Efficient Numerical Simulation of Advection Diffusion Systems

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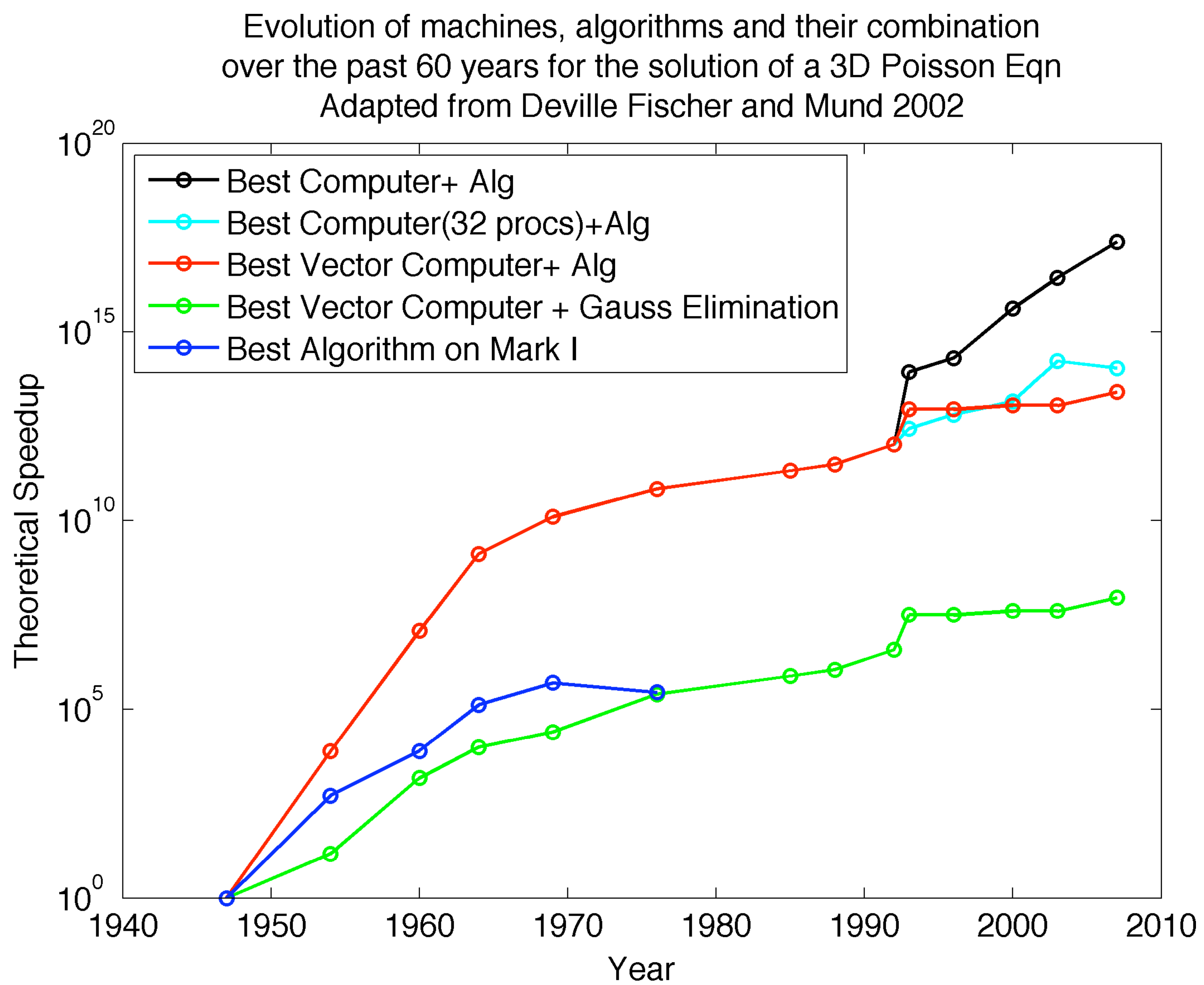
OUTLINE

- History of machine and algorithmic speedup
- Introduction to Advection-Diffusion Systems
- Choice of Numerical Discretization
- Development of Numerical Solvers
- Results
- Conclusion/Future Directions

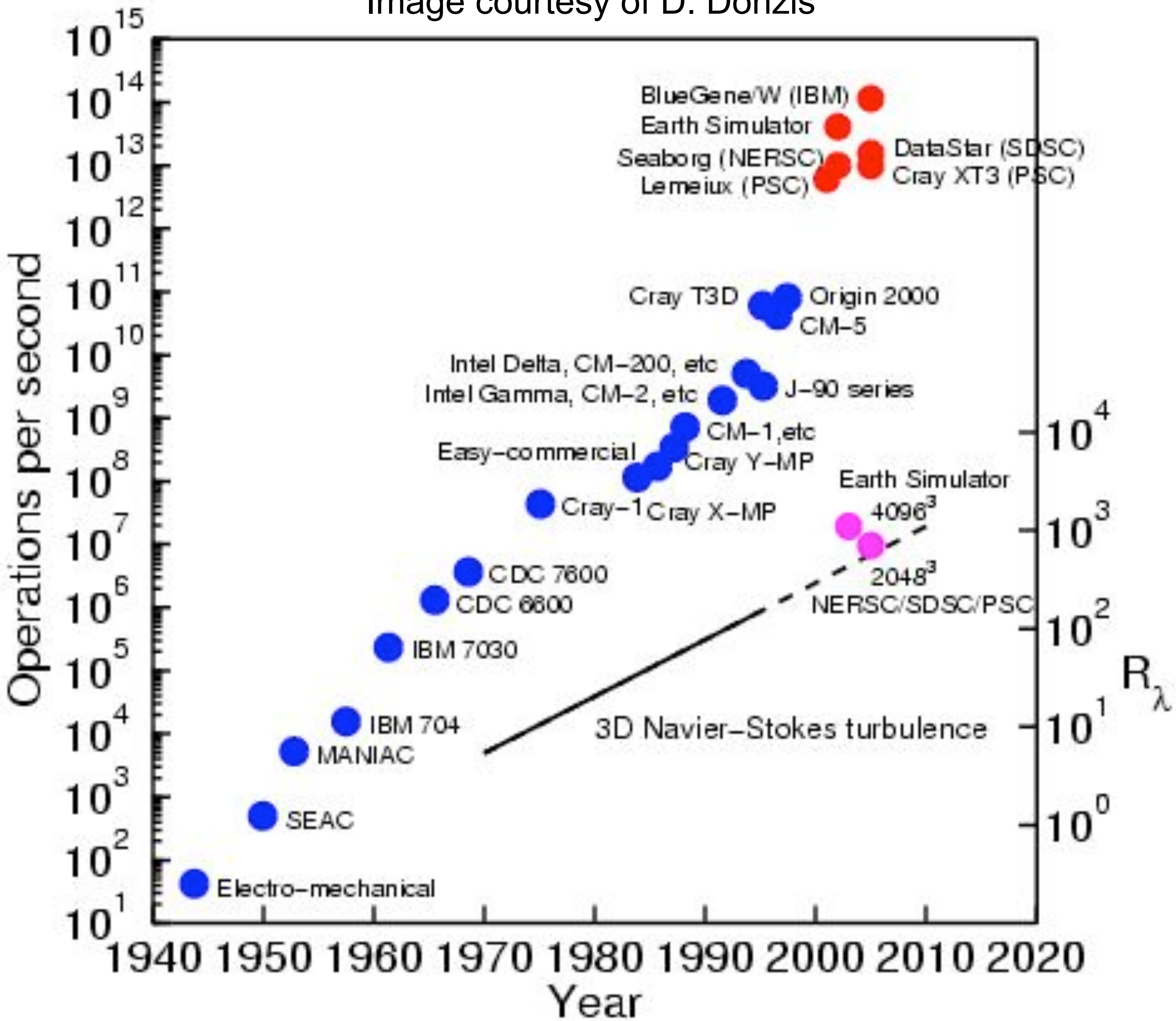
Motivation - Efficient Solvers



Faster machines and computational algorithms can dramatically reduce simulation time. (Centuries to milliseconds).

Motivation - Efficient Solvers

Image courtesy of D. Donzis



Simulating complex flows doesn't scale as well.

Motivation - Efficient Solvers

Complexity of Modern Linear Solvers

		Serial	Parallel
FFT	Direct	nlogn	logn
Multigrid	Iterative	n	(logn)^2
GMRES	Iterative	n	n
Lower Bound		n	logn

Model - Steady Advection Diffusion

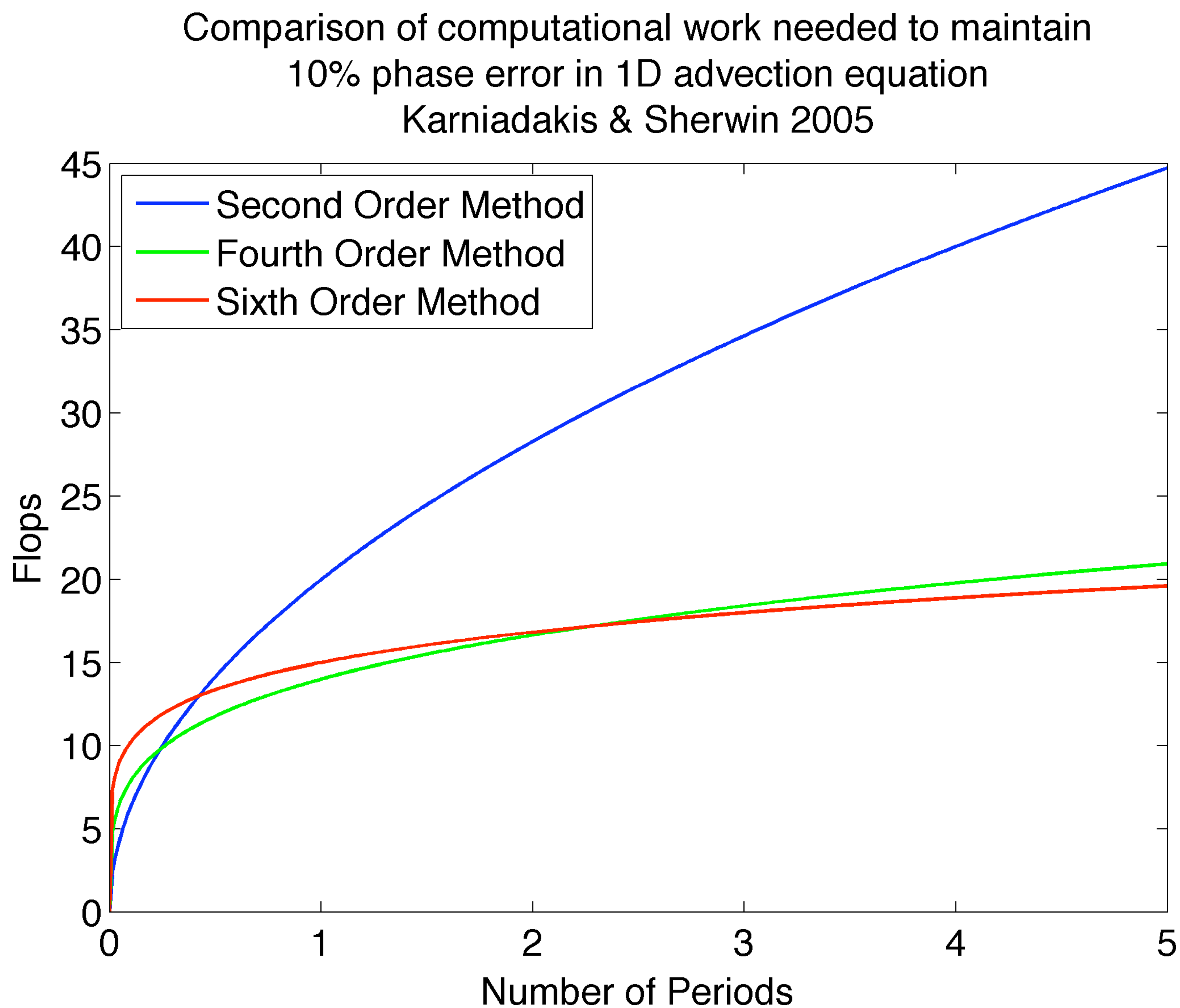
$$-\epsilon\nabla^2 u + (\vec{w}\cdot\nabla)u = f$$

Inertial and viscous forces occur on disparate scales causing **sharp flow features** which:

- require fine numerical grid resolution
- cause poorly conditioned non-symmetric discrete systems.

These properties make solving the discrete systems **computationally expensive**.

Motivation - Efficient Solvers & Discretization

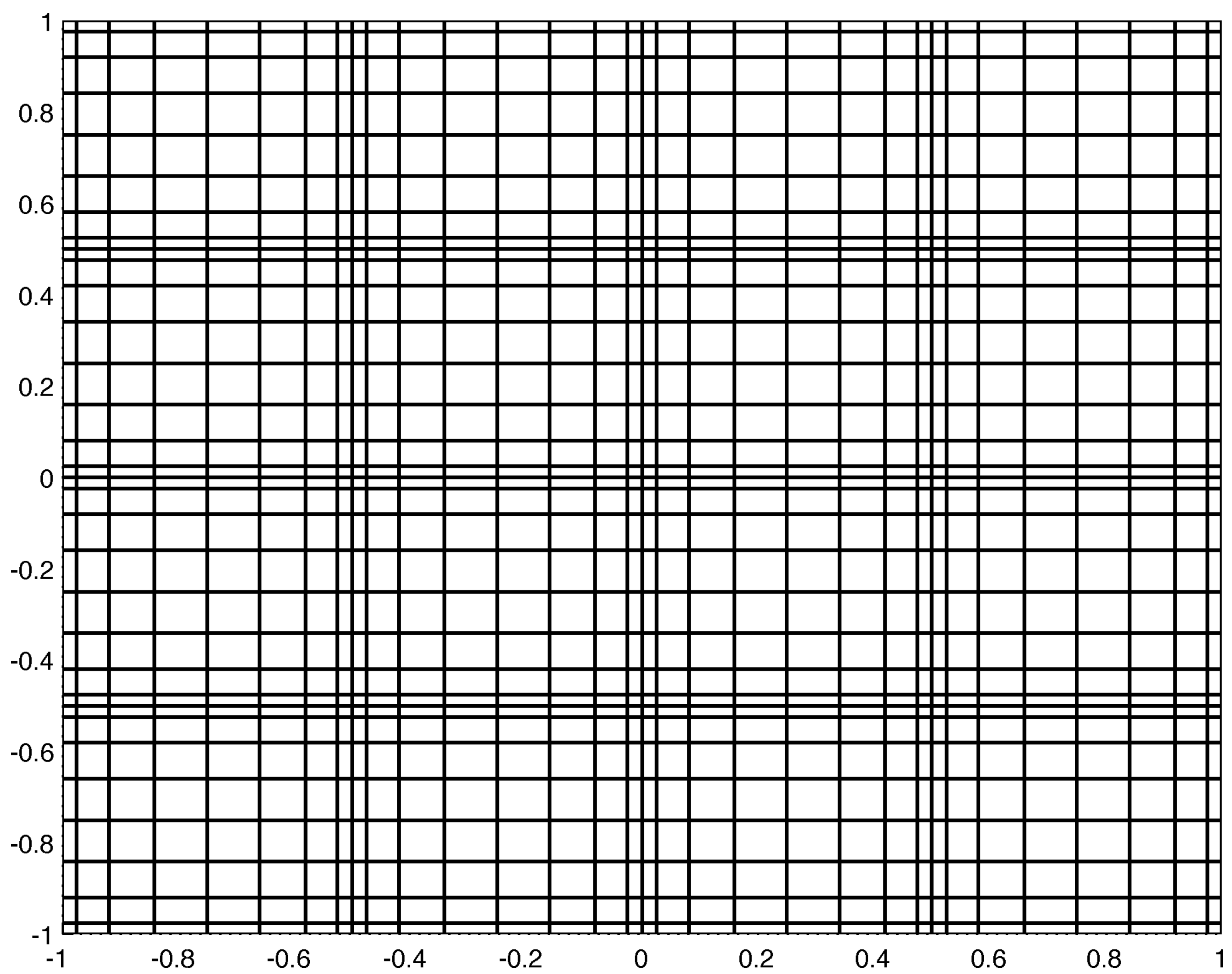


High order methods are accurate & efficient.

Methods - Spectral Element Discretization

Spectral elements provide:

- **flexible geometric boundaries**
- **large volume to surface ratio**
- **low storage requirements**



Methods - Spectral Element Discretization

The discrete system of advection-diffusion equations are of the form:

$$F(\vec{w})u = Mf$$

When w is constant in each direction on each element we can use

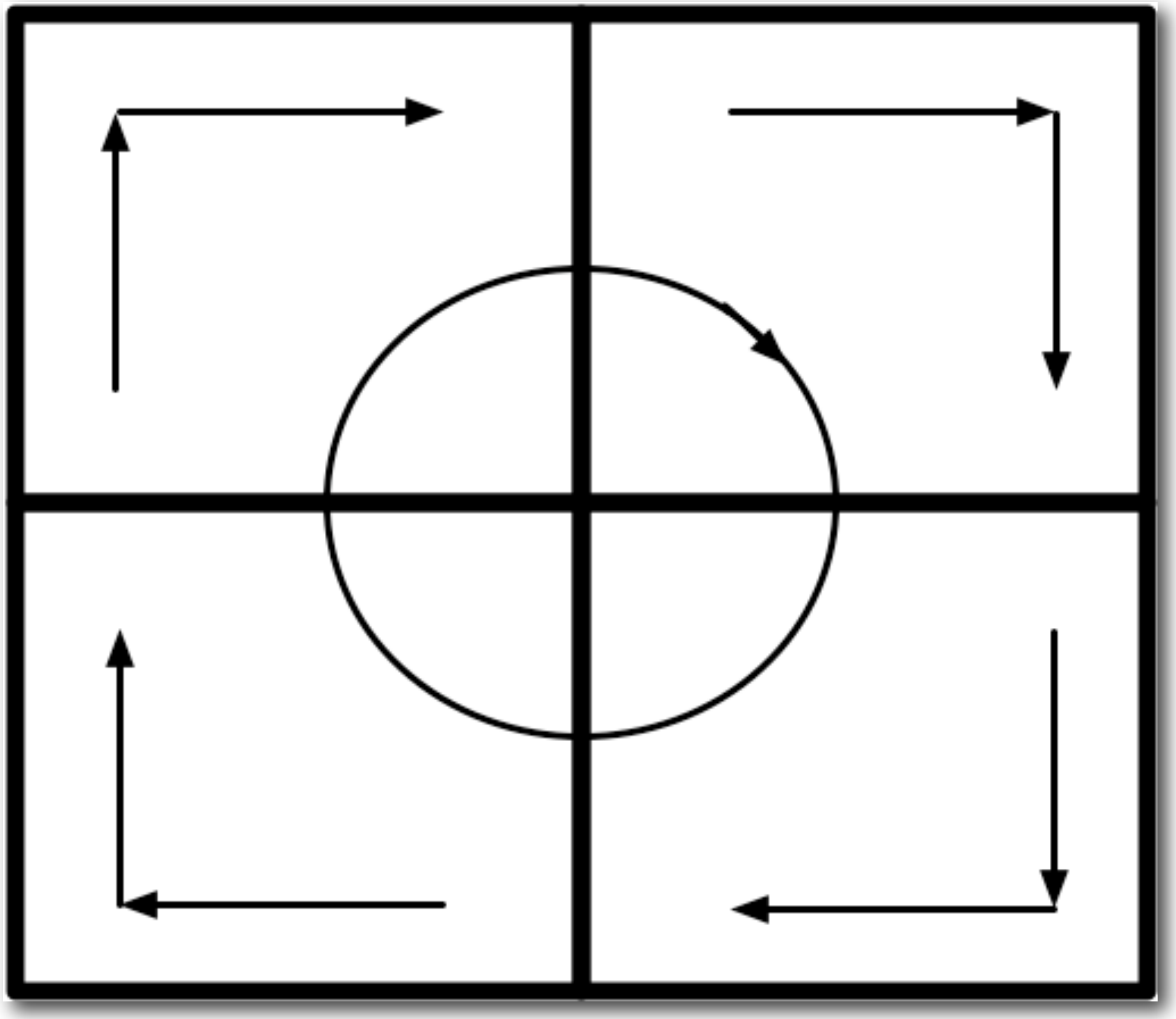
- **Fast Diagonalization & Domain Decomposition as a solver.**

$$\tilde{F} = \hat{M} \otimes \hat{F}(w_x) + \hat{F}(w_y) \otimes \hat{M}$$

Methods - Spectral Element Discretization

Otherwise, we can use this as a **Preconditioner** for an iterative solver such as GMRES

$$F(\vec{w})P_F^{-1}P_F u = Mf$$



Methods -Tensor Products

What does $\otimes$ mean?

Suppose $A_{k \times l}$ and $B_{m \times n}$

The Kronecker Tensor Product

$$C_{km \times ln} = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1l}B \\ a_{21}B & a_{22}B & \dots & a_{2l}B \\ \vdots & \vdots & & \vdots \\ a_{k1}B & a_{k2}B & \dots & a_{kl}B \end{pmatrix}.$$

Matrices of this form have properties that make computations **very efficient** and **save lots of memory**!

Methods - Fast Diagonalization

Matrix-vector multiplies $(A \otimes B)\vec{u} = BUA^T$ done in $O(n^3)$ flops instead of $O(n^4)$

Fast Diagonalization Property

$$C = A \otimes B + B \otimes A$$

$$V^T AV = \Lambda, \qquad V^T BV = I$$

$$C = (V \otimes V)(I \otimes \Lambda + \Lambda \otimes I)(V^T \otimes V^T)$$

$$C^{-1} = (V \otimes V)(I \otimes \Lambda + \Lambda \otimes I)^{-1}(V^T \otimes V^T)$$

Only need an inverse of a **diagonal matrix!**

Methods - Solver & Preconditioner

We use Flexible GMRES with a **preconditioner** based on:

- **Local constant wind approximations**
- **Fast Diagonalization**
- **Domain Decomposition**

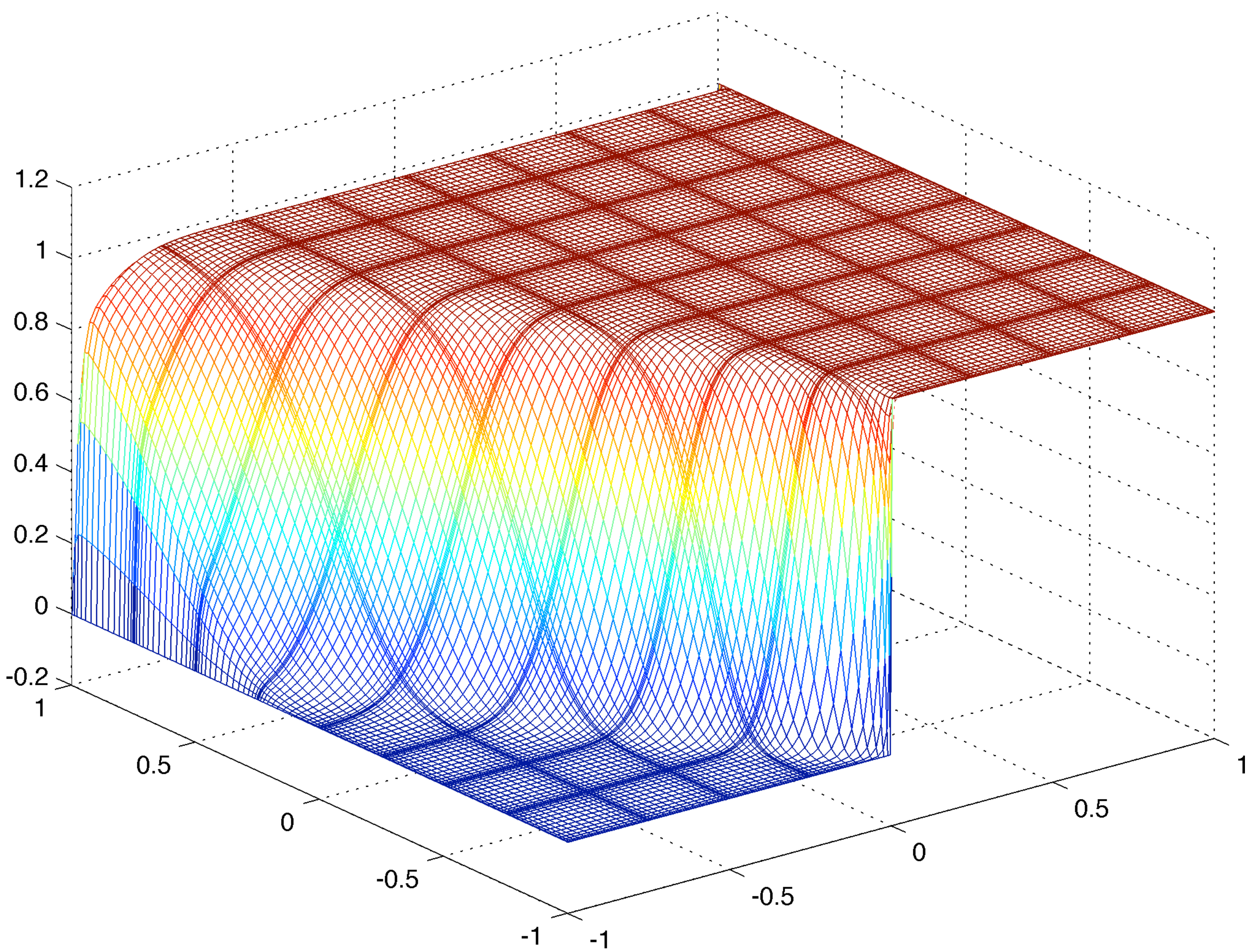
$$F(\vec{w})P_F^{-1}P_F u = Mf$$

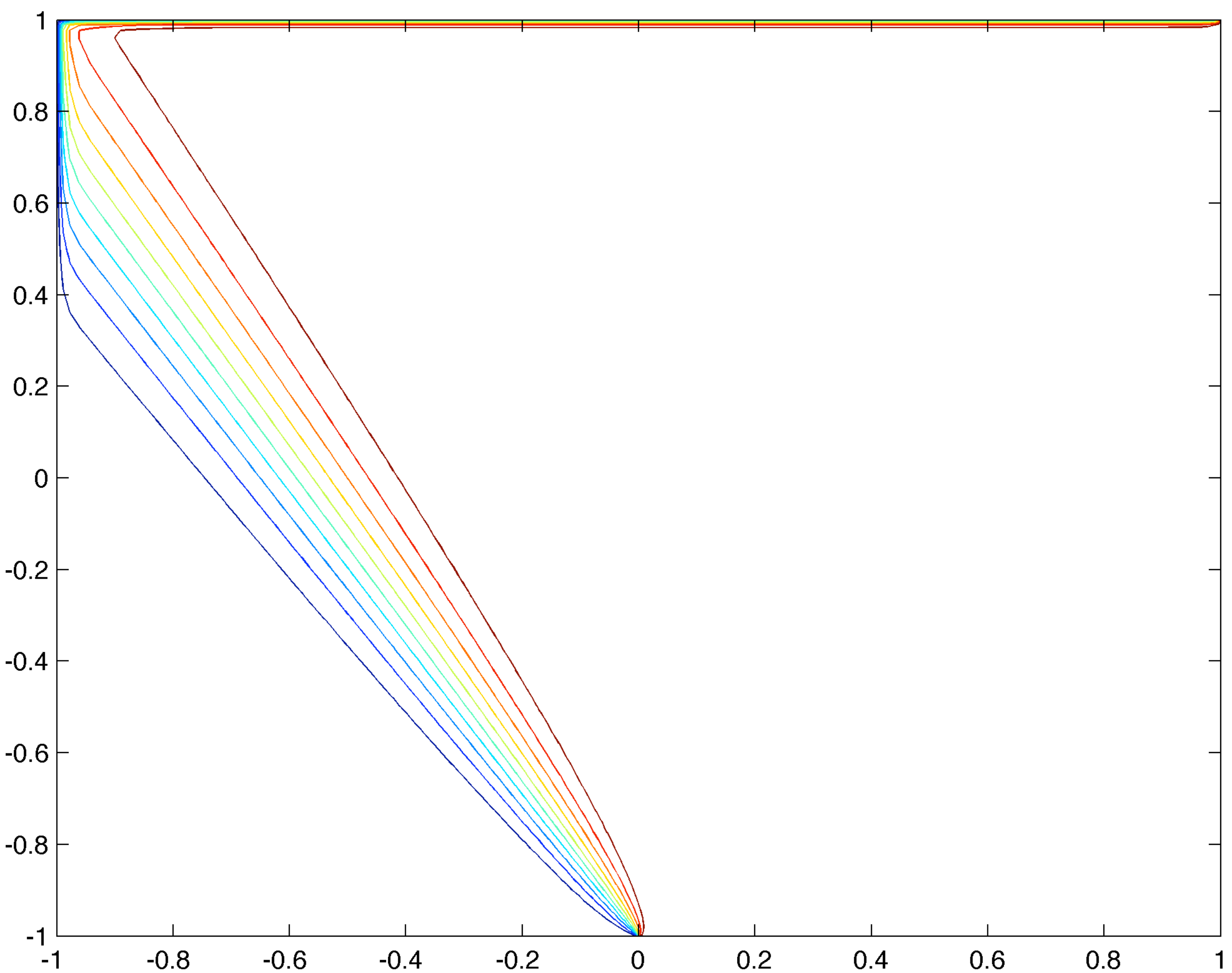
$$P_F^{-1} = R_0^T \tilde{F}_0^{-1}(\bar{w}_0)R_0 + \sum_{e=1}^{N} R_e^T \tilde{F}_e^{-1}(\bar{w}^e)R_e$$

$$\tilde{F}_e^{-1} = (\hat{M}^{-1/2} \otimes \hat{M}^{-1/2})(S \otimes T)(\Lambda \otimes I + I \otimes V)^{-1}(S^{-1} \otimes T^{-1})(\hat{M}^{-1/2} \otimes \hat{M}^{-1/2})$$

Solver Results - Constant Wind

$$\vec{w} = 200(-sin(\frac{\pi}{6}), cos(\frac{\pi}{6}))$$

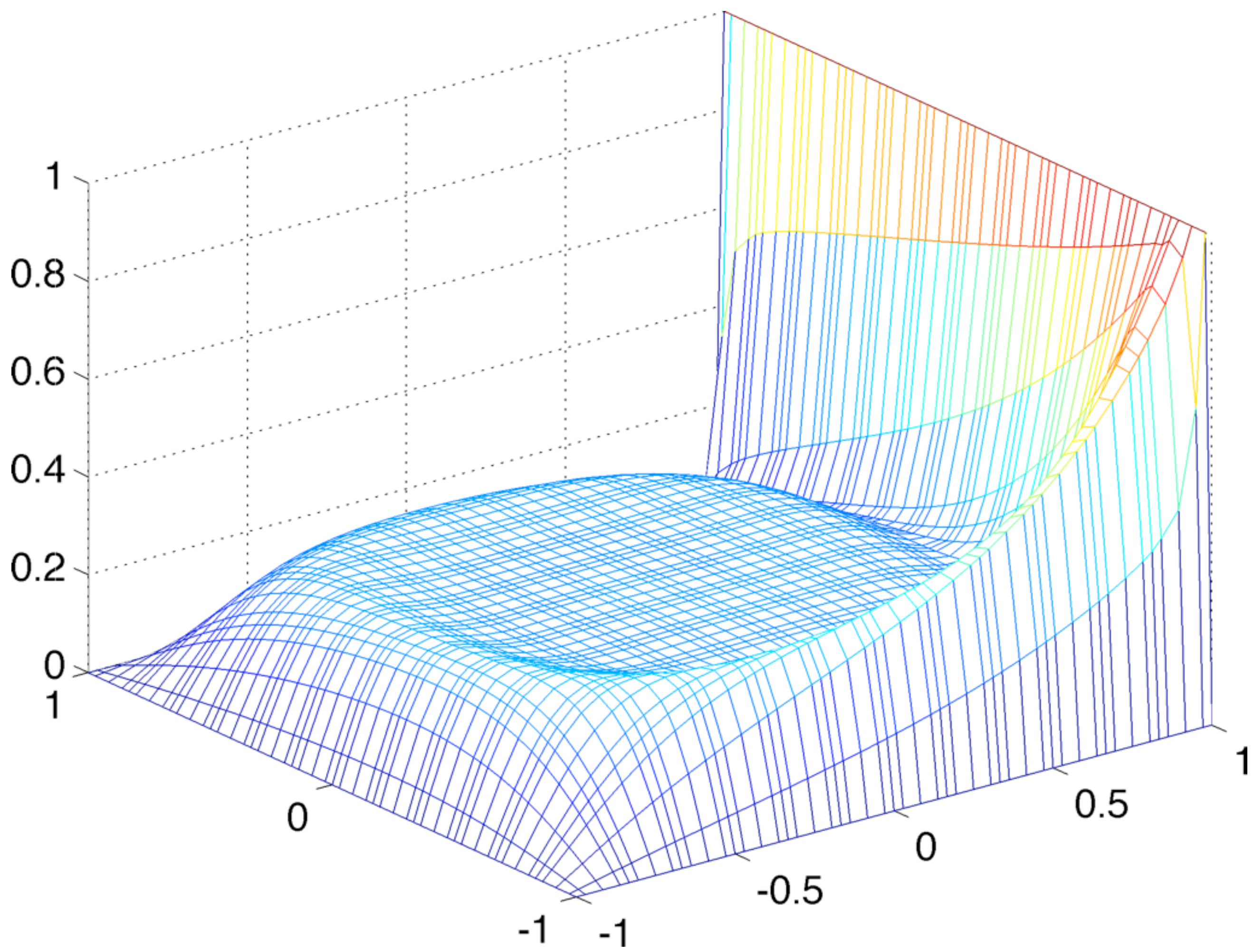


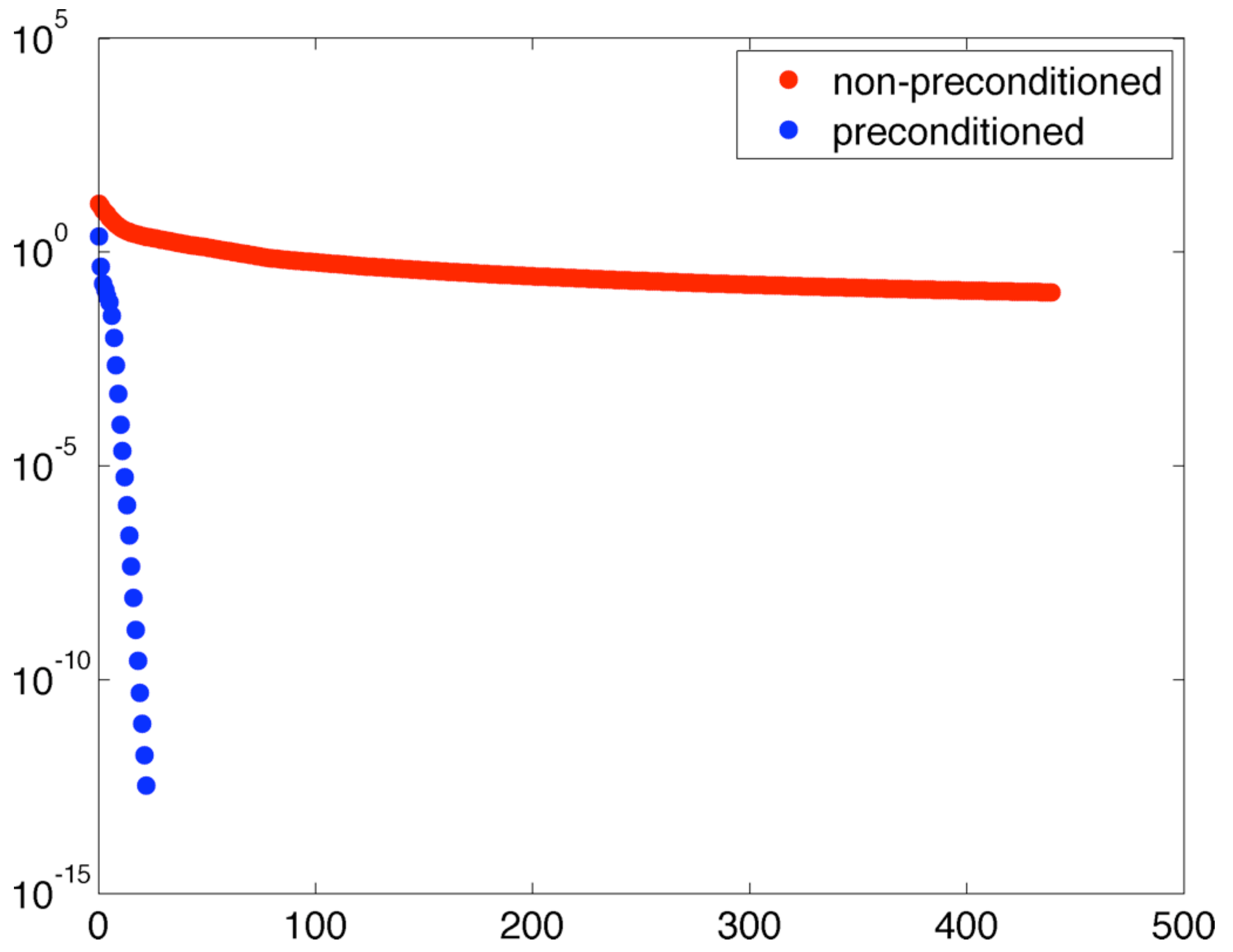


Solution and contour plots of a steady advection-diffusion flow. Via Domain Decomposition & Fast Diagonalization. Interface solve takes 150 steps to obtain 10^-5 accuracy.

Preconditioner Results - Recirculating Wind

$$\vec{w} = 200(y(1-x^2), -x(1-y^2))$$





Hot plate at wall forms internal boundary layers.

Residual Plot above.

•(P + 1) [1 2 0 N + (P + 1)] additional flops per step

Conclusions/Future Directions

Coupling Fast Diagonalization & Domain Decomposition provides an efficient solver for the advection-diffusion equation.

- Precondition Interface Solve
- Coarse Grid Solve (multilevel DD)
- Multiple wind sweeps
- Time dependent flows
- 2D & 3D Navier-Stokes
- Apply to study of complex flows

References

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