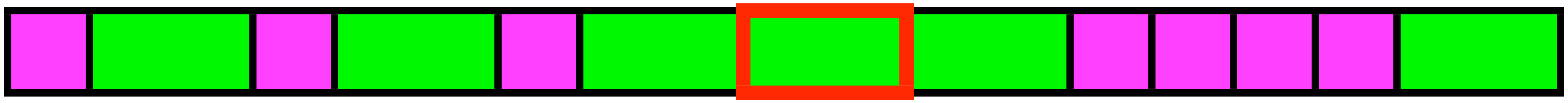

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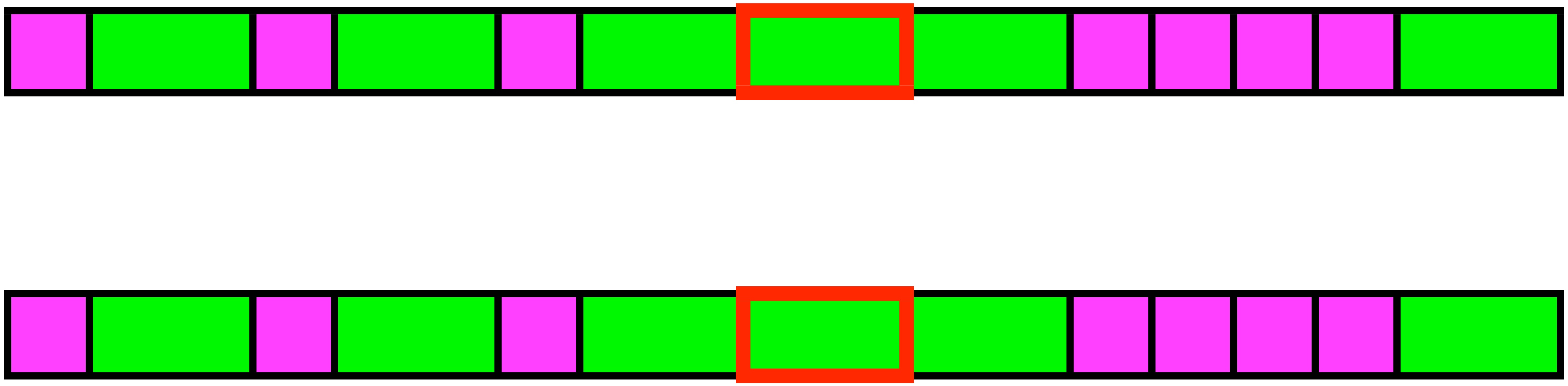

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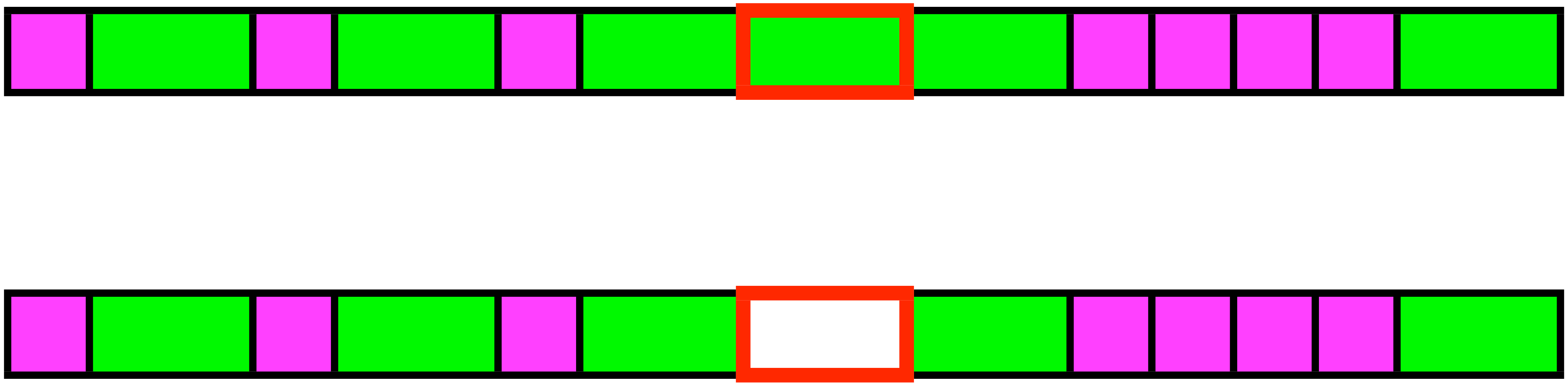

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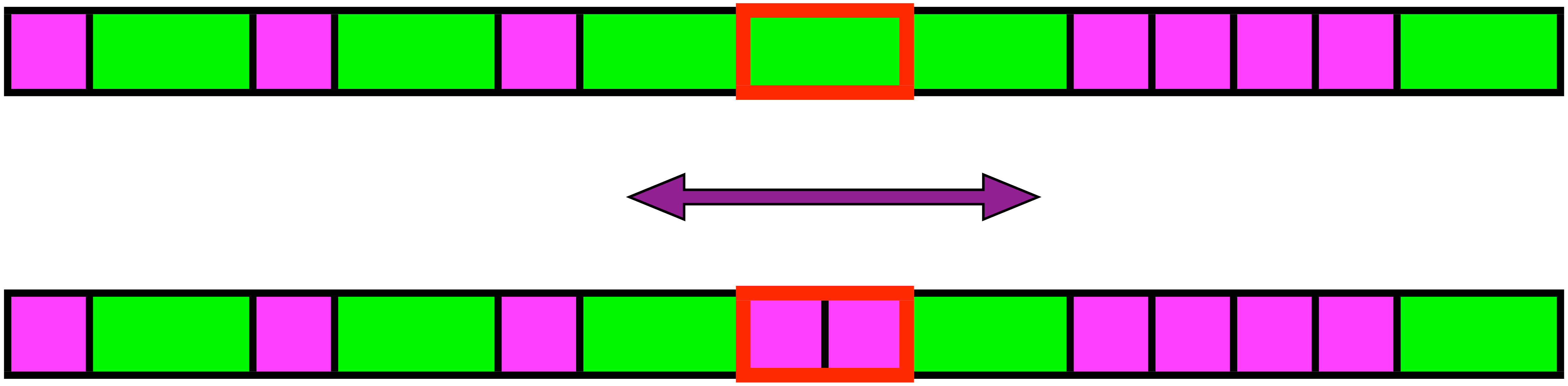

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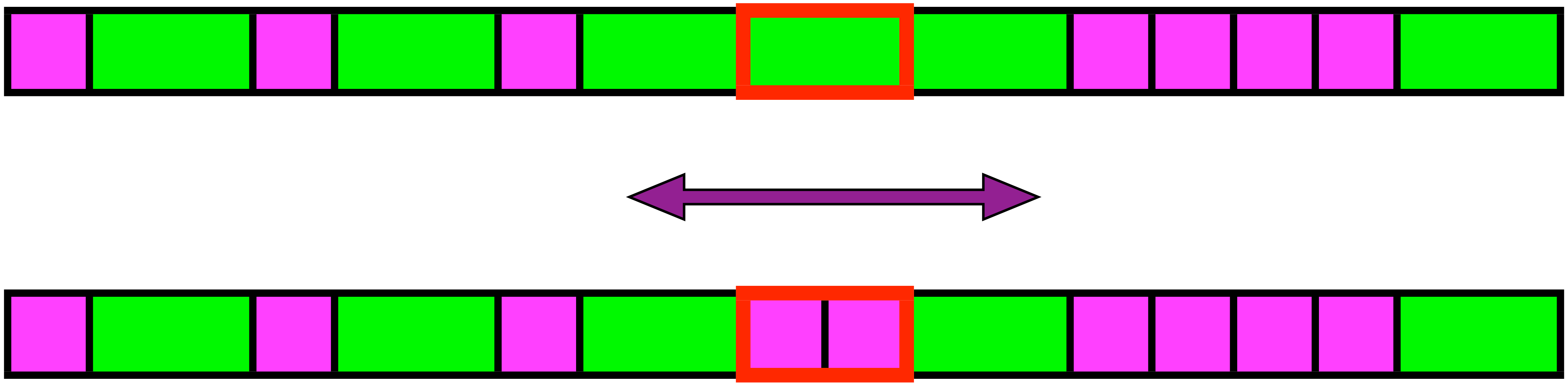
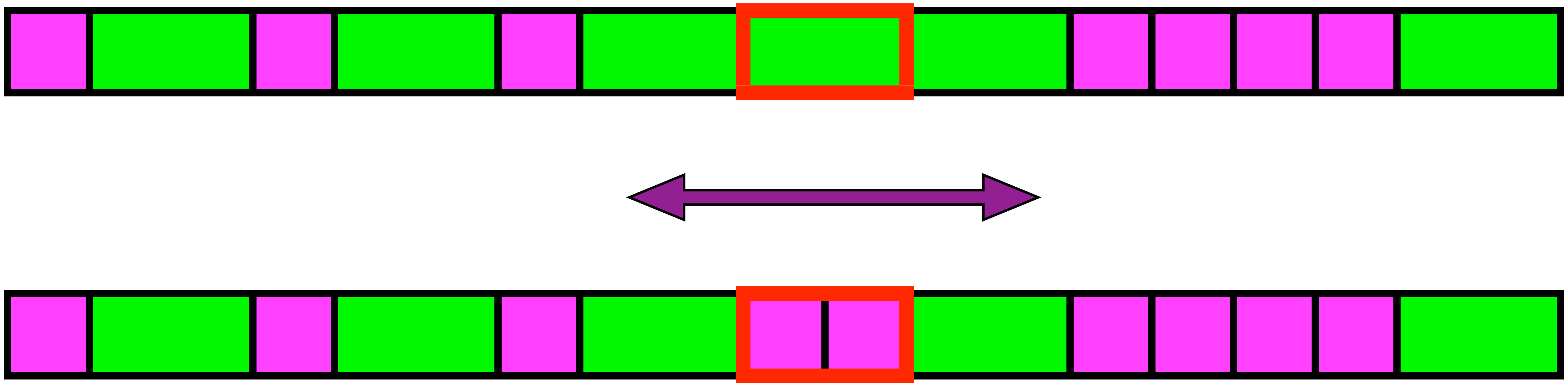


Involution. $(sd)^j dY \longleftrightarrow (sd)^j ssY$

What about ...



Involution. $(sd)^j dY \longleftrightarrow (sd)^j ssY$

Length is unchanged.
Number of dominoes changes by ± 1.

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Description. Tilings of length n.

Involution. $(sd)^j dY \longleftrightarrow (sd)^j ssY$

Exception.

$$\sum_{k\geq 0}\binom{n-k}{k}(-1)^k = \begin{array}{r} 1 \ \text{ if } \ n \equiv 0 \text{ or } 1 \pmod 6 \\ 0 \ \text{ if } \ n \equiv 2 \text{ or } 5 \pmod 6 \\ -1 \ \text{ if } \ n \equiv 3 \text{ or } 4 \pmod 6 \end{array}$$

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Exception. Every n has at most one exception.

$$\sum_{k\geq 0}\binom{n-k}{k}(-1)^k = \begin{cases} 1 & \text{if } n\equiv 0 \text{ or } 1 \pmod 6 \\ 0 & \text{if } n\equiv 2 \text{ or } 5 \pmod 6 \\ -1 & \text{if } n\equiv 3 \text{ or } 4 \pmod 6 \end{cases}$$

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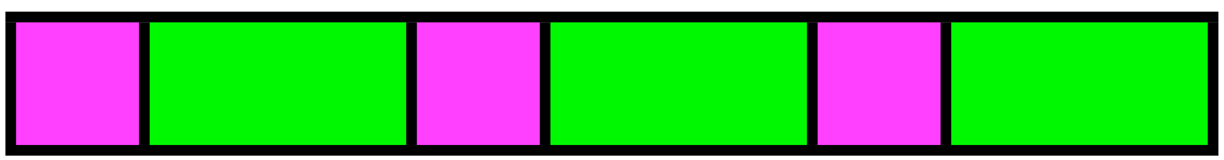
$X = (sd)^j$  $n = 3j$

$$\sum_{k\geq 0}\binom{n-k}{k}(-1)^k = \begin{matrix} 1 & \text{if} & n\equiv 0 \text{ or } 1 \pmod 6 \\ 0 & \text{if} & n\equiv 2 \text{ or } 5 \pmod 6 \\ -1 & \text{if} & n\equiv 3 \text{ or } 4 \pmod 6 \end{matrix}$$

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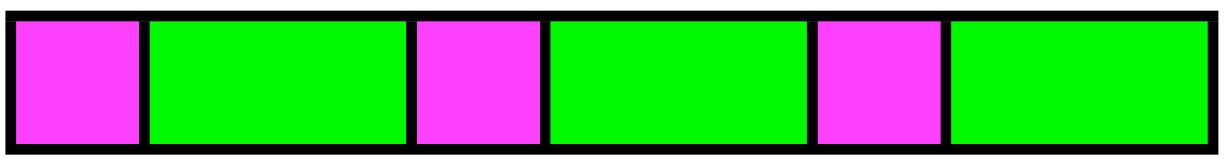
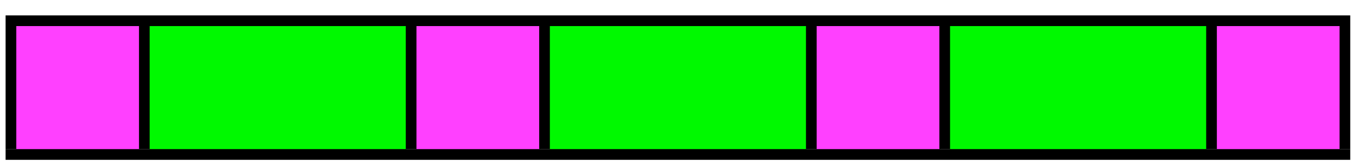
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

$$\sum_{k\geq 0}\binom{n-k}{k}(-1)^k = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 6 \\ 0 & \text{if } n \equiv 2 \text{ or } 5 \pmod 6 \\ -1 & \text{if } n \equiv 3 \text{ or } 4 \pmod 6 \end{cases}$$

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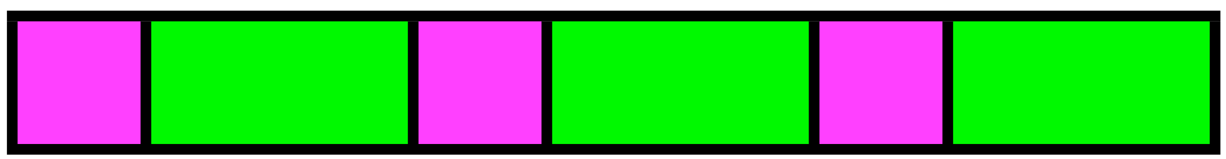
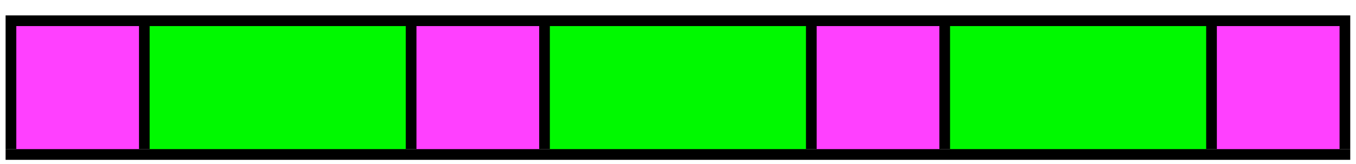
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

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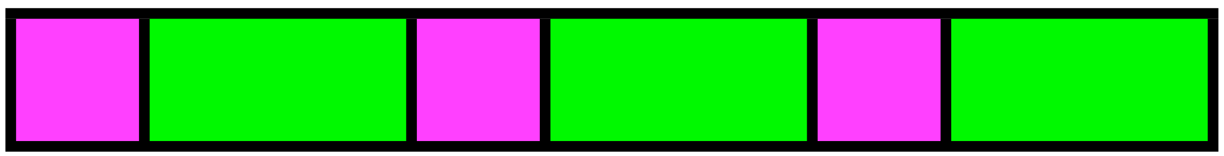
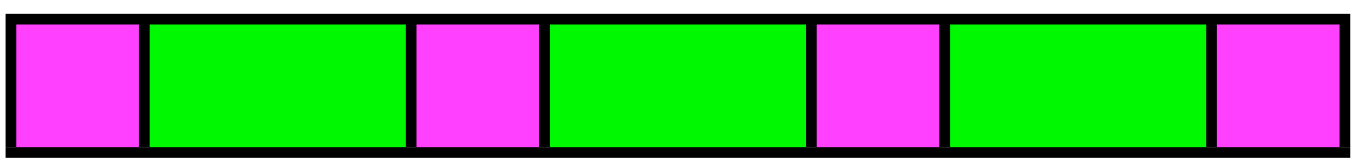
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

Thus, if $n = 3j + 2$, $(n \equiv 2 \text{ or } 5 \pmod 6)$

then there are no exceptions.

$$\sum_{k\geq 0} \binom{n-k}{k} (-1)^k = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 6 \\ 0 & \text{if } n \equiv 2 \text{ or } 5 \pmod 6 \\ -1 & \text{if } n \equiv 3 \text{ or } 4 \pmod 6 \end{cases}$$ ✔

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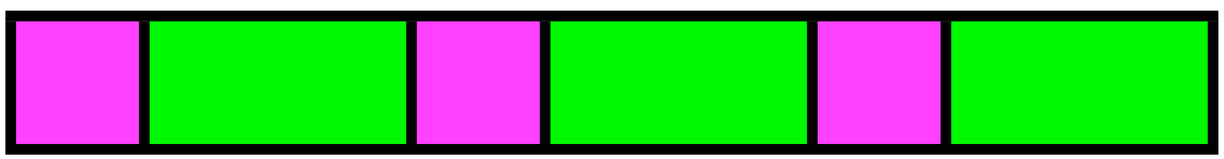

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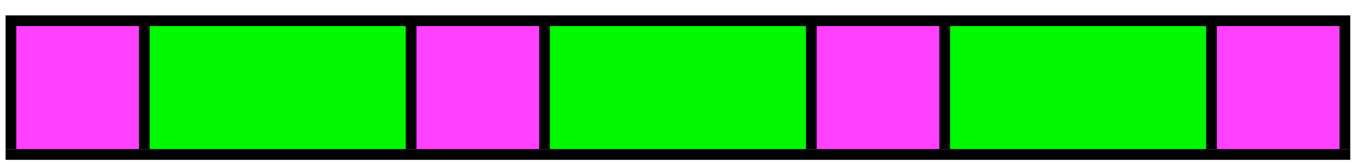

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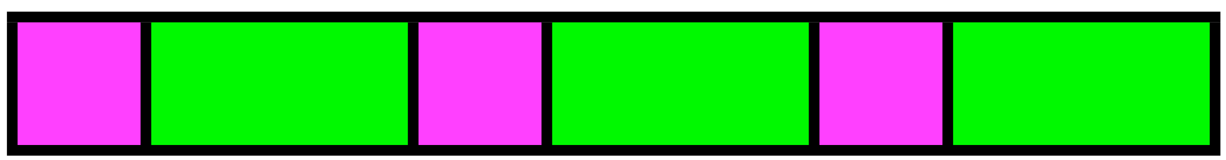
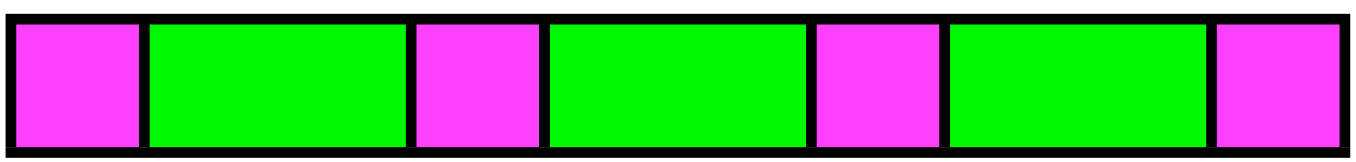
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

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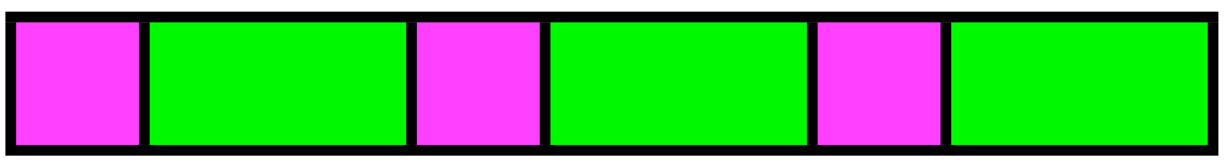
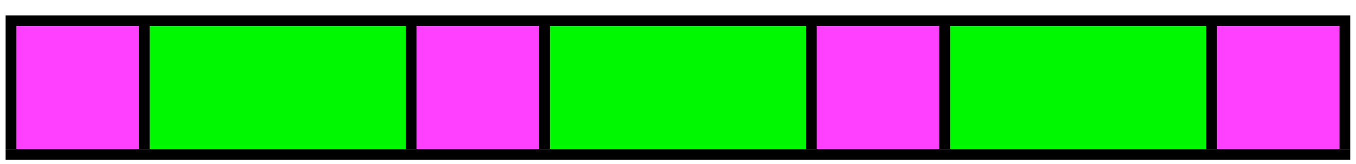
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

If n = 6j or 6j+1, then there is one exception

$$\sum_{k\geq 0}\binom{n-k}{k}(-1)^k = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 6 \\ 0 & \text{if } n \equiv 2 \text{ or } 5 \pmod 6 \\ -1 & \text{if } n \equiv 3 \text{ or } 4 \pmod 6 \end{cases}$$ ✔

Description. Tilings of length n.

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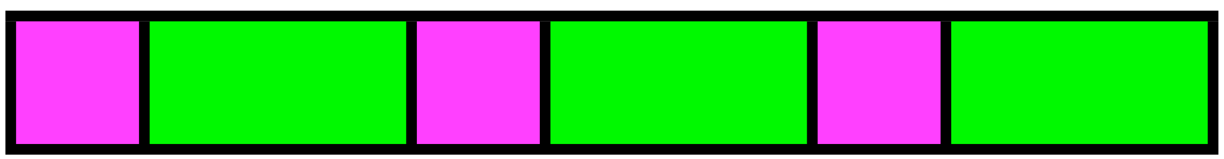
Exception. Every n has at most one exception.

$X = (sd)^j$



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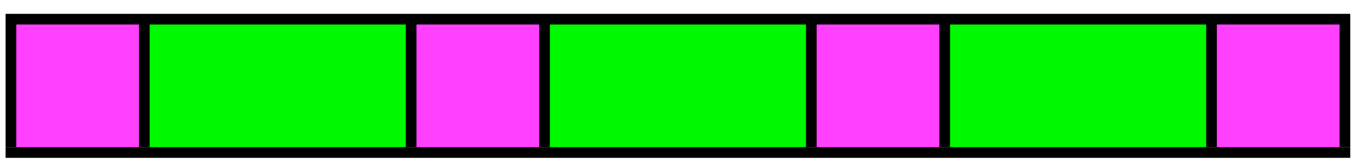
$X = (sd)^j s$



$n = 3j + 1$

If n = 6j or 6j+1, then there is one exception

$X = (sd)^{2j}$ or $X = (sd)^{2j} s$

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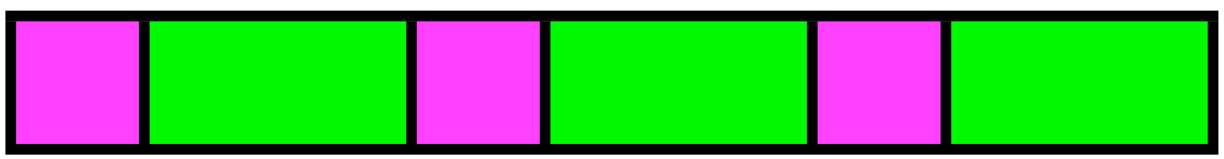
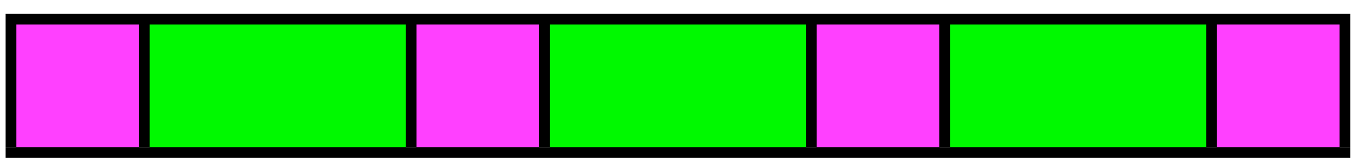
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

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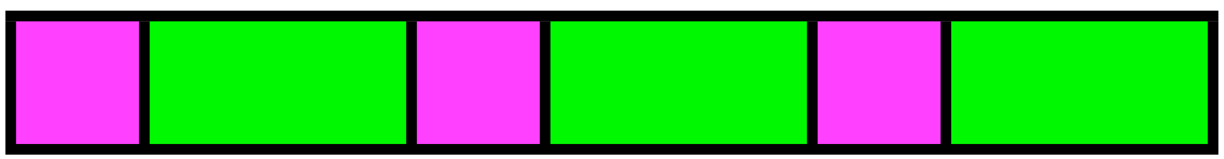
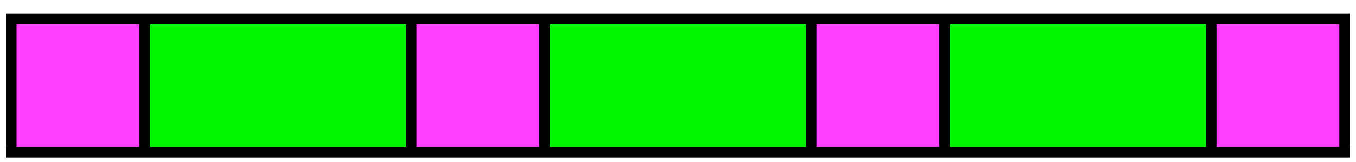
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

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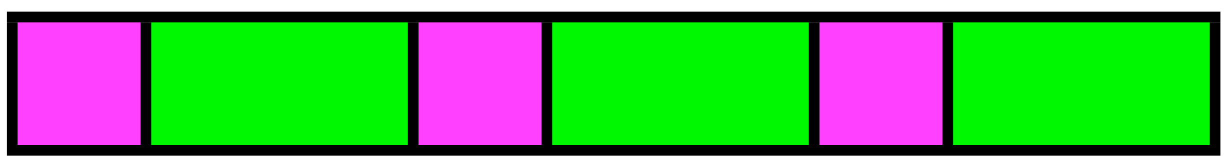

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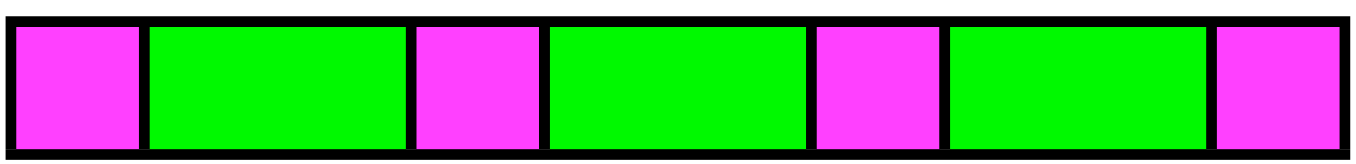

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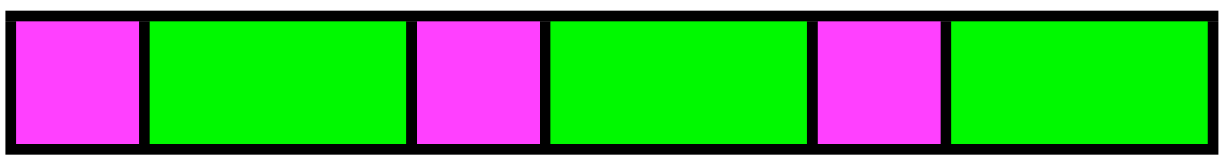
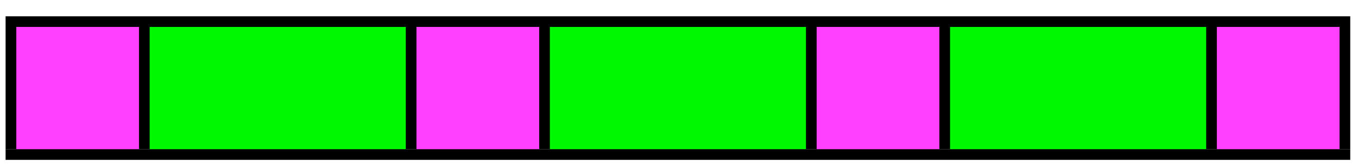
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

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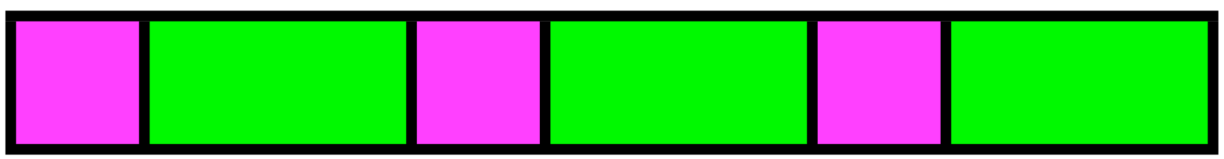

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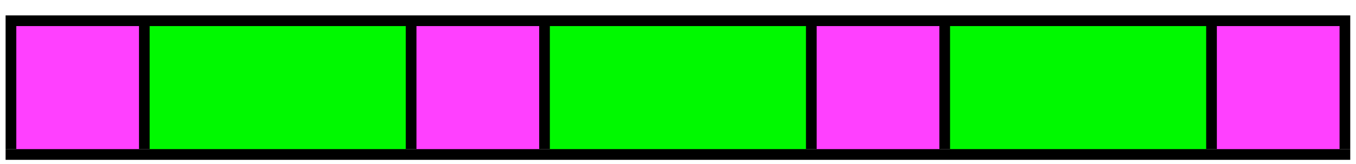

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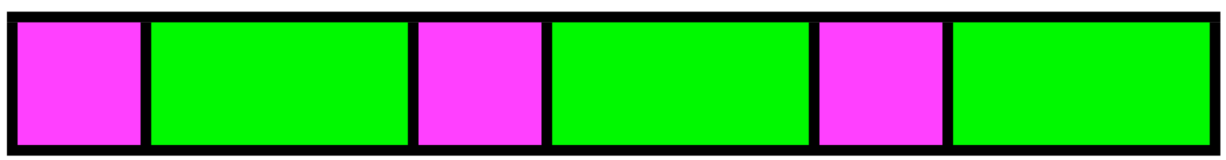

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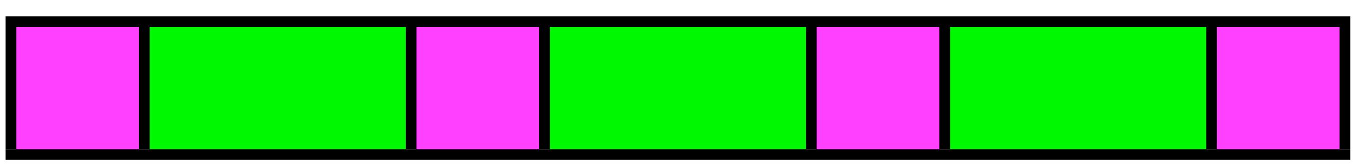

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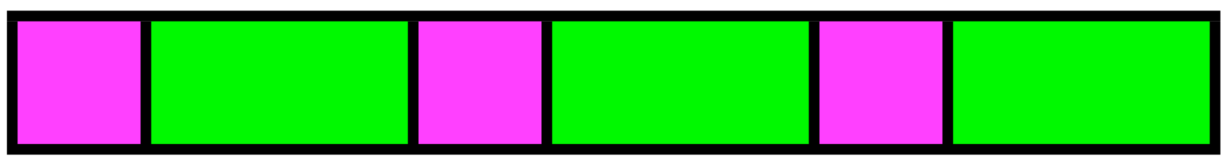
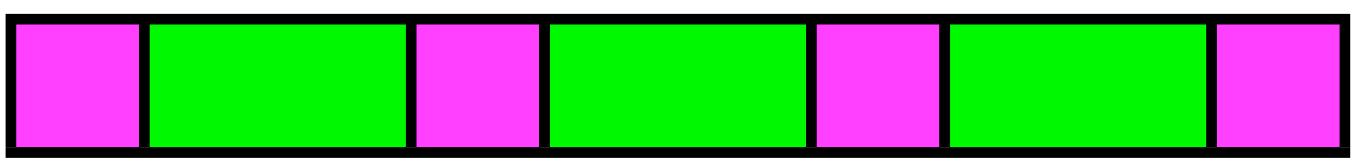
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j+1$

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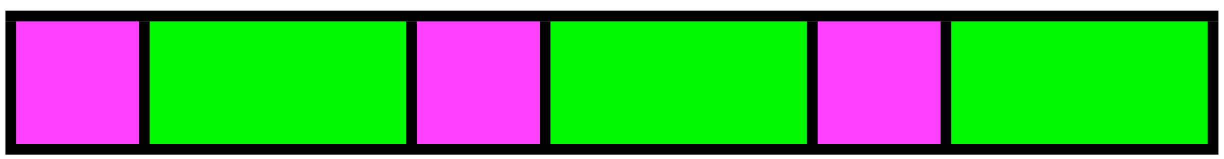
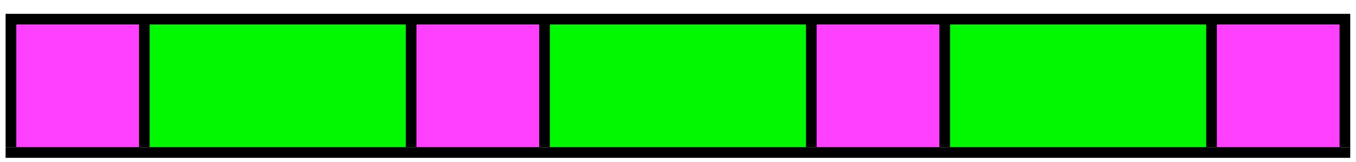
$X = (sd)^j$  $n = 3j$

$X = (sd)^j s$  $n = 3j + 1$

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where $A_0 = 1$, $A_1 = s$, and for $n \geq 2$,

$$A_n = sA_{n-1} + dA_{n-2}.$$

P.I.E. can D.I.E.!

Any combinatorial problem that can be solved by the principle of inclusion-exclusion can also be solved by D.I.E.

Derangements

For $n \geq 1$,

$D_n =$ number of *derangements* of $\{1, 2, \ldots, n\}$
is the number of ways to arrange 1,2,...,n so that no number is in its natural position.

Example: 2 1 4 3 is a derangement.

But 4 1 3 2 is not a derangement.

3 is a fixed point.

Derangements

n=1	n=2	n=3	n=4
none	21	231	2143
		312	2341
			2413
			3142
			3412
			3421
			4123
			4312
			4321

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$D_1 = 0$ $D_2 = 1$ $D_3 = 2$ $D_4 = 9$

Identity: For $n \geq 1$, $D_n = \sum_{k=0}^{n} \frac{n!}{k!}(-1)^k$

Example. n=4:

$$\begin{aligned} D_4 &= \frac{4!}{0!} - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} \\ &= 24 - 24 + 12 - 4 + 1 \\ &= 9 \end{aligned}$$

Identity: For $n \geq 1,\ D_n = \sum_{k=0}^{n} \frac{n!}{k!}(-1)^k$

Description. What does n!/k! count?

Example: n = 9, k = 6

$\frac{9!}{6!} = 9 \times 8 \times 7$ counts

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$\frac{n!}{k!}$ counts words of length $n-k$ using different letters from {1,2,..., n}.

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$\frac{n!}{k!}$ counts words of length $n-k$ using different letters from {1,2,..., n}.

$\sum_{k=0}^{n} \frac{n!}{k!}$ counts words of any length using different elements from {1,2,...,n}.

Identity: For $n \geq 1$, $D_n = \sum_{k=0}^{n} \frac{n!}{k!}(-1)^k$

Description. Words using different letters from {1,...,n}.

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Try to pair up words whose lengths have opposite parity.

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Toggle the number 1, if possible.

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Example.

2358 $\longleftrightarrow$ 12358

492 $\longleftrightarrow$ 1492

$\varnothing$ $\longleftrightarrow$ 1

23456789 $\longleftrightarrow$ 123456789

Identity: For $n \geq 1$, $D_n = \sum_{k=0}^{n} \frac{n!}{k!}(-1)^k$

Description. Words using different letters from {1,...,n}.

What if Rule 1 doesn't apply? Say X = 31459. Then try...

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Involution (Rule 2). Toggle the number 2, if possible. That is,

if 2 is missing from X, then insert 2 in position 2.
if 2 is in X in position 2, then remove it from X.

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Example. 31459 $\longleftrightarrow$ 321459

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Example. 31459 $\longleftrightarrow$ 321459

Note: By inserting or removing 2 in 2nd position, 1 cannot suddenly become a fixed point of X.

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Example. Suppose X = 3145926

Identity: For $n \geq 1,\ D_n = \sum_{k=0}^{n} \frac{n!}{k!}(-1)^k$

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Example. Suppose X = 3145926 m=7

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3145926 ⟷ 32145927 6

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The derangements of {1,2,...,n}!

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Note: n!/k! counts words of length n - k.

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Each derangement has length n, so k = 0.
Thus, all derangements are counted positively.

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☺

Back to Trigonometry: Up-Down permutations

The permutation

2 7 1 8 3 9 4 6 5

is an example of an up-down (or zig-zag) permutation since the numbers alternatiely go up then down.

$$a_1 < a_2 > a_3 < a_4 > a_5 < a_6 > \cdots$$

Let U_n be the number of up-down permutations of length n.

n = 0: $\varnothing$ $U_0 = 1$

n = 1: 1 $U_1 = 1$

n = 2: 12 $U_2 = 1$

n = 3: 132 231 $U_3 = 2$

n = 4: 1324 1423 2314 2413 3412 $U_4 = 5$

n = 5: 13254 14253 14352 15243
15342 23154 24153 24351
25143 25341 34152 34251
35142 35241 45132 45231 $U_5 = 16$

Also, $U_6 = 61$, $U_7 = 272$, ...

No exact formula for U_n, but it has recurrence

$$2U_{n+1} = \sum_{k\geq 0} \binom{n}{k} U_k U_{n-k}$$

and asymptotic formula

$$U_n \sim \frac{2^{n+2} n!}{\pi^{n+1}}$$

and exponential generating function

$$\begin{aligned} U(x) &= \sum_{n\geq 0} U_n \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \frac{16x^5}{5!} + \frac{61x^6}{6!} + \frac{272x^7}{7!} + \cdots \end{aligned}$$

Remarkably,

$$U(x) = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \frac{16x^5}{5!} + \frac{61x^6}{6!} + \frac{272x^7}{7!} + \cdots$$

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In fact,

$$\begin{aligned} U_{\text{even}}(x) &= 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots \\ &= \sec x \end{aligned}$$

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$$\begin{aligned} U_{\text{odd}}(x) &= x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \cdots \\ &= \tan x \end{aligned}$$

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$$(\cos x)U_{\text{even}}(x) = 1$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(U_0 + U_2\frac{x^2}{2!} + U_4\frac{x^4}{4!} + U_6\frac{x^6}{6!} + \cdots\right) = 1$$

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Show: For even n > 0,

$$U_{\text{even}}(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$$
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$$(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots)(U_0 + U_2\frac{x^2}{2!} + U_4\frac{x^4}{4!} + U_6\frac{x^6}{6!} + \cdots) = 1$$

Constant term = U_0 = 1.

All odd terms have coefficient of zero.

Show: For even n > 0, $\sum_{k\geq 0} \binom{n}{2k} U_{n-2k}(-1)^k = 0.$

Identity: For even n > 0, $\sum_{k\geq 0} \binom{n}{2k} U_{n-2k}(-1)^k = 0.$

This has a beautiful D.I.E. proof, due to Ira Gessel.

$T_0(x) = 1$

The Chebyshev Polynomials

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What do Chebyshev Polynomials Count?

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What do Fibonacci Numbers Count?

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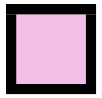
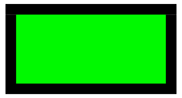
Q: How many ways to tile a *1* × *n* board with squares and dominoes ?

What do Fibonacci Numbers Count?

Q: How many ways to tile a $1 \times n$ board with squares  and dominoes  ?

$n = 1$

$n = 2$

$n = 3$

$n = 4$

$n = 5$

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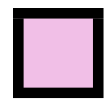
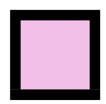
Q: How many ways to tile a *1* × *n* board with squares  and dominoes  ?

$n = 1$  1 way

$n = 2$  2 ways

$n = 3$

$n = 4$

$n = 5$

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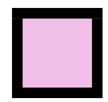
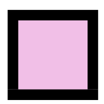
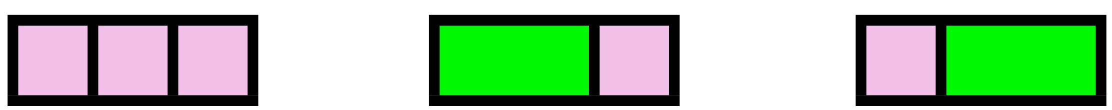
Q: How many ways to tile a $1 \times n$ board with squares  and dominoes  ?

$n = 1$  1 way

$n = 2$  2 ways

$n = 3$  3 ways

$n = 4$

$n = 5$

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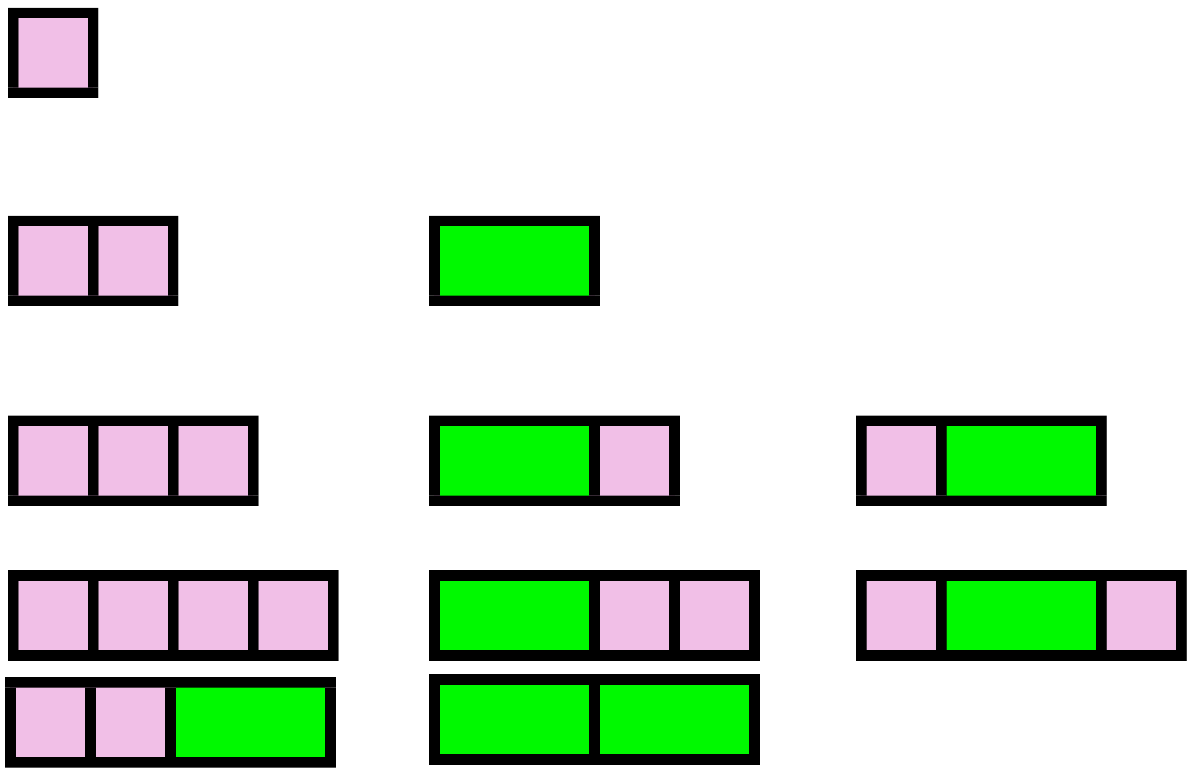

$n = 1$ — 1 way

$n = 2$ — 2 ways

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$n = 5$ — 8 ways

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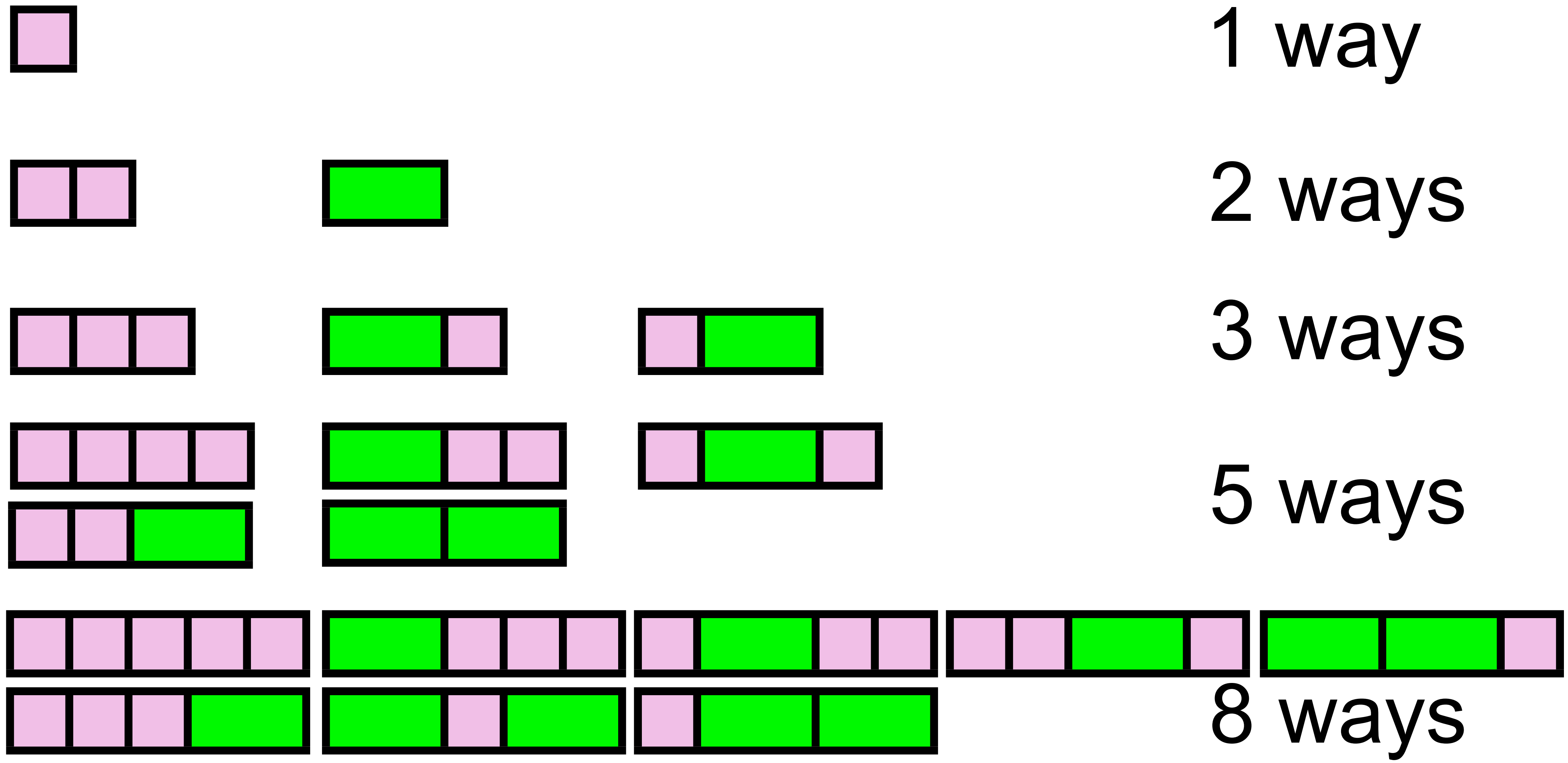

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$n = 5$ 8 ways

A: The n-th Fibonacci number!

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2x\, T_{n-1}(x) -1\, T_{n-2}(x).$$

What does $T_n(x)$ count?

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Answer: Weighted Tilings

x	2x	2x	2x	2x	2x	2x	2x
1	2	3	4	.	.	.	*n*

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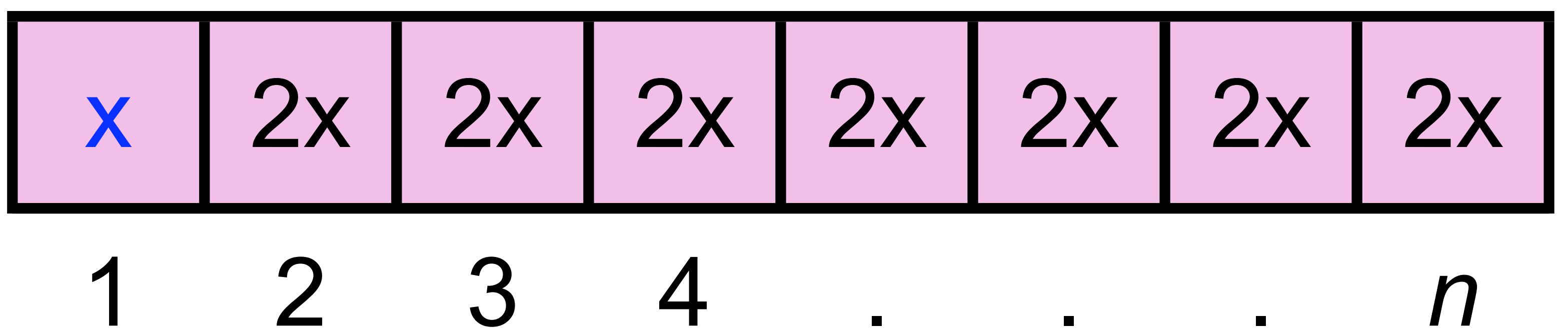
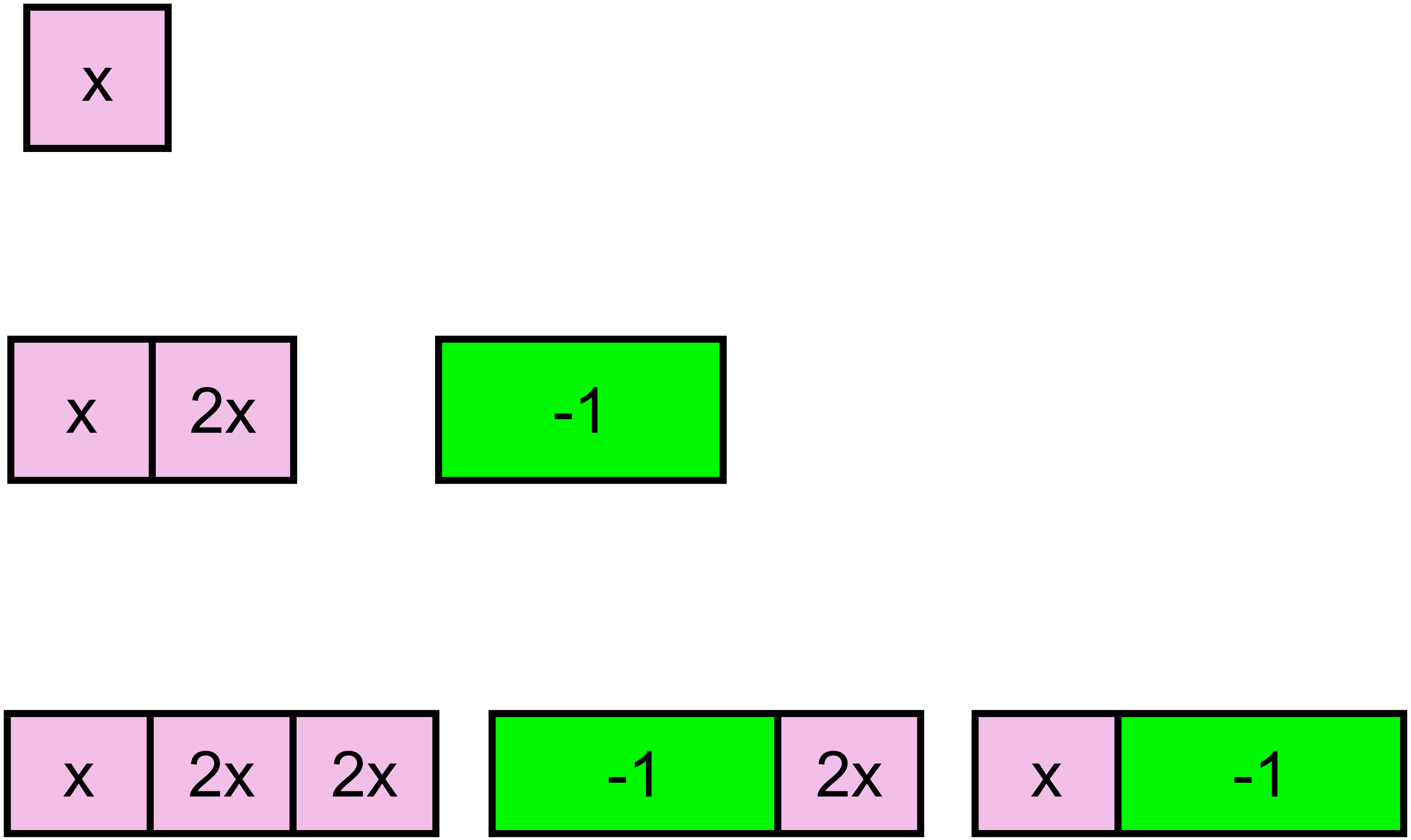
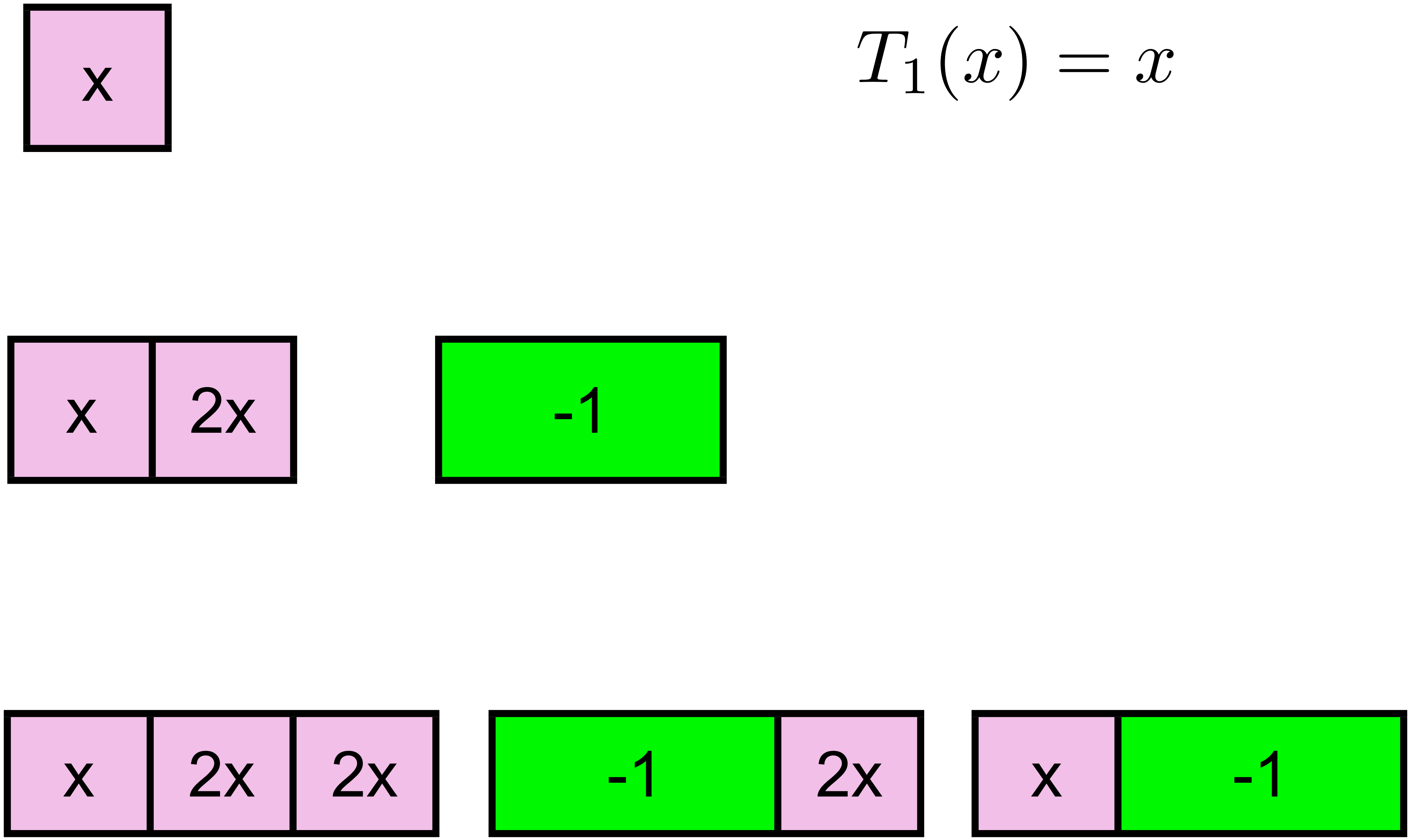
Answer: Weighted Tilings

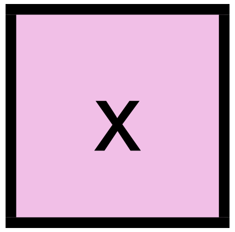


A square on cell 1 has weight *x*.
All other squares have weight *2x*.
All dominoes have weight -1.
The weight of a tiling is the *product* of the weights of its tiles.


x
x
2x
-1
x
2x
2x
-1
2x
x
-1

$$T_1(x) = x$$





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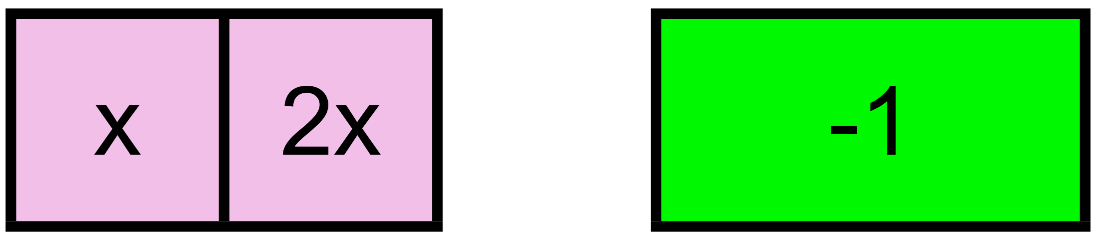
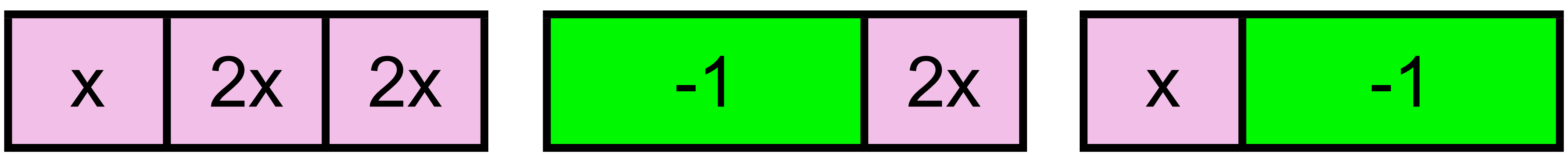
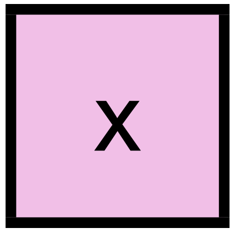


$2x^2$ -1





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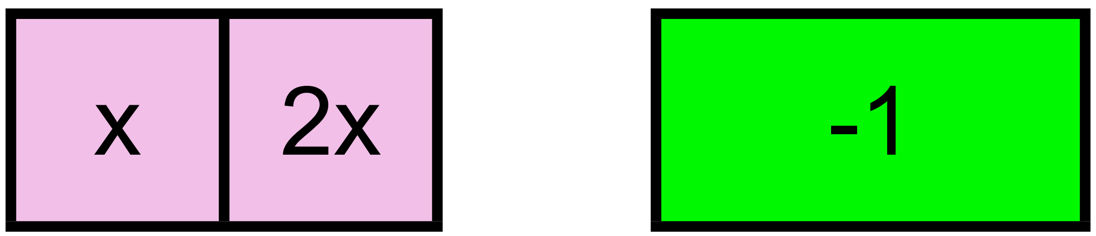
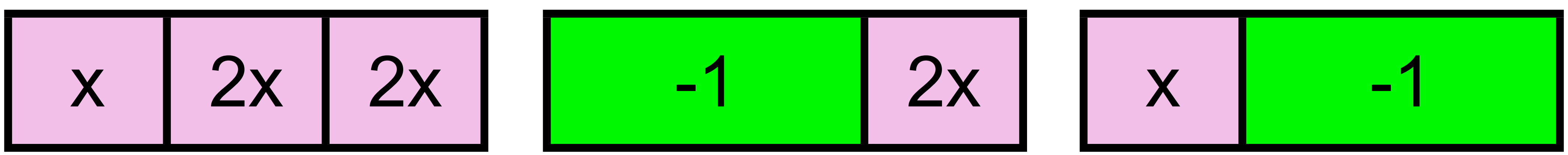
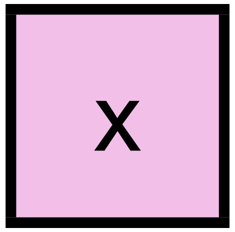


$2x^2$ $\quad$ -1

$$T_2(x) = 2x^2 - 1$$





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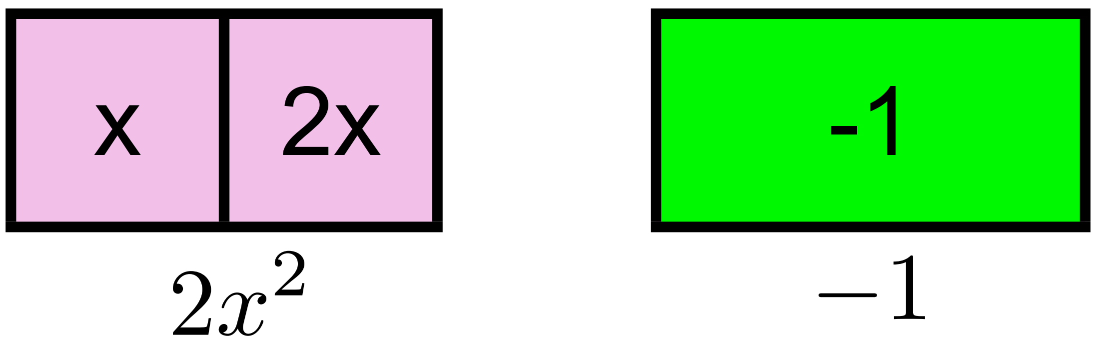

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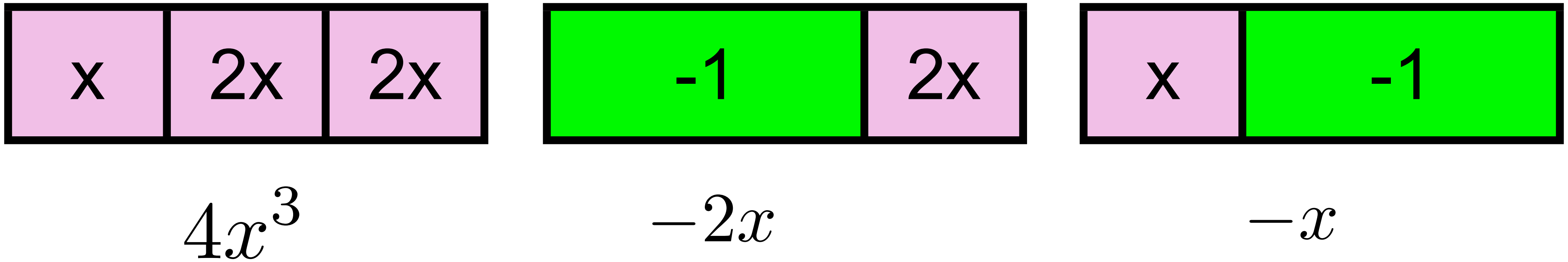

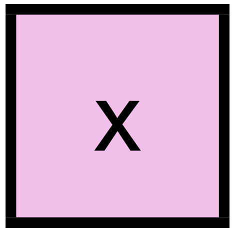

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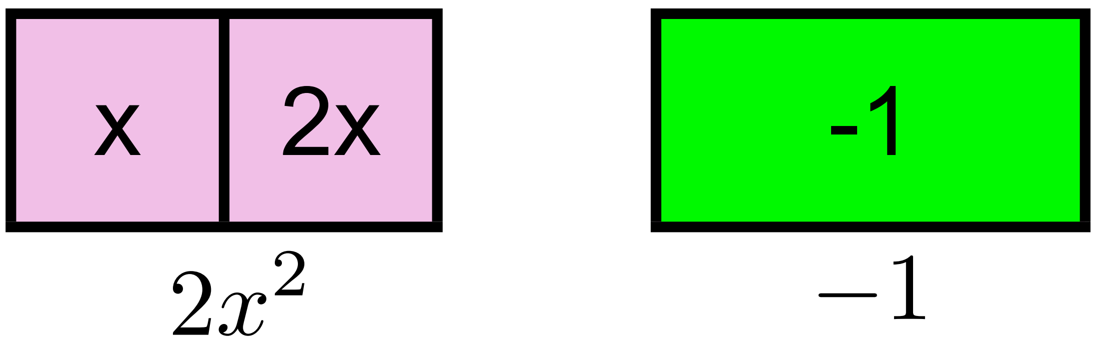

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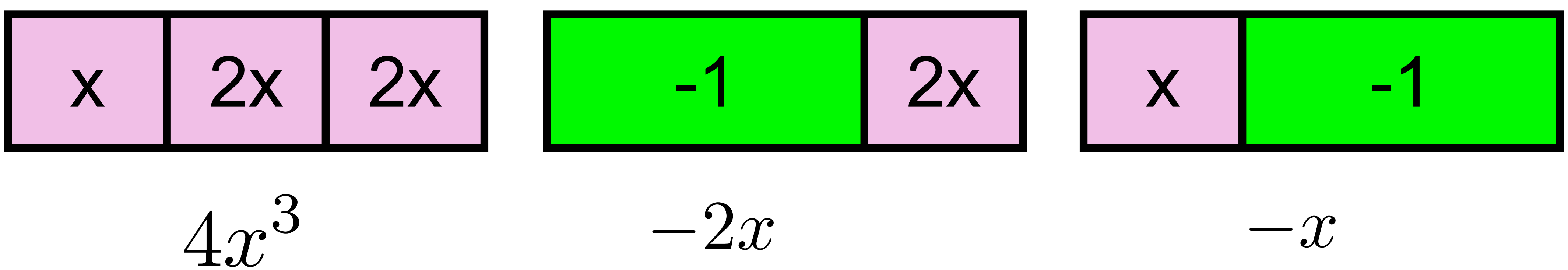

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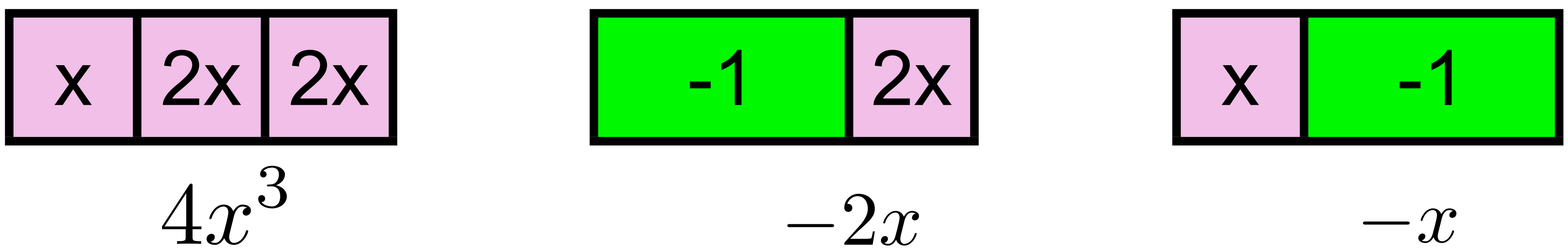

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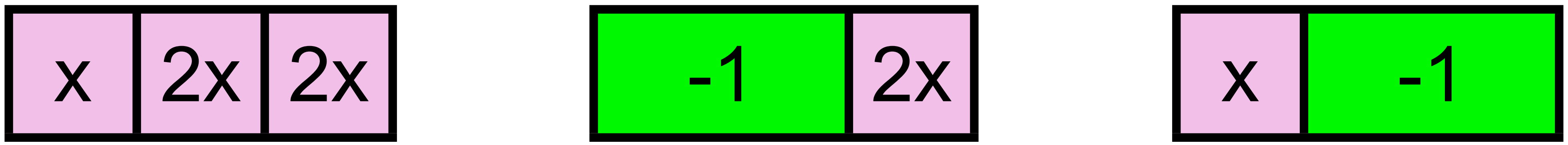

$4x^3$ $-2x$ $-x$

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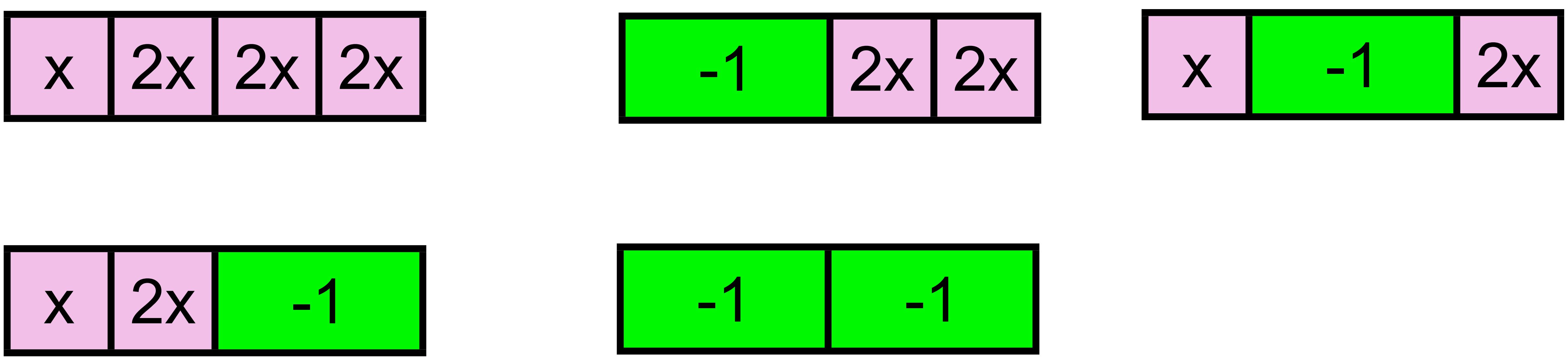

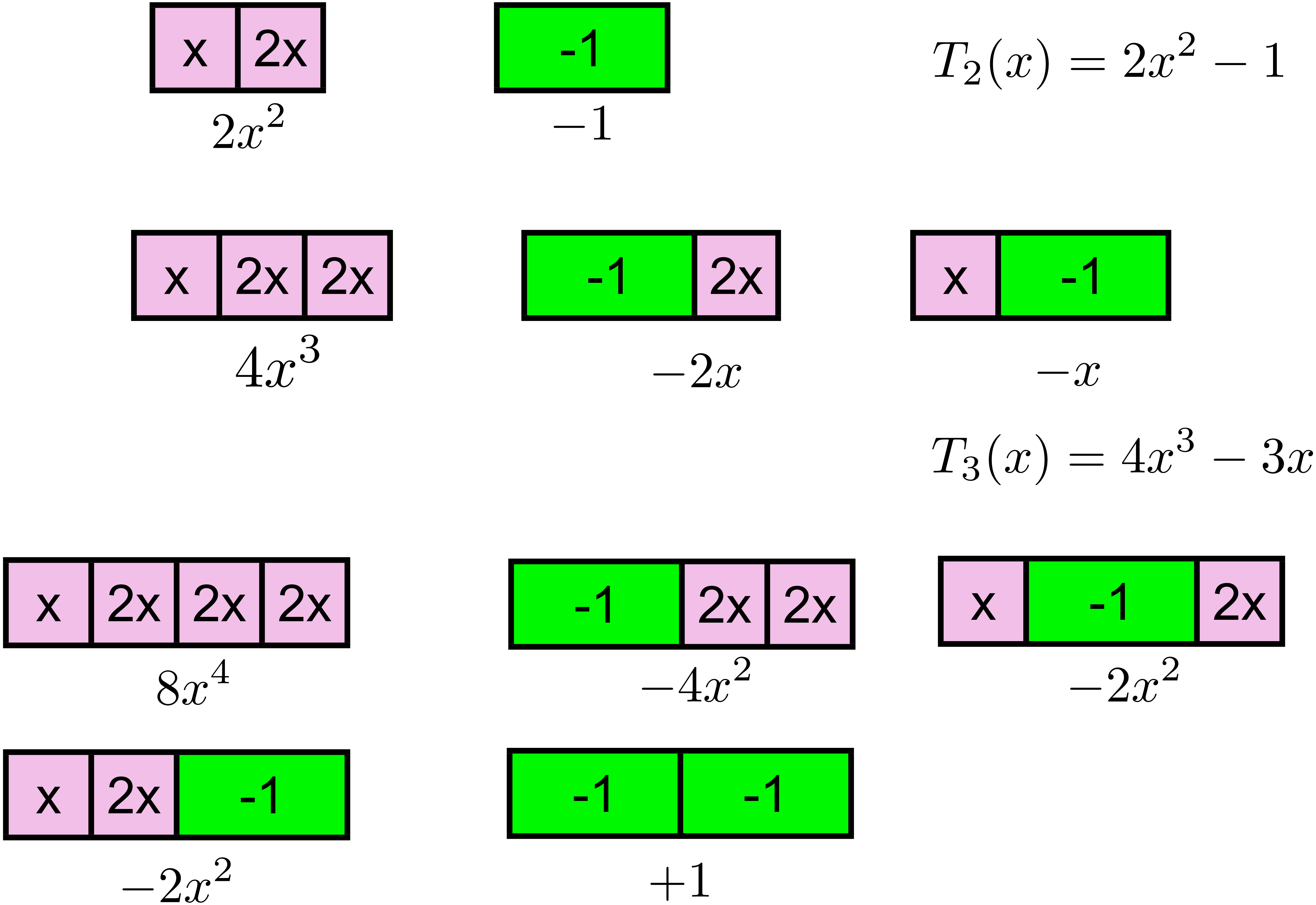

$$T_2(x) = 2x^2 - 1$$

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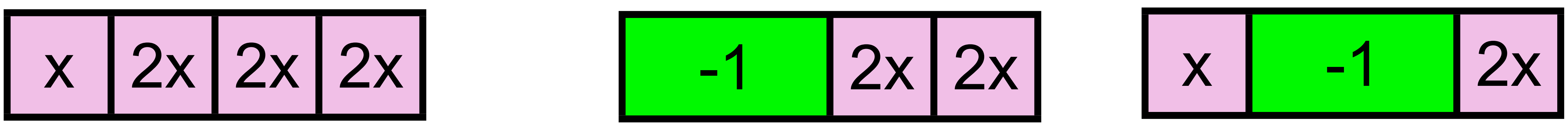
$4x^3$ $\quad$ $-2x$ $\quad$ $-x$

$$T_3(x) = 4x^3 - 3x$$

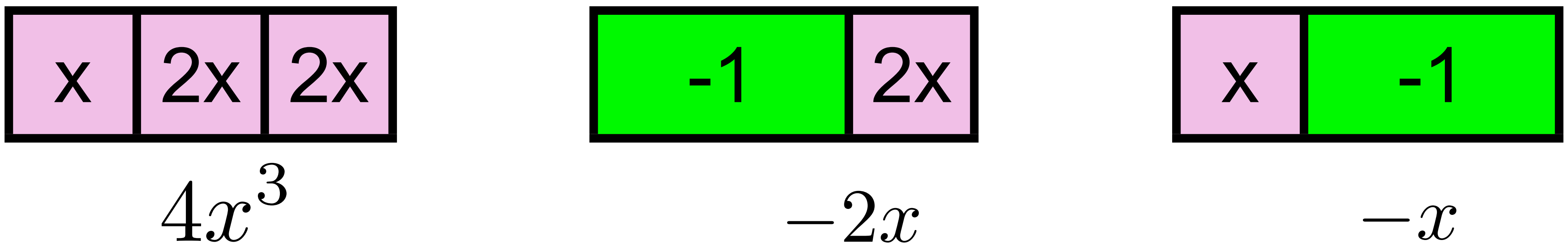


$8x^4$ $\quad$ $-4x^2$ $\quad$ $-2x^2$

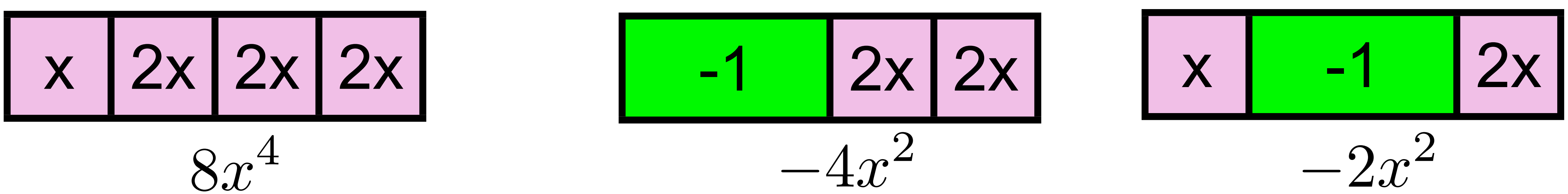
$-2x^2$ $\quad$ $+1$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_2(x) = 2x^2 - 1$$



$$T_3(x) = 4x^3 - 3x$$



$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_4(x) = 2xT_3(x) - T_2(x)$$

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

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Combinat**o**ria**l** Proof:

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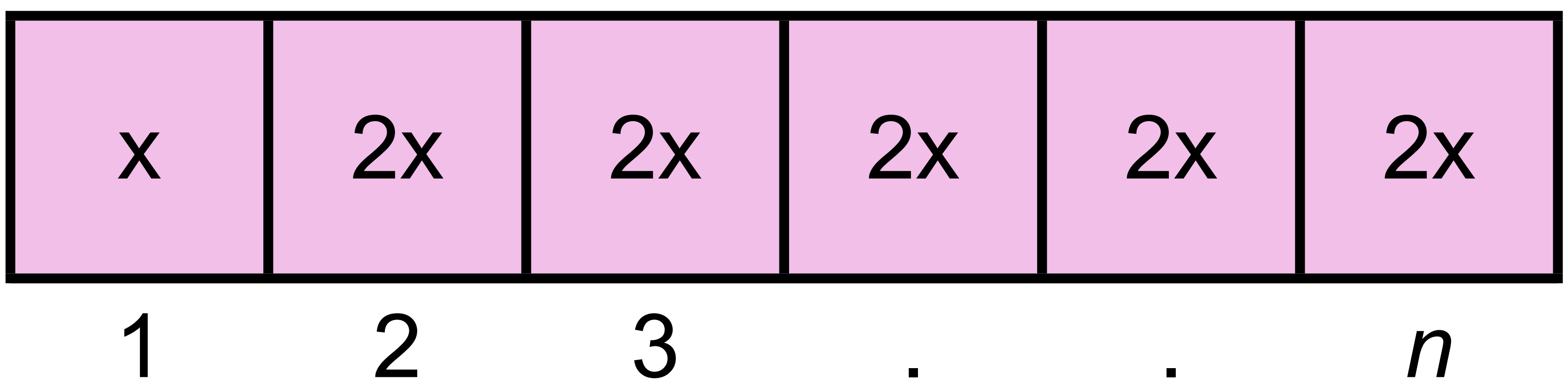
Combinat**o**rial Proof:

$T_n(x)$ counts weighted tilings of



Theorem: $T_n(\cos\theta) = \cos(n\theta)$

Combinat**o**rial Proof:

$T_n(x)$ counts weighted tilings of

x	2x	2x	2x	2x	2x
1	2	3	.	.	*n*

So $T_n(\cos\theta)$ counts weighted tilings of

$\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$
1	2	3	.	.	*n*

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

Combina**t**o**r**ia**l** Proof:

So $T_n(\cos\theta)$ counts weighted tilings of

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$\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$
1	2	3	.	.	*n*

Now what?

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

Combinat**o**ria**l** Proof:

So $T_n(\cos\theta)$ counts weighted tilings of

$\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$
1	2	3	.	.	*n*

Euler to the rescue!

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1	2	3	.	.	*n*

Euler to the rescue!

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

Combinatorial Proof:

$T_n(\cos\theta)$ counts weighted tilings of

$\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$	$2\cos\theta$
1	2	3	.	.	*n*

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$\frac{e^{i\theta} + e^{-i\theta}}{2}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$
1	2	3	.	.	*n*

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

Combinat**o**ria**l** Proof:

$T_n(\cos\theta)$ counts weighted tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	*n*

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

Combinat**o**ria**l** Proof:

$T_n(\cos\theta)$ counts weighted tilings of

$\frac{e^{i\theta} + e^{-i\theta}}{2}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$	$e^{i\theta} + e^{-i\theta}$
1	2	3	.	.	*n*

Next, we add color

Theorem: $T_n(\cos\theta) = \cos(n\theta)$

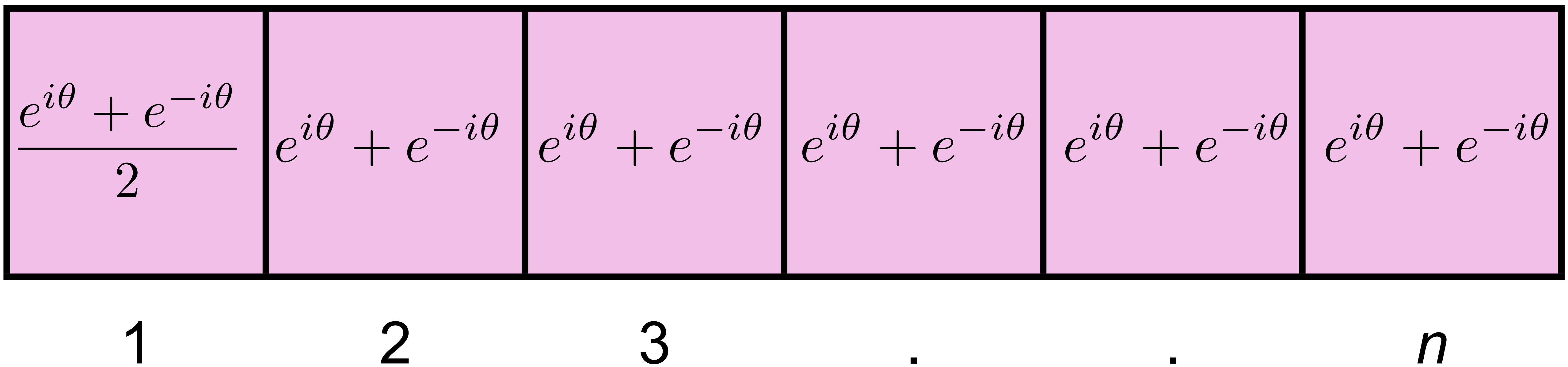
Combinatorial Proof:

$T_n(\cos\theta)$ counts weighted tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	n

Next, we add color

$T_n(\cos\theta)$ counts weighted tilings of

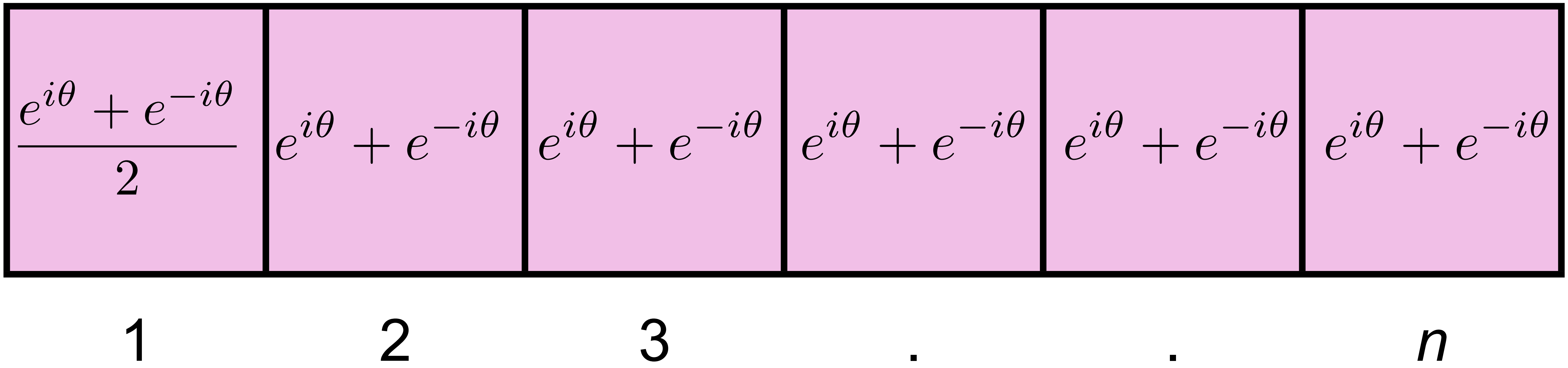


$T_n(\cos\theta)$ counts weighted tilings of

$\dfrac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	*n*

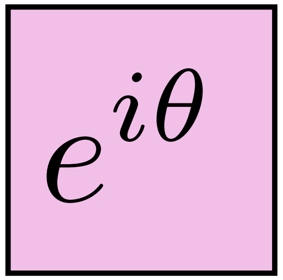
Introduce two colors of squares:

$T_n(\cos\theta)$ counts weighted tilings of



Introduce two colors of squares:

Light squares have weight $e^{i\theta}$



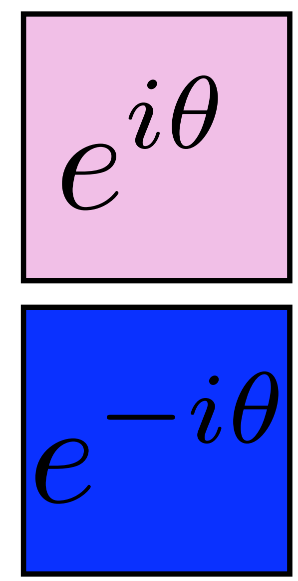
$T_n(\cos\theta)$ counts weighted tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	n

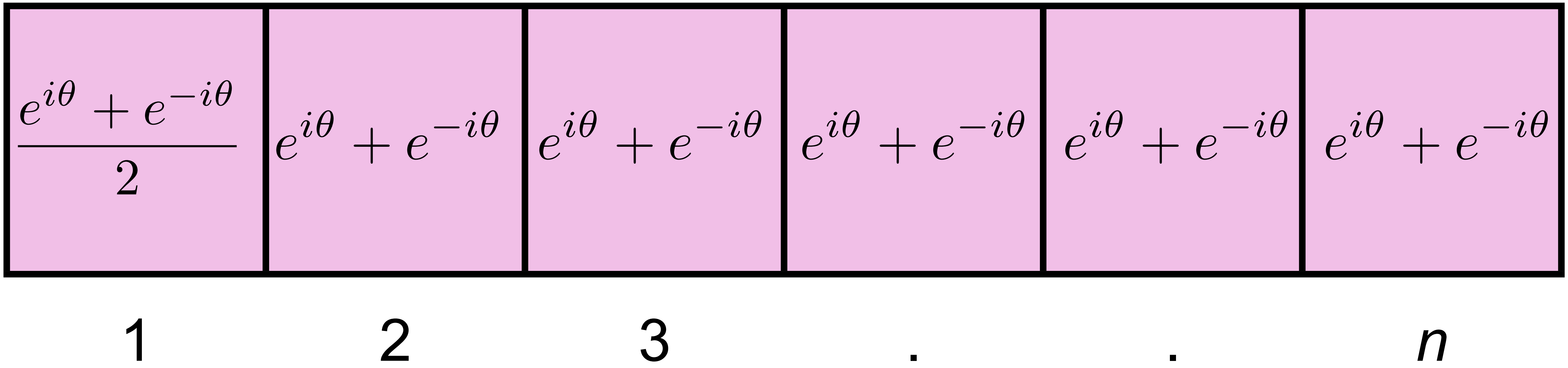
Introduce two colors of squares:

Light squares have weight $e^{i\theta}$

Dark squares have weight $e^{-i\theta}$



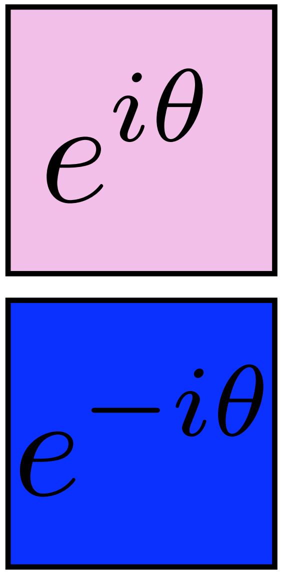
$T_n(\cos\theta)$ counts weighted tilings of



Introduce two colors of squares:

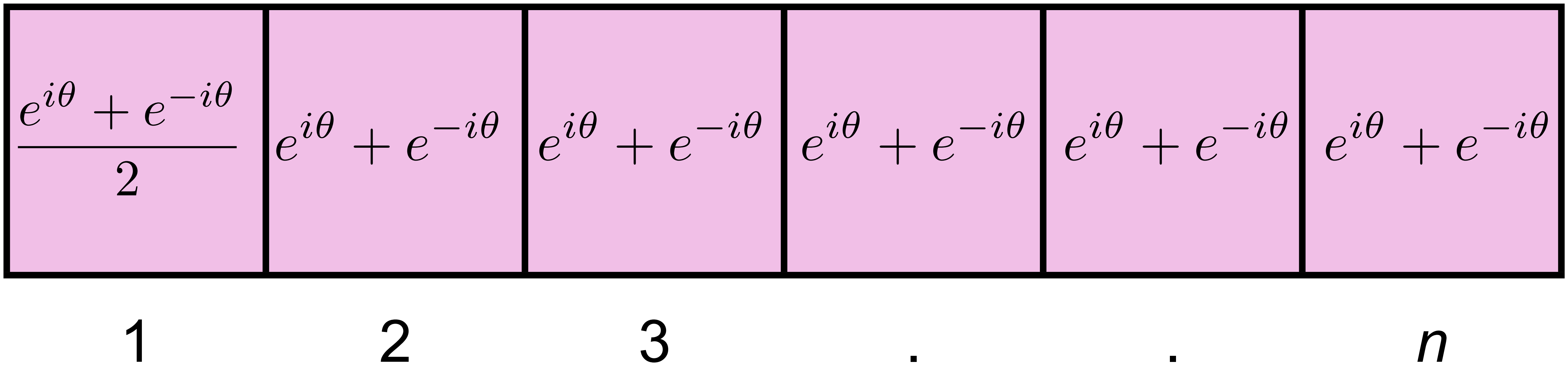
Light squares have weight $e^{i\theta}$

Dark squares have weight $e^{-i\theta}$



Exception: A square on cell 1 has weight $\frac{e^{i\theta}}{2}$ or $\frac{e^{-i\theta}}{2}$

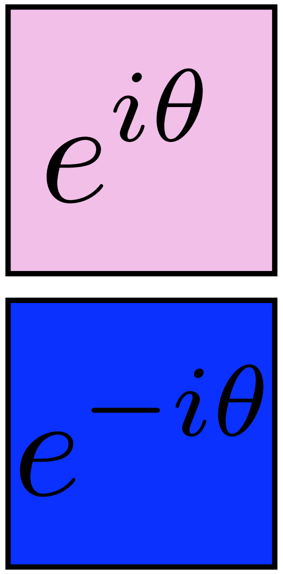
$T_n(\cos\theta)$ counts weighted tilings of



Introduce two colors of squares:

Light squares have weight $e^{i\theta}$

Dark squares have weight $e^{-i\theta}$



$T_n(\cos\theta)$ counts weighted tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	*n*

Introduce two colors of squares:

Light squares have weight $e^{i\theta}$ — $e^{i\theta}$

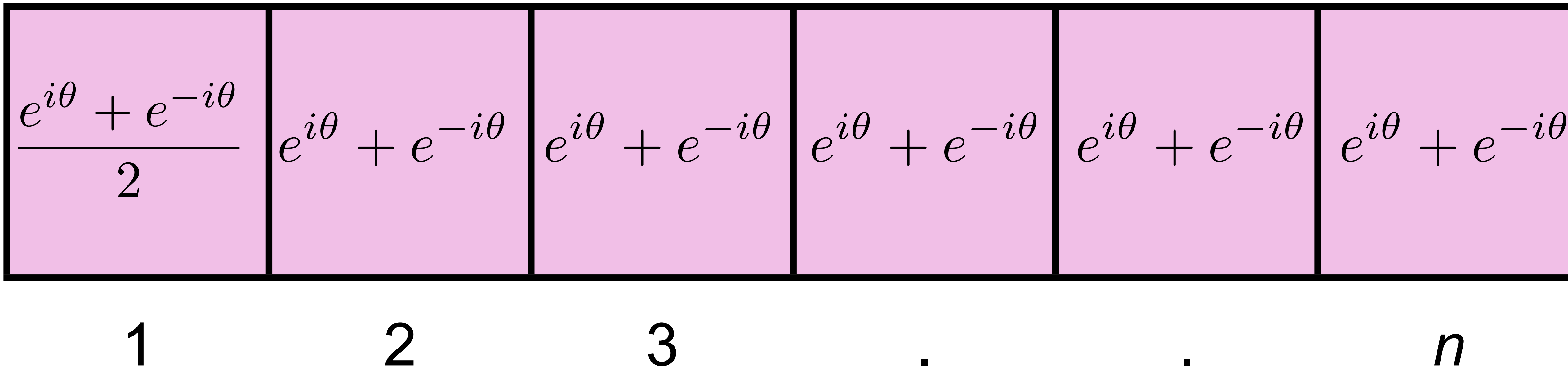
Dark squares have weight $e^{-i\theta}$ — $e^{-i\theta}$

Dominoes have weight -1 — -1

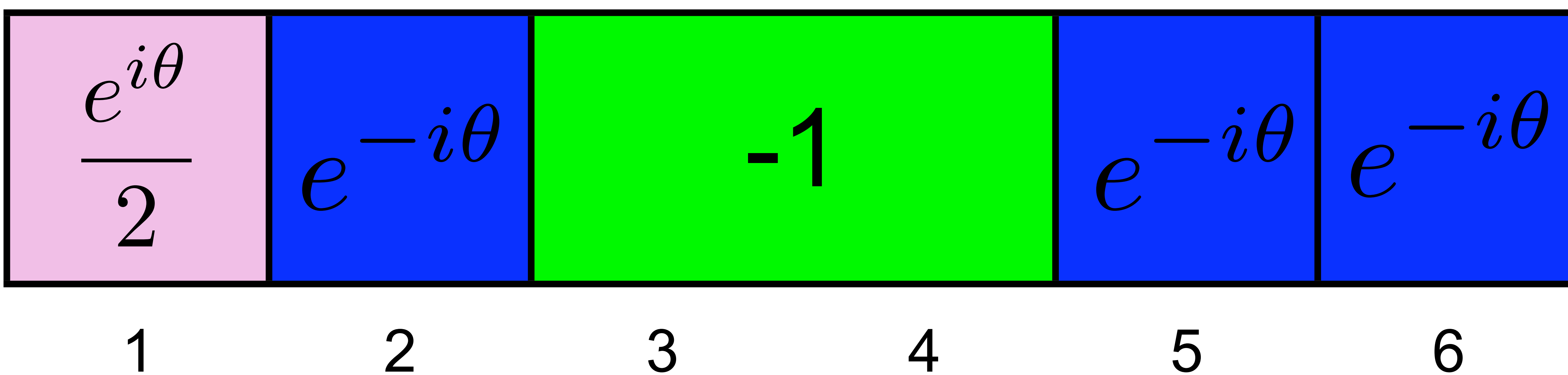
$T_n(\cos\theta)$ counts weighted colored tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	*n*

$T_n(\cos\theta)$ counts weighted colored tilings of



Example:



$T_n(\cos\theta)$ counts weighted colored tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	n

Example:

$\frac{e^{i\theta}}{2}$	$e^{-i\theta}$	-1		$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	4	5	6

has weight

$T_n(\cos\theta)$ counts weighted colored tilings of

$\frac{e^{i\theta}+e^{-i\theta}}{2}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$	$e^{i\theta}+e^{-i\theta}$
1	2	3	.	.	*n*

Example:

$\frac{e^{i\theta}}{2}$	$e^{-i\theta}$	-1		$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	4	5	6

has weight $-\frac{e^{-2i\theta}}{2}$

Another example:

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{-i\theta}$	$e^{i\theta}$	-1	$e^{i\theta}$	$e^{-i\theta}$	-1

Another example:

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{-i\theta}$	$e^{i\theta}$	-1	$e^{i\theta}$	$e^{-i\theta}$	-1

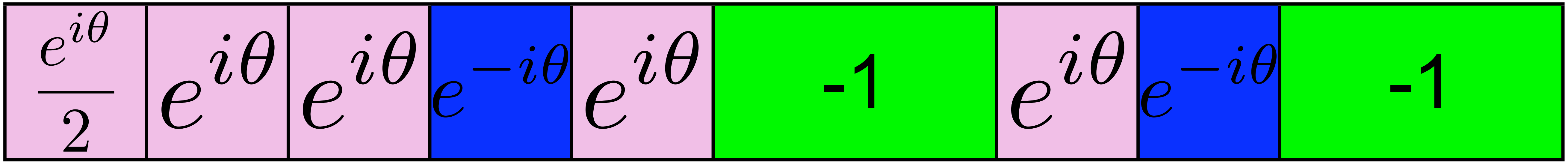
has weight

Another example:

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{-i\theta}$	$e^{i\theta}$	-1	$e^{i\theta}$	$e^{-i\theta}$	-1

has weight $\frac{e^{3i\theta}}{2}$

Another example:



has weight $\dfrac{e^{3i\theta}}{2}$

$T_n(\cos\theta)$ is the sum of the weights of all colored tilings.

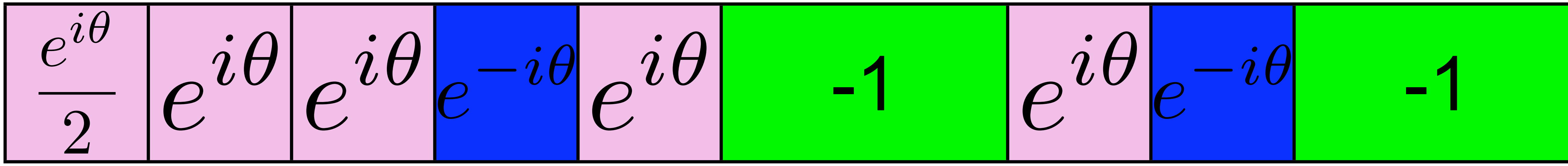
Another example:

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{-i\theta}$	$e^{i\theta}$	-1	$e^{i\theta}$	$e^{-i\theta}$	-1

has weight $\frac{e^{3i\theta}}{2}$

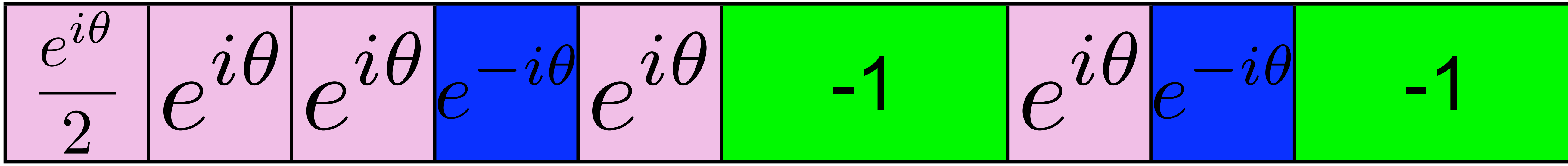
$T_n(\cos\theta)$ is the sum of the weights of all colored tilings.

We now show that this sum is almost zero!



has weight $\frac{e^{3i\theta}}{2}$

Definition: A tiling is impure if it contains a domino or if it contains two adjacent squares of opposite color.

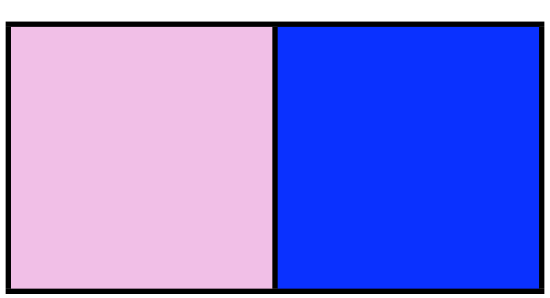


has weight $\frac{e^{3i\theta}}{2}$

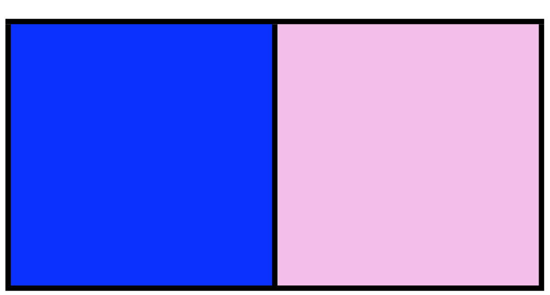
Definition: A tiling is impure if it contains a domino or if it contains two adjacent squares of opposite color.

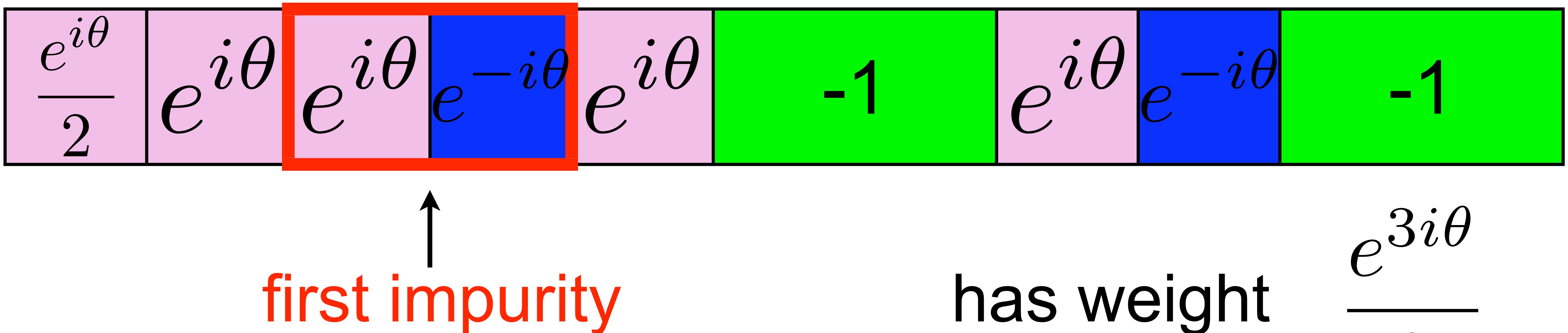
Thus a tiling has an impurity if it contains

or



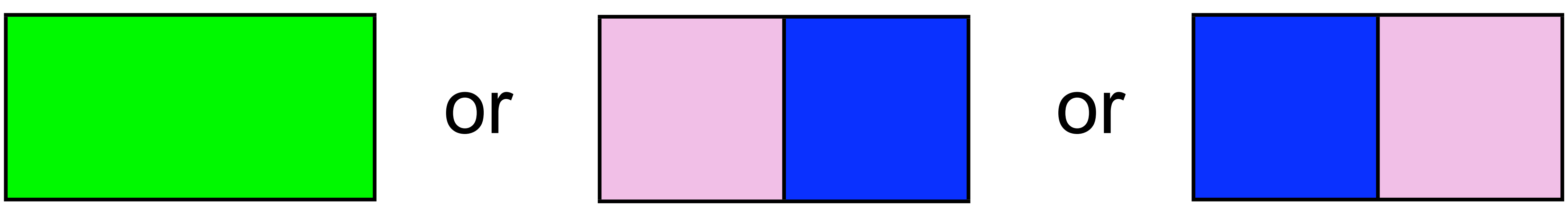
or





has weight $\frac{e^{3i\theta}}{2}$

A tiling has an impurity if it contains



Our Proof Strategy

$T_n(\cos\theta)$ counts all colored tilings of length *n*.

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We show that all the impure tilings sum to zero.

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Thus $T_n(\cos\theta)$ counts all the pure tilings.

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2 cases:

Thus $T_n(\cos\theta)$ counts all the pure tilings.

Our Proof Strategy

$T_n(\cos\theta)$ counts all colored tilings of length *n*.

We show that all the impure tilings sum to zero.

2 cases:

The tiling does not start with an impurity.

Thus $T_n(\cos\theta)$ counts all the pure tilings.

Our Proof Strategy

$T_n(\cos\theta)$ counts all colored tilings of length *n*.

We show that all the impure tilings sum to zero.

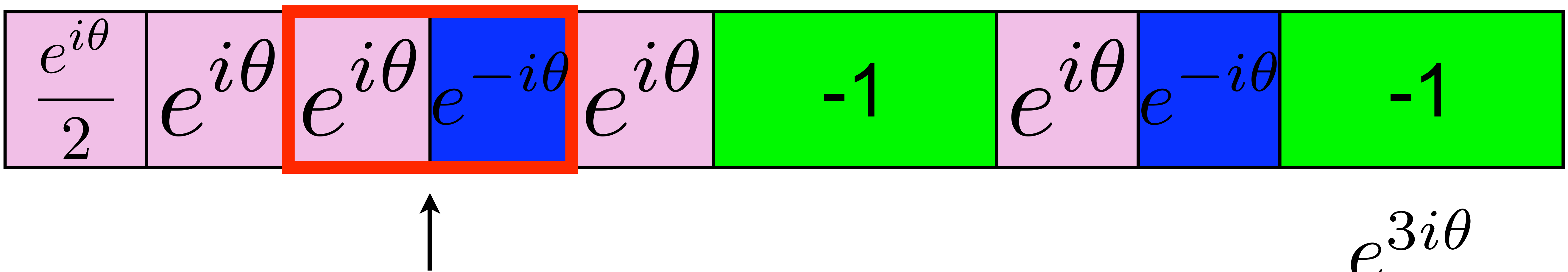
2 cases:

The tiling does not start with an impurity.

The tiling does start with an impurity.

Thus $T_n(\cos\theta)$ counts all the pure tilings.

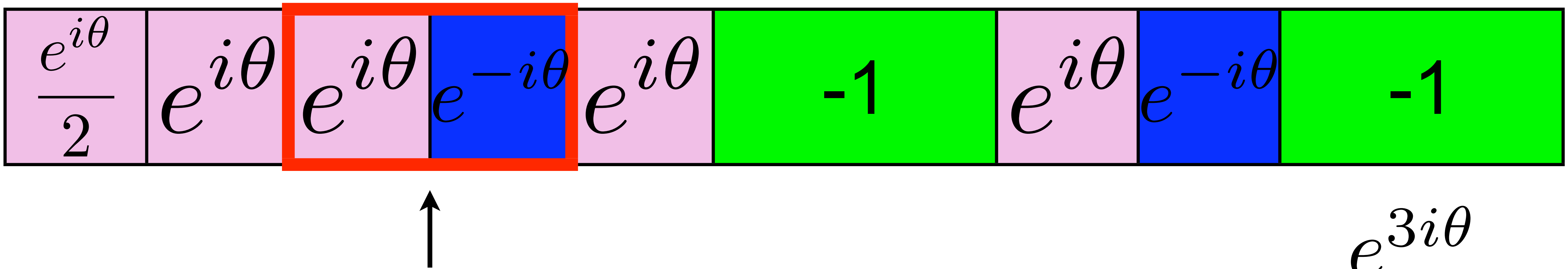
If the tiling does not start with an impurity



first impurity

has weight $\frac{e^{3i\theta}}{2}$

If the tiling does not start with an impurity

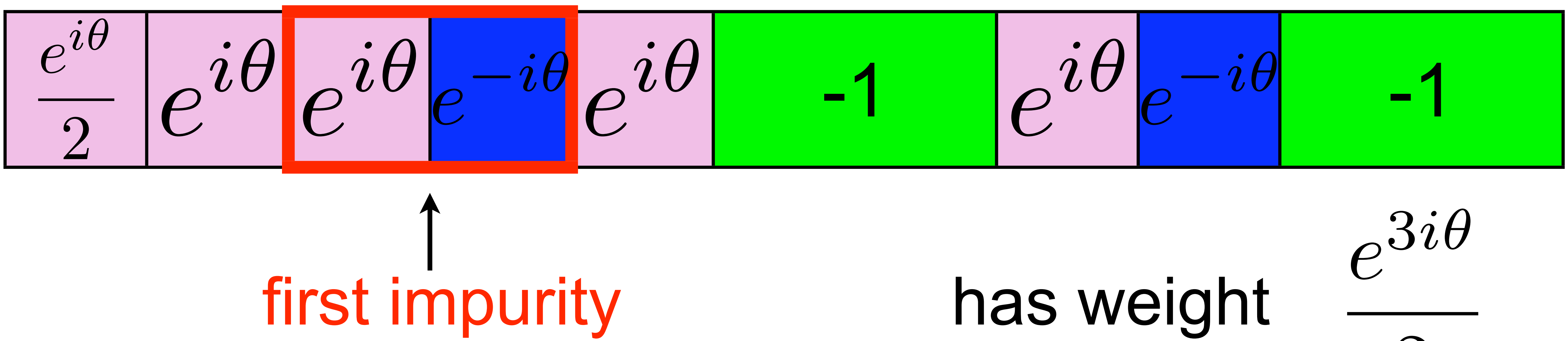


first impurity

has weight $\frac{e^{3i\theta}}{2}$

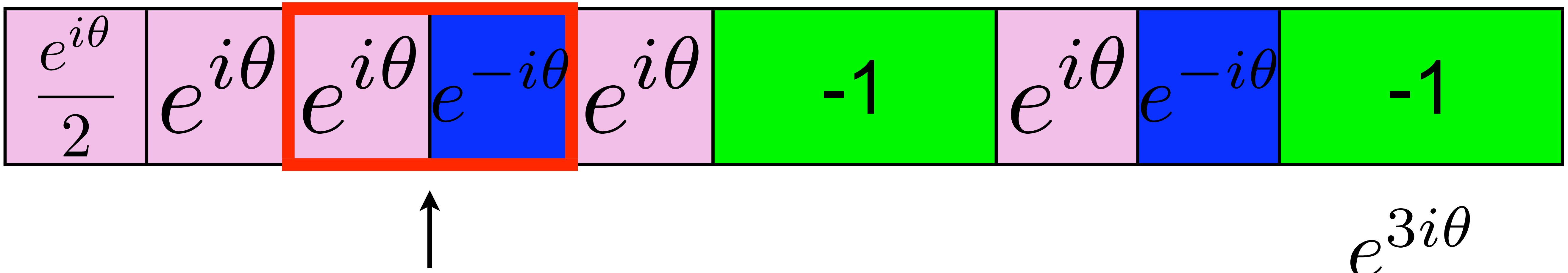
Find a mate of opposite weight!

If the tiling does not start with an impurity



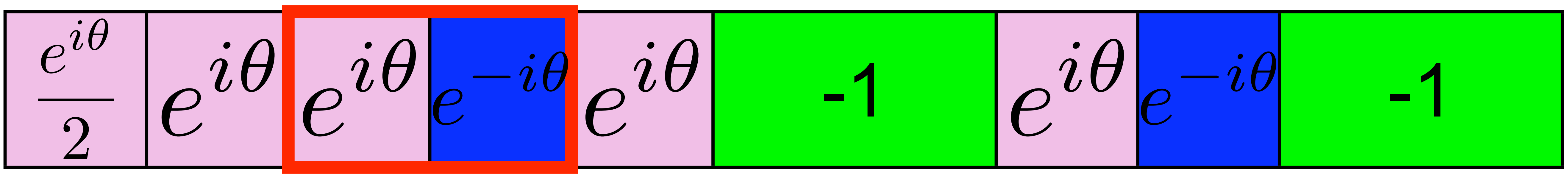
has weight $\frac{e^{3i\theta}}{2}$

If the tiling does not start with an impurity

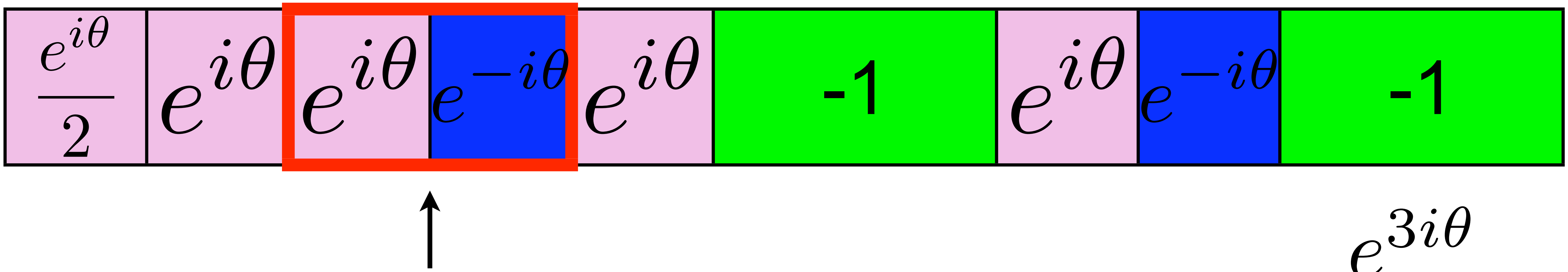


first impurity

has weight $\frac{e^{3i\theta}}{2}$

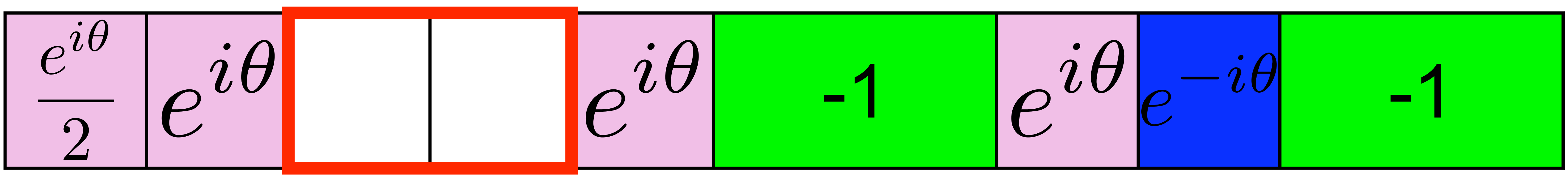


If the tiling does not start with an impurity



first impurity

has weight $\frac{e^{3i\theta}}{2}$



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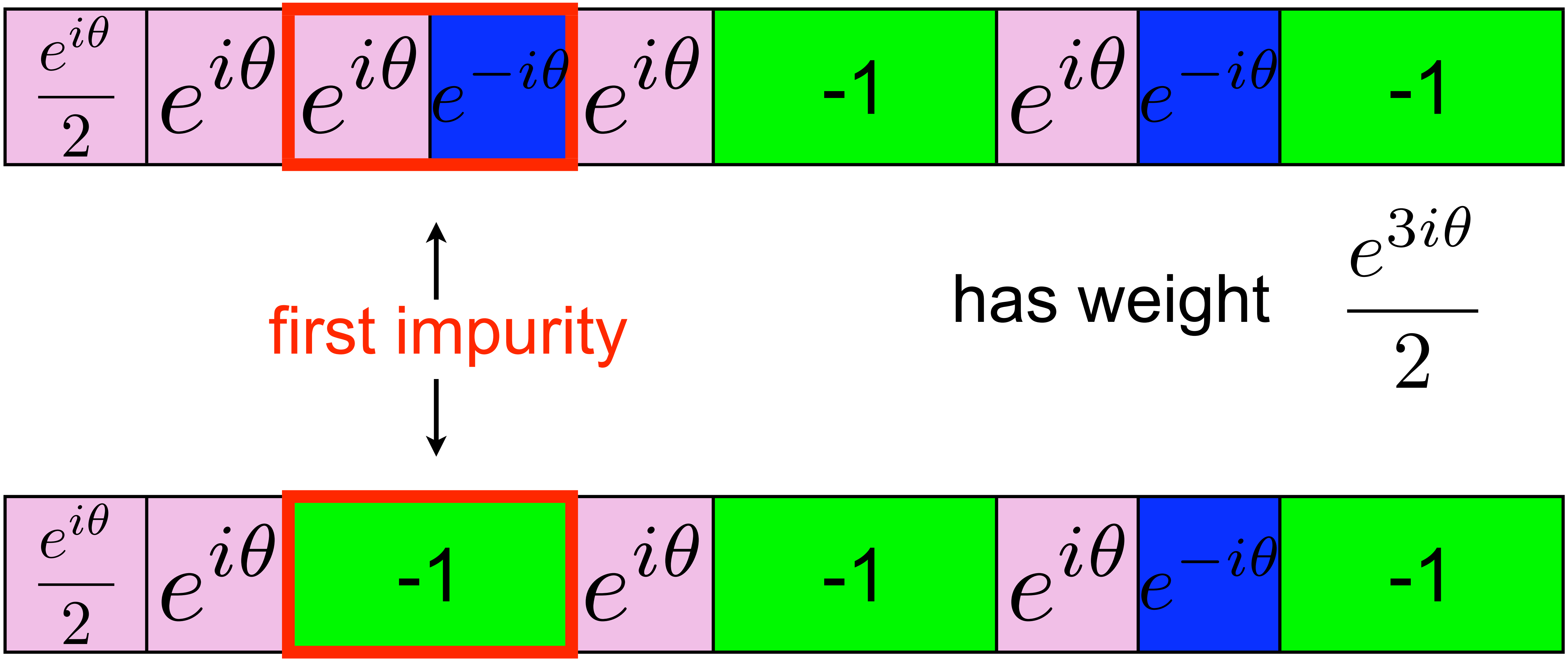

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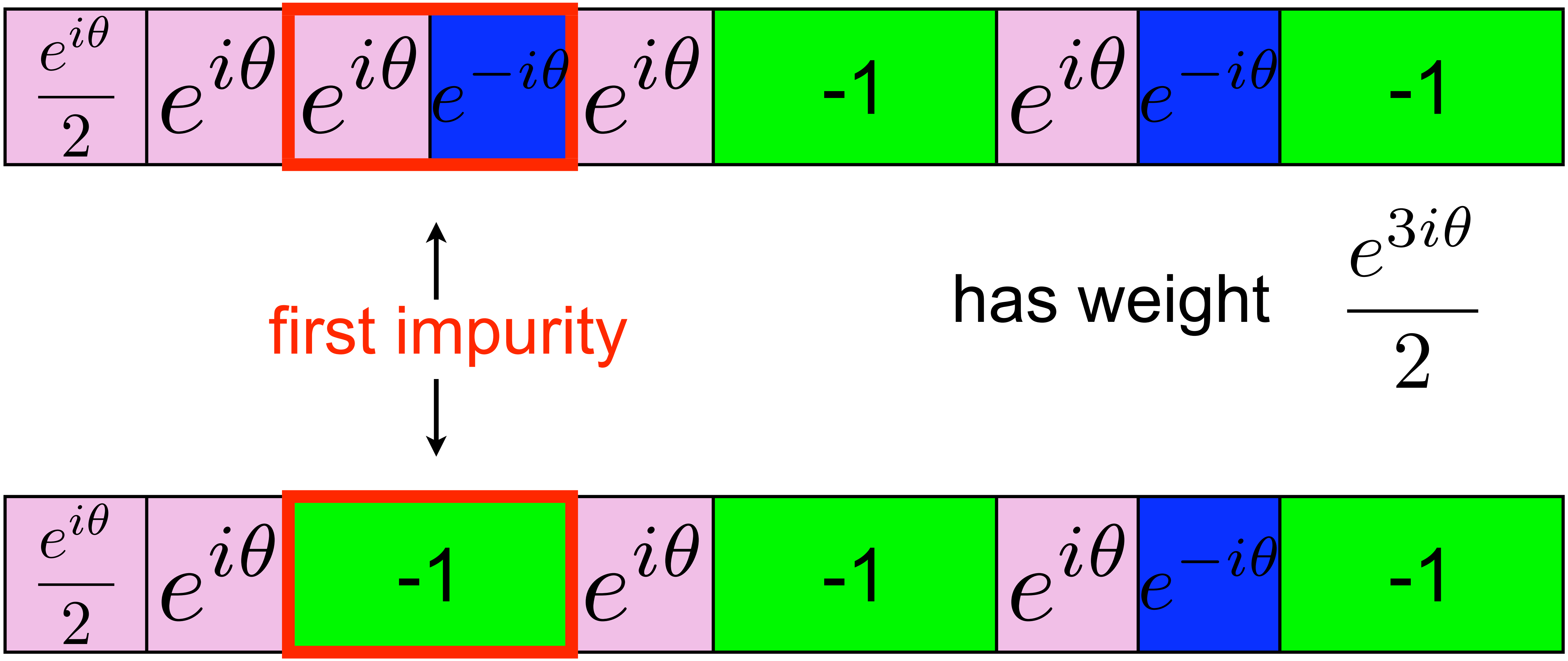

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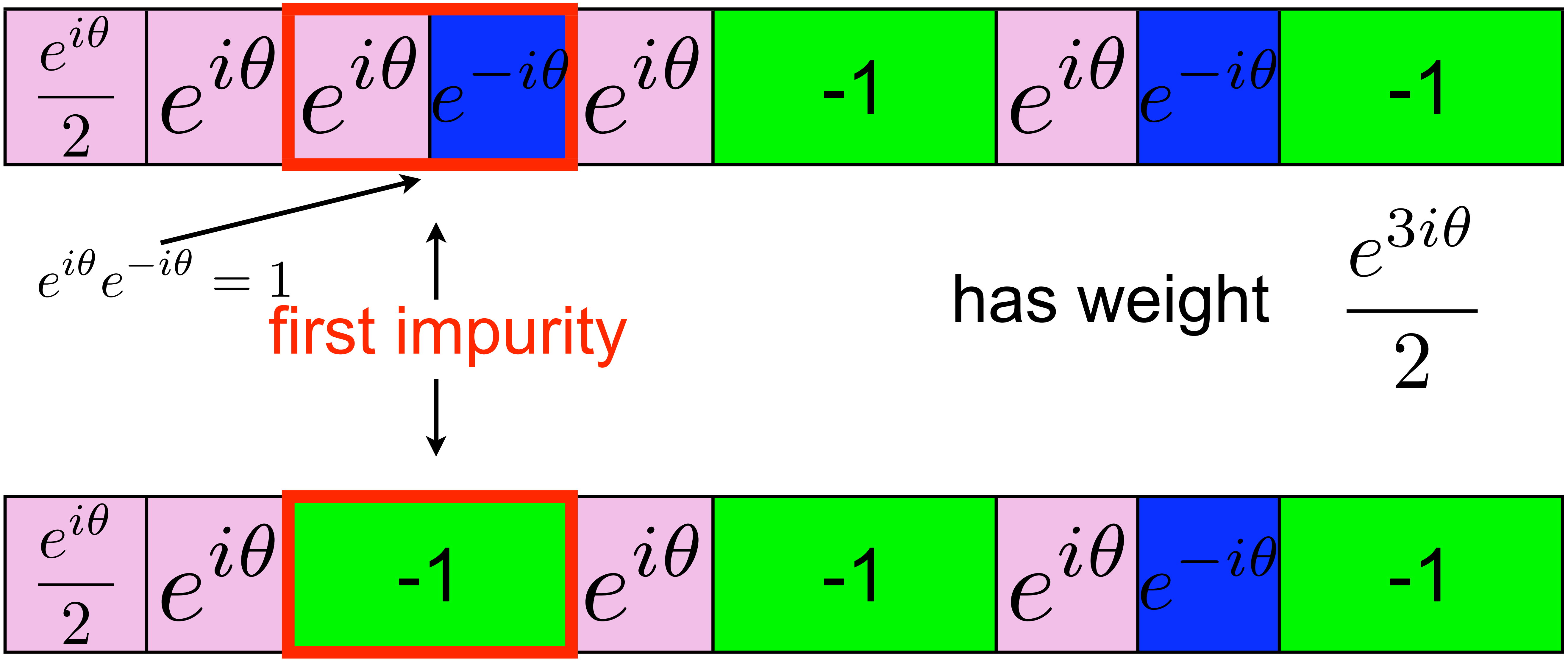

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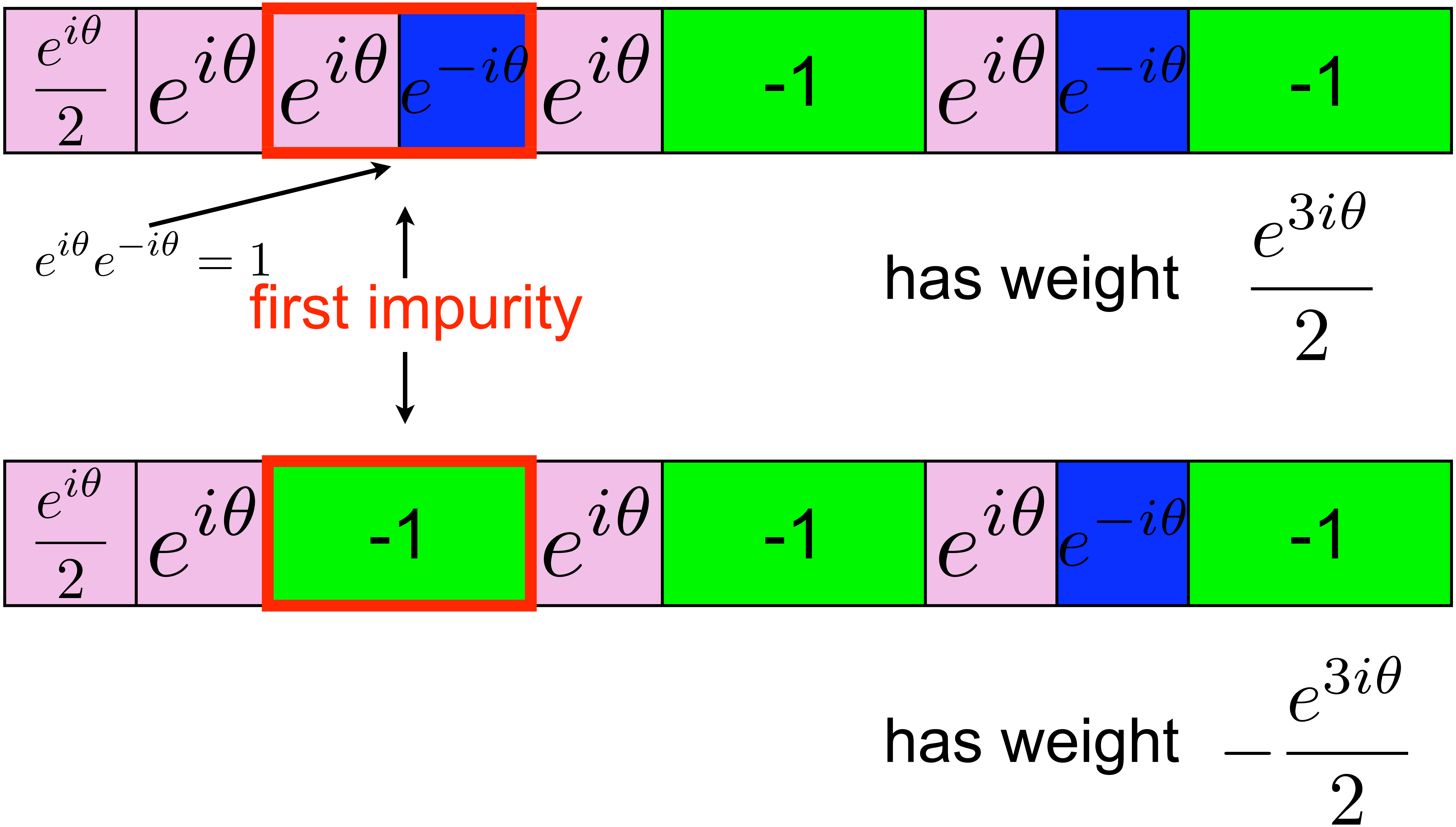

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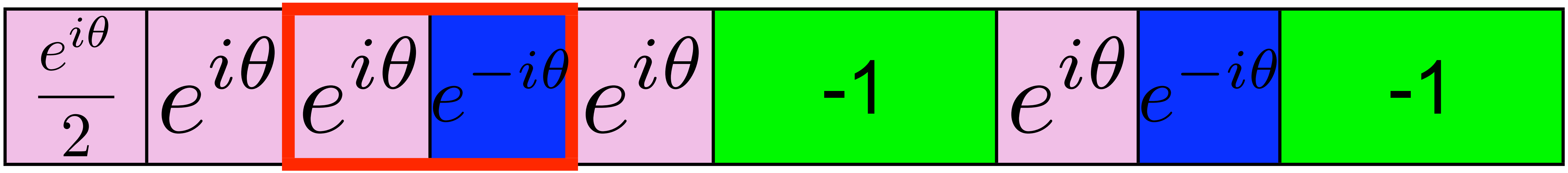
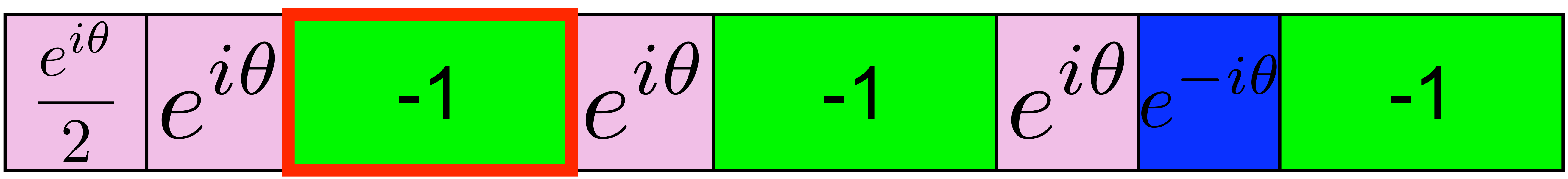


$e^{i\theta}e^{-i\theta} = 1$

first impurity

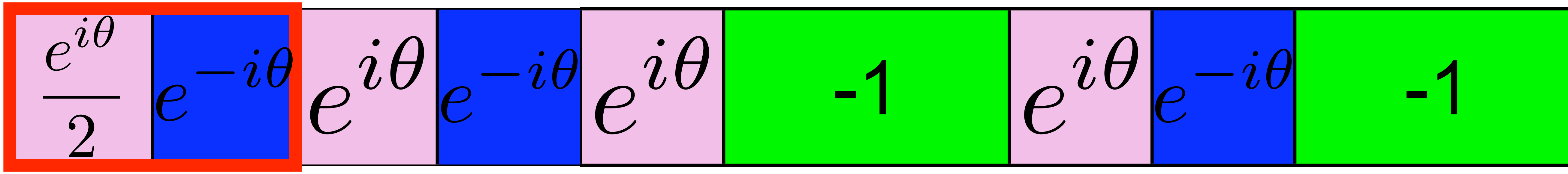
has weight $\frac{e^{3i\theta}}{2}$



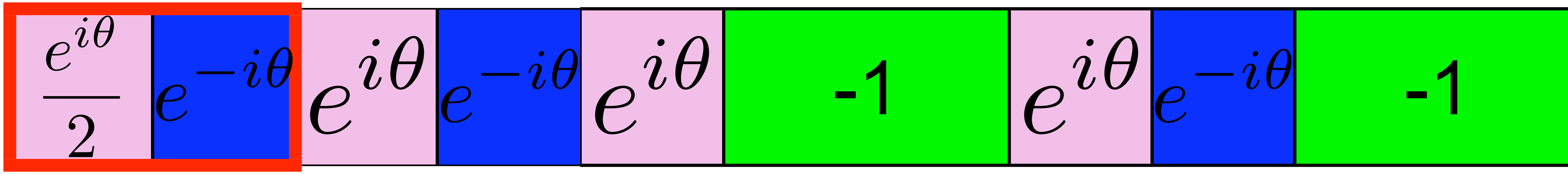
has weight $-\frac{e^{3i\theta}}{2}$

Their weights sum to zero!

If the tiling starts with an impurity

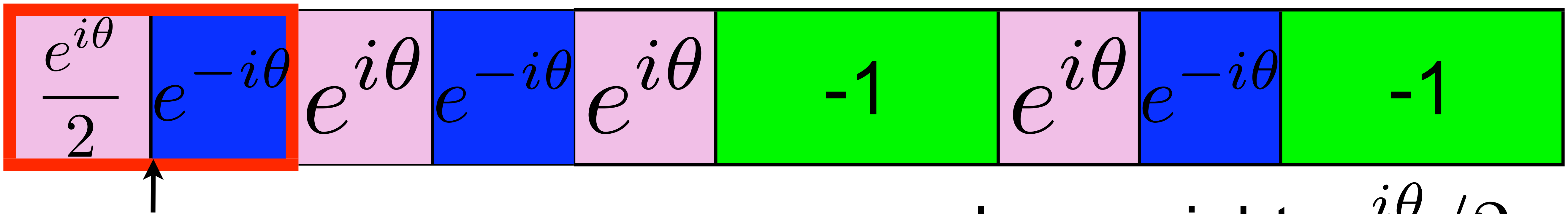


If the tiling starts with an impurity



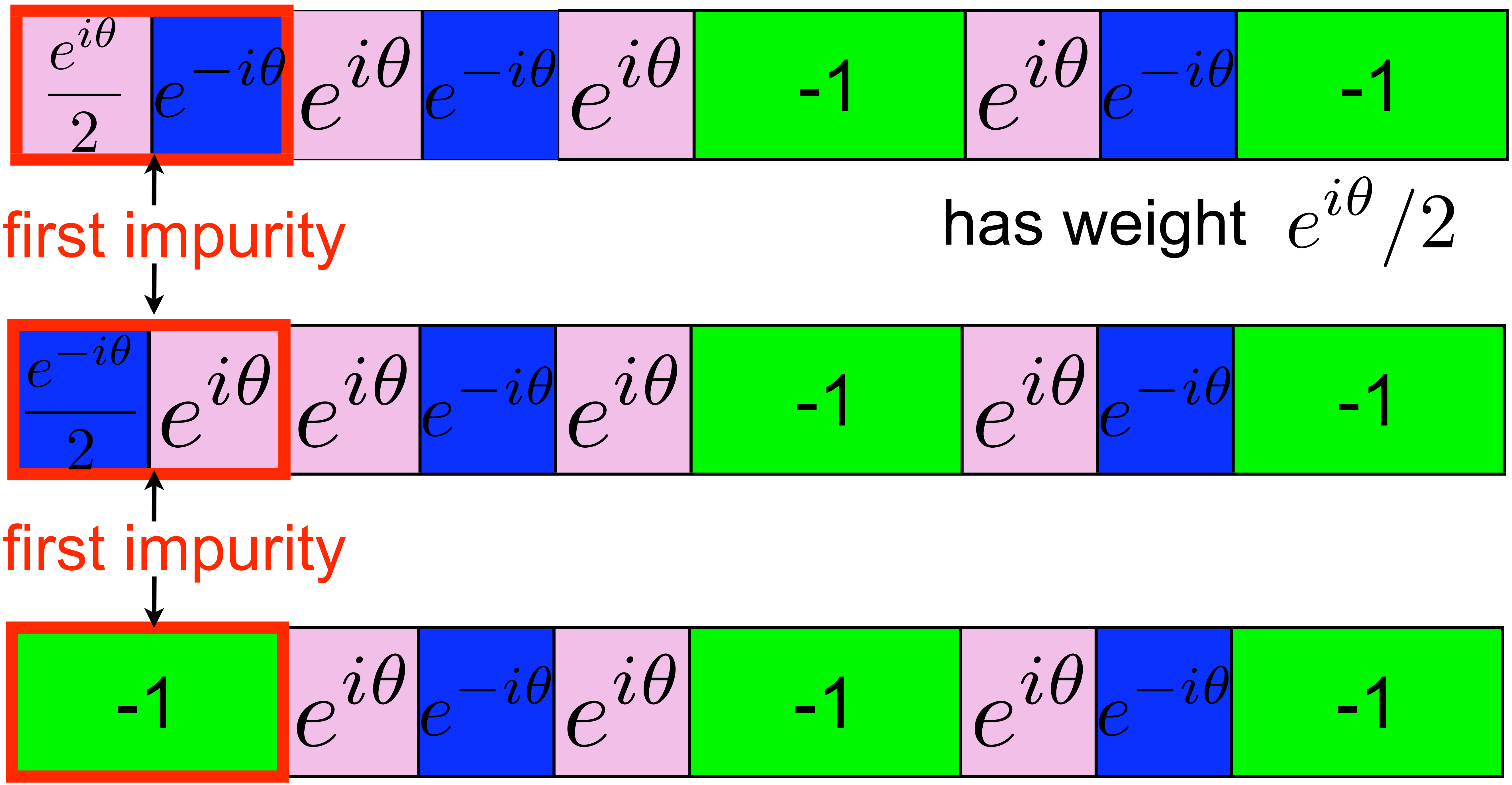
Find a trio that sums to zero!

If the tiling starts with an impurity



has weight $e^{i\theta}/2$

If the tiling starts with an impurity



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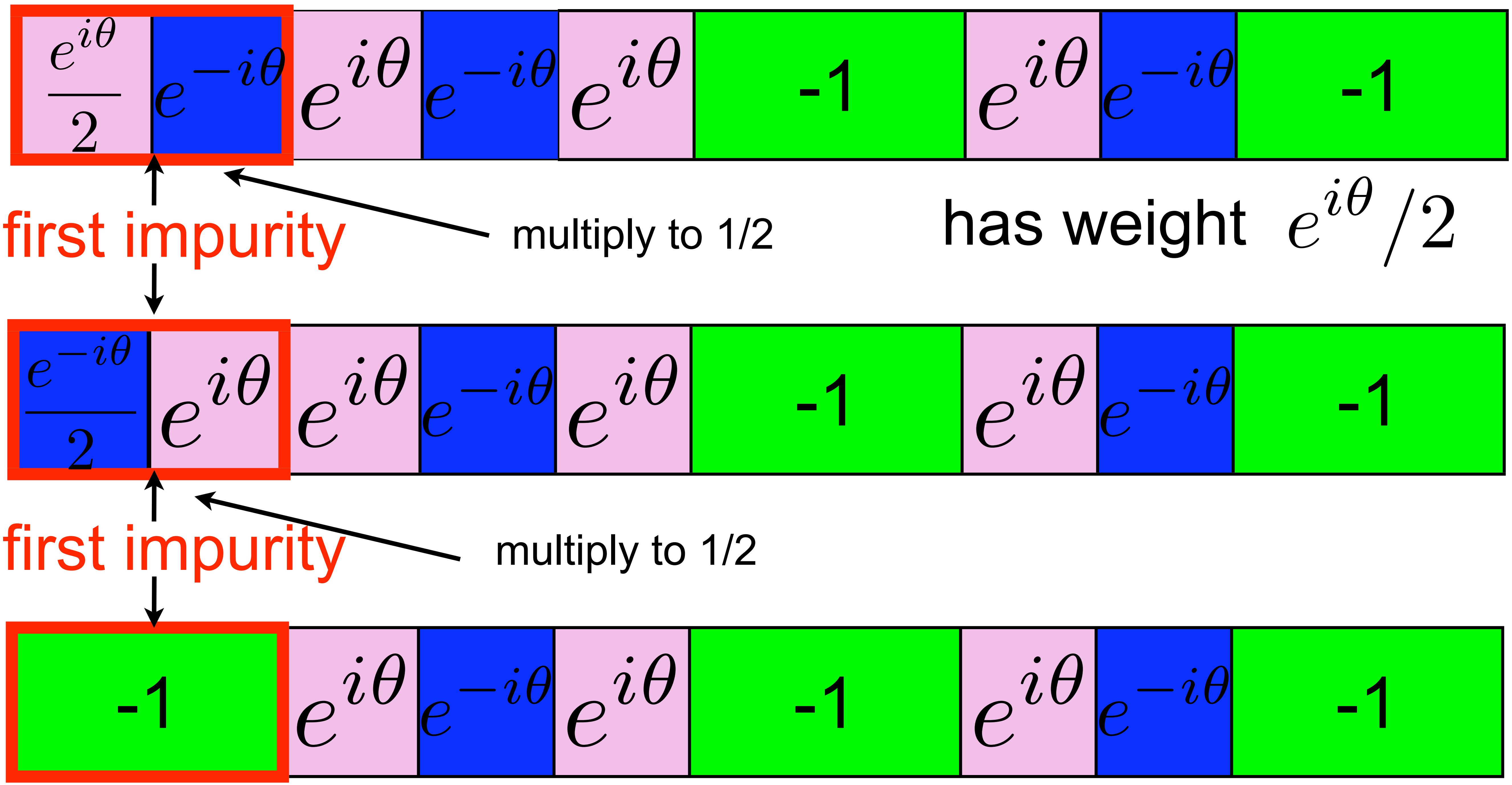
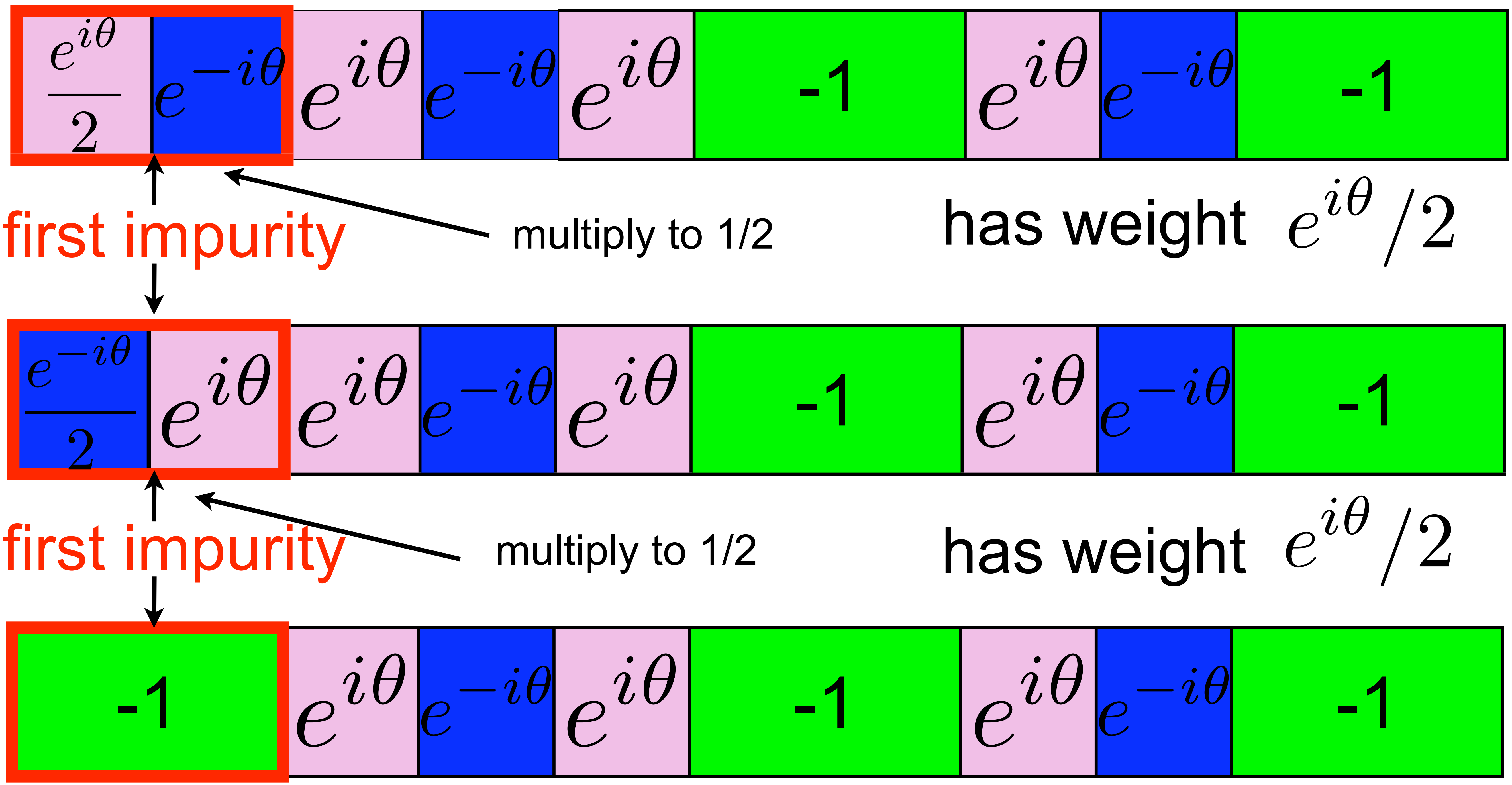


If the tiling starts with an impurity



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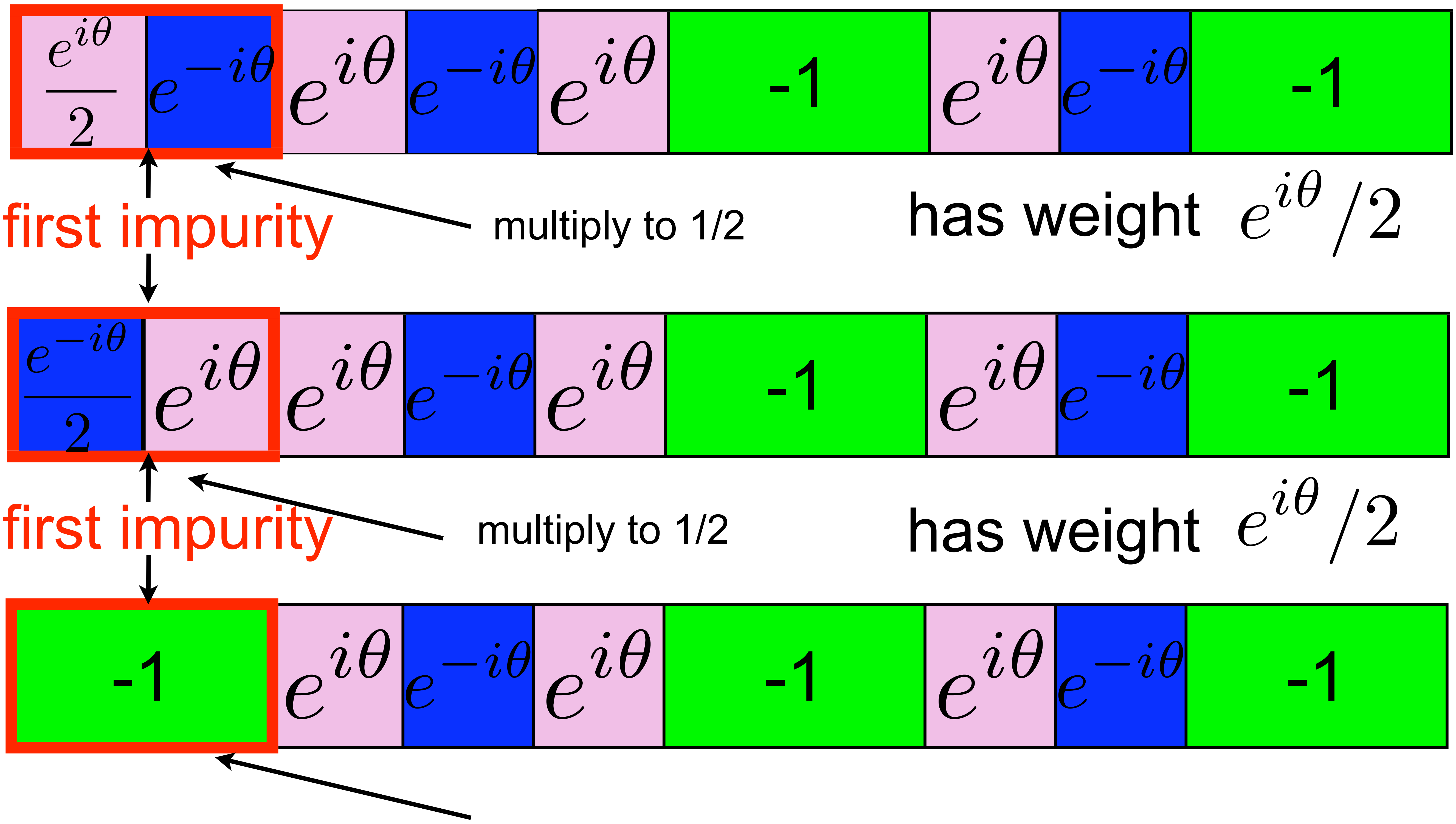

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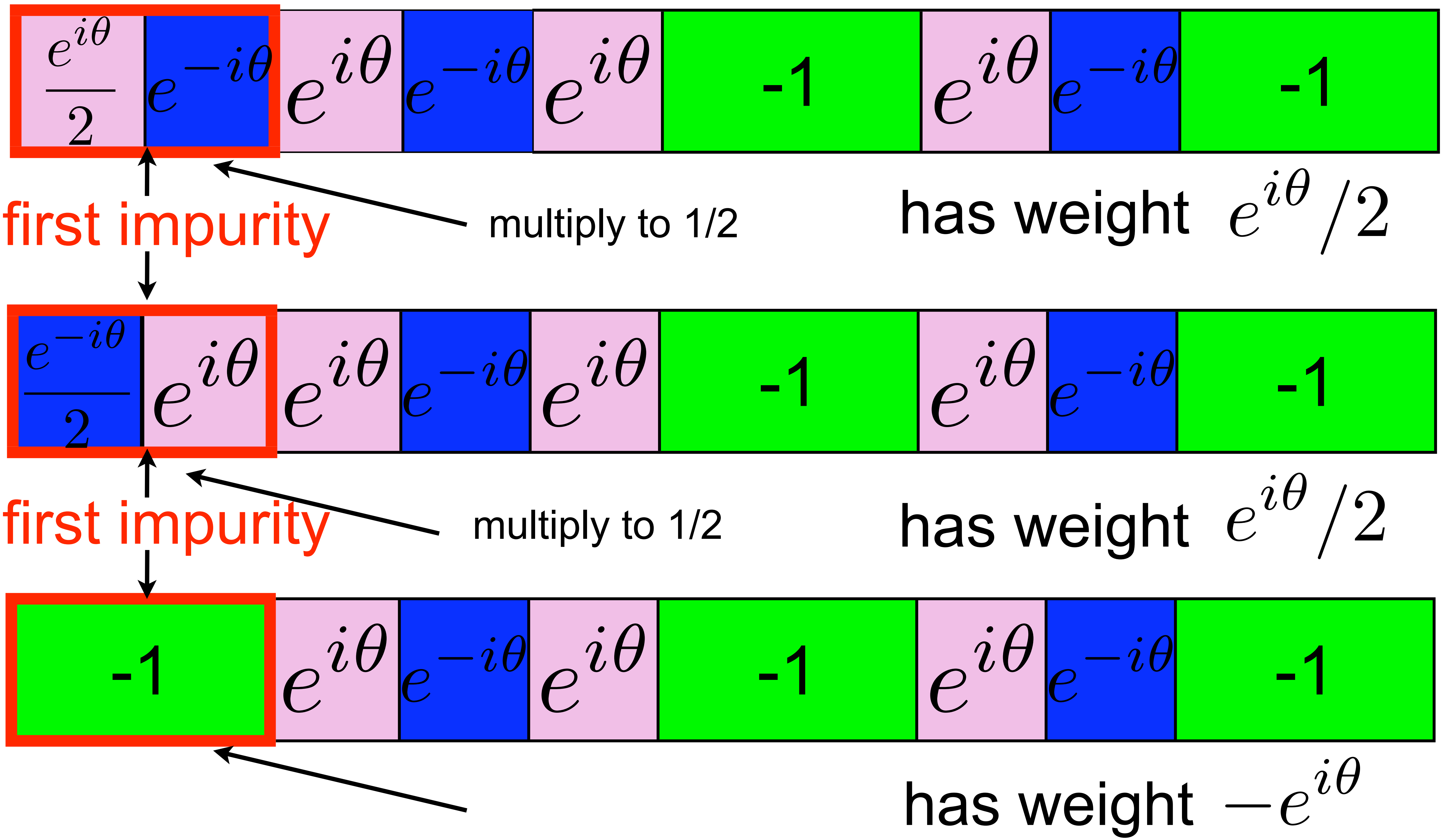

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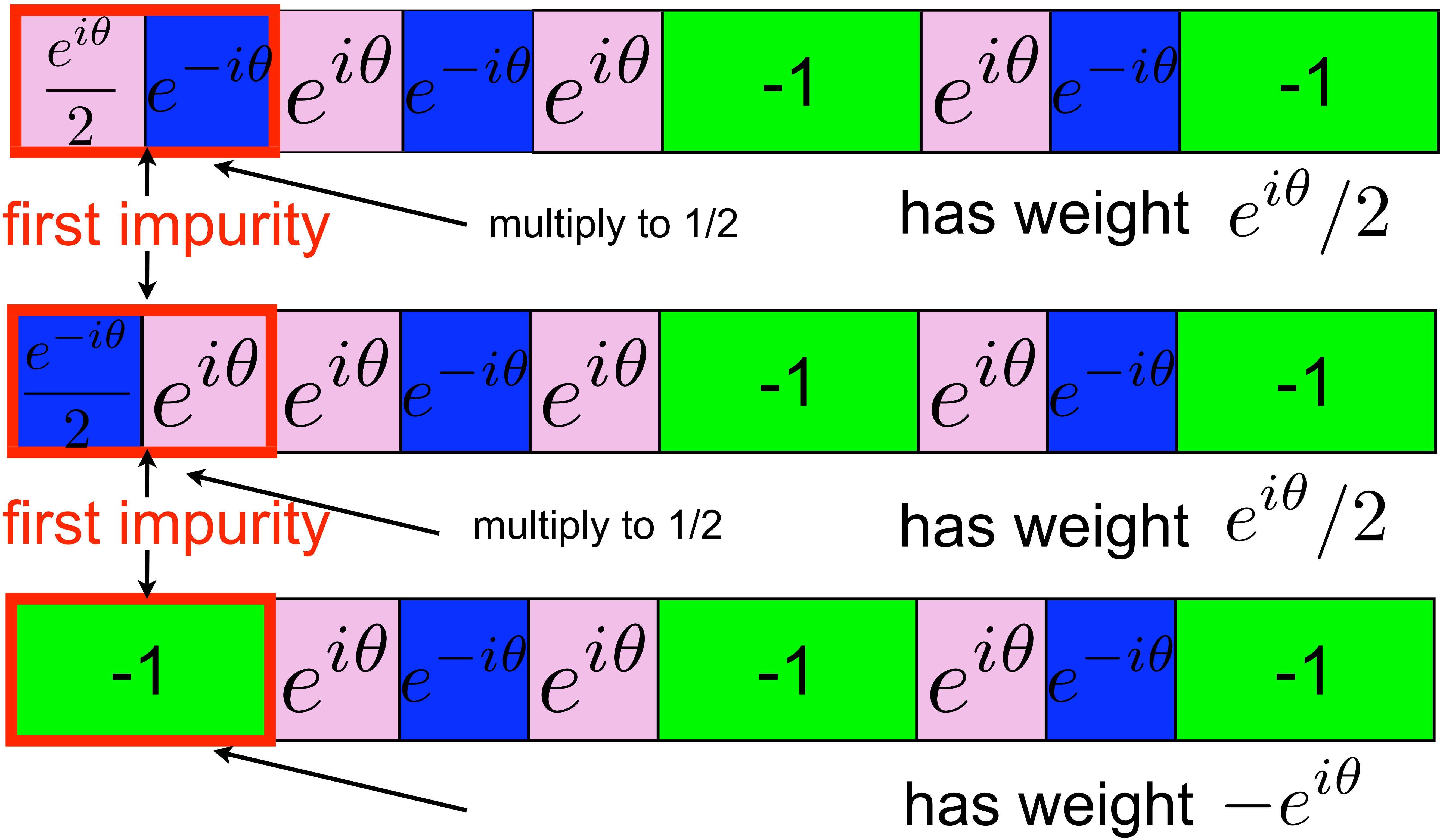

Their weights sum to zero!

Summary

- Every impure tiling belongs to exactly one pair or trio that sums to zero.
- Thus $T_n(\cos\theta)$ is the sum of the weights of all the pure tilings.

What are the pure tilings?

What are the pure tilings?

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$
1	2	3	.	.	*n*

What are the pure tilings?

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$
1	2	3	.	.	n

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	.	.	n

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$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$
1	2	3	.	.	n

$$e^{in\theta}/2$$

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	.	.	n

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1	2	3	.	.	n

$$e^{in\theta}/2$$

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	.	.	n

$$e^{-in\theta}/2$$

What are the pure tilings?

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$
1	2	3	.	.	n

$$e^{in\theta}/2$$

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	.	.	n

$$e^{-in\theta}/2$$

$$\text{Thus } T_n(\cos\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

What are the pure tilings?

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$
1	2	3	.	.	n

$$e^{in\theta}/2$$

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	.	.	n

$$e^{-in\theta}/2$$

$$\text{Thus } T_n(\cos\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \cos n\theta.$$

What are the pure tilings?

$\frac{e^{i\theta}}{2}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$	$e^{i\theta}$
1	2	3	.	.	n

$e^{in\theta}/2$

$\frac{e^{-i\theta}}{2}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$	$e^{-i\theta}$
1	2	3	.	.	n

$e^{-in\theta}/2$

$$\text{Thus } T_n(\cos\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \cos n\theta.$$