Sufficiently Class for Global (in time) Solutions to the 3d-Navier-Stokes Equations

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Introduction

A well-known unsolved problem is to identify a set of initial velocities that will allow global in time solutions to the three-dimensional Navier-Stokes equations. (These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces.) A related problem is to provide conditions under which we can be assured that the numerical approximation of these equations, used in a variety of fields from weather prediction to submarine design, have only one solution. In this talk, I will discuss an approach which allows us to prove that there exists unique global in time solutions to the Navier-Stokes equations for a bounded domain in R^3.

Basic Ideas **(for the non-mathematician)**

If $-\mathbf{A}$ is a positive definite symmetric matrix then all its eigenvalues are positive (so that all the eigenvalues of $\mathbf{A}$ are negative). It follows that the solution to the initial-value problem $\partial_t \mathbf{u}(t,\mathbf{x})=\mathbf{A}\mathbf{u}(t,\mathbf{x})$, $\mathbf{u}(0,\mathbf{x})=\mathbf{u}^0(\mathbf{x})$, has a unique solution given by $\mathbf{u}(t,\mathbf{x})=\exp\{t\mathbf{A}\}\mathbf{u}^0(\mathbf{x})$, which is exists for all $t\geq 0$. The operator $T(t)=\exp\{t\mathbf{A}\}$, satisfies $T(t+s)=T(t)T(s)$ (semigroup property), and $\|T(t)\|\leq 1$ (contraction).

If we replace $\mathbf{A}$ by a general nonlinear partial differential operator (on an appropriate space of functions), the same basic idea goes though, with negative eigenvalues of the matrix replaced by the notion of dissipative for operators, which means that $\langle \mathbf{A}\mathbf{u},\mathbf{u}\rangle\leq 0$ for all $\mathbf{u}$ in the domain of the operator.

General Background

Let where Ω be a bounded open domain in R^3. The global in time classical Navier-Stokes initial-value problem is to find functions $\mathbf{u}$: $[0,T]\times\Omega \to \mathrm{R}^3$, p: $[0,T]\times\Omega \to \mathrm{R}$ such that

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla p=f(t) \text{ and } \nabla\cdot\mathbf{u}=0 \text{ in } (0,T)\times\Omega,$$

(1)

$$\mathbf{u}(t,\mathbf{x})=0 \text{ in } (0,T)\times\partial\Omega \text{ and } \mathbf{u}(0,\mathbf{x})=\mathbf{u}^0(\mathbf{x}) \text{ in } \Omega.$$

These equations describe the time evolution of the fluid velocity $\mathbf{u}(t,\mathbf{x})$ and the pressure p of an incompressible homogeneous Newtonian fluid with constant viscosity coefficient ν, in terms of a given initial velocity $\mathbf{u}^0(\mathbf{x})$ and given external body forces $f(t,\mathbf{x})$.

Technical Background

Definition 1 Let $\mathbf{H} = \{\mathbf{u} \in (\mathrm{L}^2[\Omega])^3 : \nabla \cdot \mathbf{u} = 0\}$, then the operator $\boldsymbol{J}$,

defined on $\mathbf{H}$ is said to be:

a) 0-dissapative if $\langle \boldsymbol{J}\mathbf{u}, \mathbf{u} \rangle \leq 0, \ \ \forall \mathbf{u} \in \mathbf{H}$,

b) Dissipative if $\langle \boldsymbol{J}\mathbf{u} - \boldsymbol{J}\mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \leq 0, \ \ \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}$,

c) Strongly dissipative if
$\langle \boldsymbol{J}\mathbf{u} - \boldsymbol{J}\mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \leq -k\|\mathbf{u} - \mathbf{v}\|^2, \ k > 0, \ \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}$,

d) Uniformly dissipative if $\langle \boldsymbol{J}\mathbf{u} - \boldsymbol{J}\mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \leq -a(\|\mathbf{u} - \mathbf{v}\|)\|\mathbf{u} - \mathbf{v}\|$,
$a(0) = 0$, and $a(s) \to \infty$.

Theorem 2 (Browder) Let B, be a closed, bounded, convex subset of H. If J maps B into H is strongly dissipative then the range of J, $Ran(\boldsymbol{J}) \supset \mathbf{B}$.

<u>Theorem 3</u> (Crandall-Liggett) If $\mathbf{A}(t),\ t\in[0,\infty)$ is a family of closed, densely defined operators on $\mathbf{B}$, with $\mathbf{D}(\mathbf{A}(t))=\mathbf{D}$ independent of t and $Ran(\mathbf{I}-\alpha\mathbf{A}(t))\supset\mathbf{B}$, then $\mathbf{A}(t)$ is the generator of a contraction semigroup.

<u>Theorem 4</u> (Miyadera) Let $\mathbf{A}(t),\ t\in I=[0,\infty)$ be a closed, densely defined family of operators on $\mathbf{H}$, with $\mathbf{D}(\mathbf{A}(t))=\mathbf{D}$ independent of t and $Ran(\mathbf{I}-\alpha\mathbf{A}(t))\supset\mathbf{B}$. If $\mathbf{B}_+\subset\mathbf{D}\cap\mathbf{H}$ is a closed convex set (in an appropriate topology) and:

1. ***The operator $\mathbf{A}(t)$ is the generator of a contraction semigroup for each t.***

2. ***The function $\mathbf{A}(t)\mathbf{u}$ continuous in both variables on $I\times\mathbf{B}_+$.***

Then for every $\mathbf{u}^0\in\mathbf{B}_+$, the problem $\partial_t\mathbf{u}=\mathbf{A}(t)\mathbf{u}(t,\mathbf{x})$, $\mathbf{u}(t,0)=\mathbf{u}^0$ has a unique solution $\mathbf{u}(t,\mathbf{x})$ in $\mathbf{C}^1(I,\mathbf{B}_+)$.

Preliminaries

Let $\mathrm{P}:\left(\mathbf{L}^2[\Omega]\right)^3 \to \mathbf{H}$ **(projection), let** $\mathbf{A} = -\mathrm{P}\Delta$ **(StokeÕs operator),** $\mathrm{P}\nabla = -\mathbf{A}^{1/2}\mathbf{R}$**, where** $\|\mathbf{R}\| = 1$ **and** $\mathbf{C}(\mathbf{u},\mathbf{u}) = \mathrm{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$**. We can now write the equations as:**

$$\frac{\partial \mathbf{u}}{\partial t} = -\nu \mathbf{A}\mathbf{u} - \mathbf{C}(\mathbf{u},\mathbf{u}) + \mathrm{P}\mathbf{f}(t,\mathbf{x}), \qquad \frac{\partial \mathbf{u}}{\partial t} = \nu \mathbf{A}^{1+\delta}\mathbf{J}(\mathbf{u},t)\ ,$$

(3)

$$\mathbf{J}(\mathbf{u},t) = -\mathbf{A}^{-\delta}\mathbf{u} - \nu^{-1}\mathbf{A}^{-(1+\delta)}\mathbf{C}(\mathbf{u},\mathbf{u}) + \nu^{-1}\mathbf{A}^{-(1+\delta)}\mathrm{P}\mathbf{f}(t).$$

Theorem 5 **If** Ω **is of class** $\mathbf{C}^3$ **and** $\delta > 1/4$ **then**

1. $\left|\left\langle \mathbf{A}^{-(1+\delta)}\mathbf{C}(\mathbf{u},\mathbf{v}),\mathbf{w}\right\rangle\right| \le c\|\mathbf{u}\|\|\mathbf{v}\|\|\mathbf{w}\|$.
2. $\left|\left\langle \mathbf{C}(\mathbf{u},\mathbf{v}),\mathbf{w}\right\rangle\right| \le c\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|\|\mathbf{A}\mathbf{v}\|\|\mathbf{w}\|$.
3. $\|\mathbf{C}(\mathbf{u},\mathbf{v})\| \le c\|\mathbf{A}\mathbf{u}\|\|\mathbf{A}\mathbf{v}\|$.

Main Results

Theorem 6 Let $f(t)$ be continuous, with $f = \sup_{t \geq 0} \|\mathrm{P}f(t)\| < \infty$. Then there exists a positive constant $\mathbf{u}_+$, depending only on $f(t)$, $\mathbf{A}$, Ω and ν, such that for all $\mathbf{u}$, with $\|\mathbf{u}\| \leq \mathbf{u}_+$, $J(\cdot, t)$ is strongly dissipative.

Proof (outline)

1. $J(\cdot, t)$ is zero dissipative:

$$\left\langle J(\mathbf{u}, t), \mathbf{u} \right\rangle = -\left\| \mathbf{A}^{-\delta/2} \mathbf{u} \right\|^2 - \nu^{-1} \left\langle \mathbf{A}^{-(1+\delta)} \mathbf{C}(\mathbf{u}, \mathbf{u}), \mathbf{u} \right\rangle + \nu^{-1} \left\langle \mathbf{A}^{-(1+\delta)} \mathrm{P} f(t), \mathbf{u} \right\rangle$$

$$\leq -\lambda_0^{-\delta/2} \|\mathbf{u}\|^2 + \nu^{-1} c \|\mathbf{u}\|^3 + \nu^{-1} \lambda_1^{-(1+\delta)} f \|\mathbf{u}\| \leq 0$$

$$\mathbf{u}_{\pm} = (1/2) \nu^{-1} c \lambda_0^{-\delta} [1 \pm \sqrt{1 - \gamma}], \quad \gamma = (4 c \lambda_0^{2\delta} f) / (\nu^2 \lambda_1^{(1+\delta)}) < 1.$$

2. $J(\cdot,t)$ **is uniformly dissipative** (eg, $J(\cdot,t)\colon D(\mathbf{A}) \xrightarrow{onto} D(\mathbf{A})$): If $\mathbf{B}_{+} = \left\{ \mathbf{u},\mathbf{v} \in D(\mathbf{A}),\ \max(\|\mathbf{Au}\|,\|\mathbf{Av}\|) \le \mathbf{u}_{+} \right\}$ then

$$
\begin{aligned}
\left\langle J(\mathbf{u},t) - J(\mathbf{v},t), \mathbf{u}-\mathbf{v} \right\rangle &= -\left\| \mathbf{A}^{-\delta/2}(\mathbf{u}-\mathbf{v}) \right\|^2 \\
&\quad - \frac{1}{2}\nu^{-1} \left\langle \mathbf{A}^{-(1+\delta)}[\mathbf{C}(\mathbf{u}-\mathbf{v},\mathbf{u}) + \mathbf{C}(\mathbf{u}-\mathbf{v},\mathbf{v})], \mathbf{u}-\mathbf{v} \right\rangle \\
&\le -\lambda_0^{-\delta} \|(\mathbf{u}-\mathbf{v})\|^2 + \frac{1}{2} c \nu^{-1} \|(\mathbf{u}-\mathbf{v})\|^2 \ \ \|\mathbf{u}\| + \|\mathbf{v}\| \\
&\le -\lambda_0^{-\delta} \|(\mathbf{u}-\mathbf{v})\|^2 + c\nu^{-1} \|(\mathbf{u}-\mathbf{v})\|^2 \mathbf{u}_{+} \\
&= -\frac{1}{2}\lambda_0^{-\delta} \|(\mathbf{u}-\mathbf{v})\|^2 \ \ 1 - \sqrt{1-\gamma} \ .
\end{aligned}
$$

This shows that $J(\cdot,t)$ **is onto. Since it is closed and bounded, it generates a contraction semigroup on** $\mathbf{B}_{+}$**.**

<u>Theorem 7</u> The operator $\mathbf{A}(t) = \mathbf{A}^{(1+\delta)} J(\cdot, t)$ is closed, uniformly dissipative and jointly continuous in u and t. Furthermore, for each t and $\beta > 0$, $Ran\ \ I - \beta\mathbf{A}(t)\ \supset \mathbf{B}_{+}$, so that $\mathbf{A}(t)$ generates a contraction semigroup on $\mathbf{B}_{+}$

Proof (outline) Let $\mathbf{u}, \mathbf{v} \in \mathbf{B}_{+}$, then

$$\left\langle \mathbf{A}(\mathbf{u}, t) - \mathbf{A}(\mathbf{v}, t), \mathbf{u} - \mathbf{v} \right\rangle = -\nu \left\| \mathbf{A}^{1/2}(\mathbf{u} - \mathbf{v}) \right\|^{2}$$

$$-\frac{1}{2}\left\langle [\mathbf{C}(\mathbf{u}-\mathbf{v}, \mathbf{u}) + \mathbf{C}(\mathbf{u}-\mathbf{v}, \mathbf{v})], \mathbf{u} - \mathbf{v} \right\rangle.$$

Now use

$$\left| \left\langle [\mathbf{C}(\mathbf{u}-\mathbf{v}, \mathbf{u}) + \mathbf{C}(\mathbf{u}-\mathbf{v}, \mathbf{v})], \mathbf{u} - \mathbf{v} \right\rangle \right| \leq c \left\| \mathbf{A}^{1/2}(\mathbf{u} - \mathbf{v}) \right\| \left\| (\mathbf{u} - \mathbf{v}) \right\| \left\{ \left\| \mathbf{A}\mathbf{u} \right\| + \left\| \mathbf{A}\mathbf{v} \right\| \right\}.$$

To get:

$$
\begin{aligned}
&\left\langle \mathbf{A}(\mathbf{u},t)-\mathbf{A}(\mathbf{v},t),\mathbf{u}-\mathbf{v}\right\rangle \\
&\quad\le \left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|\left\{-\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|+\frac{1}{2}c\left\|(\mathbf{u}-\mathbf{v})\right\|\ \left\|\mathbf{A}\mathbf{u}\right\|+\left\|\mathbf{A}\mathbf{v}\right\|\ \right\} \\
&\quad\le \left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|\left\|(\mathbf{u}-\mathbf{v})\right\|\left\{-\nu\lambda_0^{-\delta}+\frac{1}{2}c\ \left\|\mathbf{A}\mathbf{u}\right\|+\left\|\mathbf{A}\mathbf{v}\right\|\ \right\} \\
&\quad\le \left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|\left\|(\mathbf{u}-\mathbf{v})\right\|\left\{-\nu\lambda_0^{-\delta}+\frac{1}{2}\nu\lambda_0^{-\delta}\ 1+\sqrt{1-\gamma}\ \right\}\le 0 \\
&= -a\left(\left\|(\mathbf{u}-\mathbf{v})\right\|\right)\left\|(\mathbf{u}-\mathbf{v})\right\|,\quad a\left(\left\|(\mathbf{u}-\mathbf{v})\right\|\right)=-\frac{1}{2}\nu\lambda_0^{-\delta}\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|\left\{-1+\sqrt{1-\gamma}\right\}.
\end{aligned}
$$

It is easy to show that $\mathbf{A}(t)\mathbf{u}$ is continuous in $\mathbf{u}$ and t. Thus, since $\mathbf{B}_+$ is closed in the graph norm, by Theorem 4, for every $\mathbf{u}^0\in\mathbf{B}_+$, the problem $\partial_t\mathbf{u}=\mathbf{A}(t)\mathbf{u}(t,\mathbf{x})$, $\mathbf{u}(t,0)=\mathbf{u}^0$ has a unique solution $\mathbf{u}(t,\mathbf{x})$ in $\mathbf{C}^1(I,\mathbf{B}_+)$.

Uniqueness of Weak Solutions

Serrin has shown that if a strong solution exists on (0,T) then there is no other weak solution on (0,T) (Serrin, "The initial value problem for the NS-equations", in Non-Linear problems, R. E. Langer, editor, U. Wisconsin Press, 1963, pages 69-98). Sell and You provides a proof in their recent Book, (G. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Math. Sciences, Vol. 143, Springer, New York, 2002).

Theorem 8 for every $\mathbf{u}^0 \in \mathbf{B}_+$, the problem $\partial_t \mathbf{u} = \mathrm{A}\,(t)\mathbf{u}(t,\mathrm{x})$, $\mathbf{u}(t,0) = \mathbf{u}^0$ has a unique weak solution $\mathbf{u}(t,\mathrm{x})$ in $\mathrm{C}^1(I,\mathbf{B}_+)$.

Discussion **In** $V=\{\mathbf{u}$, with weak first order derivatives in $\mathbf{H}\}$, **our approach is the same, with similar results, using**

$$\frac{\partial \mathbf{u}}{\partial t}=\nu \mathbf{AJ}(\mathbf{u},t),\ \mathbf{J}(\mathbf{u},t)=-\mathbf{u}-\nu^{-1}\mathbf{A}^{-1}\mathbf{C}(\mathbf{u},\mathbf{u})+\nu^{-1}\mathbf{A}^{-1}\mathrm{P}\mathbf{f}(t).$$

This case has special interest since for small initial data and body forces the NS-equations have global in time solutions. For comparison, we follow Sell and You (see Theorem 64.4 and Corollary 64.5, on pages 402-405). They require that (using $\nu=1$ **and** $f=0$**),** $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|\leq\frac{\lambda_1^{1/4}}{(216)^{1/4}c}$. **We get** $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|\leq\frac{\lambda_1^{1/4}}{c}=\mathbf{u}_+$**, so our ball is about 3.8 times larger. When** $\nu=1$ **and** $f\neq 0$**, They get that** $\sup\left\|\mathrm{P}f(t)\right\|\leq\frac{\lambda_1^{1/4}}{2(216)^{1/4}c}$**, while we have** $\sup\left\|\mathrm{P}f(t)\right\|\leq\frac{\lambda_1^{1/4}}{4c}$**.**

The case $\Omega = \mathrm{R}^3$, shows the real power of our approach. Here we introduce the Hermite-Stokes operator $\mathbf{B} = \frac{1}{2}\left(-\Delta + |\mathbf{x}|^2\right)$. We then have

$$\frac{\partial \mathbf{u}}{\partial t} = -\nu \mathbf{A}\mathbf{u} - \mathbf{C}(\mathbf{u},\mathbf{u}) + \mathrm{P}\mathbf{f}(t,\mathbf{x}), \qquad \frac{\partial \mathbf{u}}{\partial t} = \nu(\mathbf{A}\mathbf{B})^{(1+\delta)}\mathbf{J}(\mathbf{u},t),$$

$$\mathbf{J}(\mathbf{u},t) = -\mathbf{B}^{-(1+\delta)}\mathbf{A}^{-\delta}\mathbf{u} - \nu^{-1}(\mathbf{A}\mathbf{B})^{-(1+\delta)}\mathbf{C}(\mathbf{u},\mathbf{u}) + \nu^{-1}(\mathbf{A}\mathbf{B})^{(1+\delta)}\mathrm{P}\mathbf{f}(t).$$

All proofs go though with a few some modifications.