GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

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Abstract. In two recent papers ([GZ1] [GZ2]), we provided solutions to the well-known unsolved problem of constructing sufficiency classes of functions in $\mathbb{H}[\mathbb{R}^3]^3$ and $\mathbb{V}[\mathbb{R}^3]^3$, which would allow global, in time, strong solutions to the three-dimensional Navier-Stokes equations. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. In both previous papers, our solution was restricted to functions defined on a bounded open domain of class $C^3$ contained in $\mathbb{R}^3$. In this paper, we study this problem for functions defined on all of $\mathbb{R}^3$. We prove that, under appropriate conditions, there exists a positive constant $\alpha$ and a number $u_+$, depending only on the domain, the viscosity, the body forces and the eigenvalues of the “Hermite” Stokes operator (defined below) such that, for all functions in a dense set $\mathcal{D}$ contained in the closed ball $B(\mathbb{R}^3)$ of radius $(1/2)u_+$ in $\mathbb{H}[\mathbb{R}^3]^3$, the Navier-Stokes equations have unique strong solutions in $C^1((0, \infty), \mathbb{H}[\mathbb{R}^3]^3)$.

Introduction

Let $L^2[\mathbb{R}^3]^3$ be the real Hilbert space of square integrable functions on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, and let $\mathbb{H}_0[\mathbb{R}^3]^3$ be the completion of the set of functions in

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\{ \mathbf{u} \in C_0^\infty(\mathbb{R}^3)^3 : \nabla \cdot \mathbf{u} = 0 \} \) which vanish at infinity with respect to the inner product of \( L^2[\mathbb{R}^3]^3 \), and let \( V_0[\mathbb{R}^3]^3 \) be the completion of the above functions which vanish at infinity with respect to the inner product of \( H^1_0[\mathbb{R}^3]^3 \), the functions in \( H^1_0[\mathbb{R}^3]^3 \) with weak derivatives in \( (L^2[\mathbb{R}^3])^3 \). The global in time classical Navier-Stokes initial-value problem (on \( \mathbb{R}^3 \) and all \( T > 0 \)) is to find functions \( \mathbf{u} : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) and \( p : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) such that

\[
\begin{align*}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= f(t) \text{ in } (0, T) \times \mathbb{R}^3, \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } (0, T) \times \mathbb{R}^3 \text{ (in the weak sense),} \\
\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t, \mathbf{x}) &= 0 \text{ on } (0, T) \times \mathbb{R}^3, \\
\mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3.
\end{align*}
\]

(1)

The equations describe the time evolution of the fluid velocity \( \mathbf{u}(\mathbf{x}, t) \) and the pressure \( p \) of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient \( \nu \) in terms of a given initial velocity \( \mathbf{u}_0(\mathbf{x}) \) and given external body forces \( f(\mathbf{x}, t) \). (Note that our third condition, \( \lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t, \mathbf{x}) = 0 \) on \( (0, T) \times \mathbb{R}^3 \), is natural in this case since it is well-known that \( H^k_0[\mathbb{R}^3]^3 = H^k[\mathbb{R}^3]^3 \) (see Stein [S] or [SY]).

**Purpose**

Let \( \mathcal{P} \) be the (Leray) orthogonal projection of \( (L^2[\mathbb{R}^3])^3 \) onto \( H^0_0[\mathbb{R}^3]^3 \) and define the Stokes operator by: \( A \mathbf{u} = -\mathcal{P} \Delta \mathbf{u}, \) for \( \mathbf{u} \in D(A) \subset H^2_0[\mathbb{R}^3]^3 \), the domain of \( A \). Let \( B \mathbf{u} : = \frac{1}{2} \mathcal{P}( -\Delta + |\mathbf{x}|^2 ) \mathbf{u} \) for \( \mathbf{u} \in D(B) \). We call \( B \) the Hermite-Stokes operator. The purpose of this paper is to prove that there exists a number \( \mathbf{u}_+ \), depending only on \( A, B, f, \nu \) and \( \mathbb{R}^3 \), such that, for all functions in \( \mathcal{D} = D(A) \cap B(\mathbb{R}^3) \), where
$B(\mathbb{R}^3)$ is the closed ball of radius $u_+$ in $H_0(\mathbb{R}^3)$, the Navier-Stokes equations have unique strong solutions in $u \in L^\infty_{loc}([0, \infty); V_0(\mathbb{R}^3) \cap C^1((0, \infty); H_0(\mathbb{R}^3))$. 

**Preliminaries**

In terms of notation and convention, we follow Sell and You [SY]. In order to simplify notation, we let $H$ denote $H_0[\mathbb{R}^3]$ and $V$ denote $V_0[\mathbb{R}^3]$. Our use of the Fourier transform follows the definition of Rudin [RU]: 

$$\hat{\mathcal{F}}(h) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot y} h(y) dy,$$

so that no factors of $2\pi$ appear in the transform pairs. In order to simplify our proofs, we always assume that all functions $u, v$ are in $D(A)$ and, as in [GZ2], we take $c = max\{c_i\}$, where $c_i$ is one of the nine positive constants that appear on pages 363-367 in [SY]. It will also be convenient to use the fact that the norms of $V$ and $V^{-1}$ are equivalent in their respective graph norms relative to $H$.

**The Stokes Operator**

It is known that $A$ is a nonnegative linear operator which generates an analytic contraction semigroup. It follows that the fractional powers $A^{1/2}$ and $A^{-1/2}$ are well defined. Moreover, it is also known (cf., [SY], [T1]) that the norms $\|A^{1/2}u\|_H$ and $\|A^{-1/2}u\|_H$ are equivalent to the corresponding norms induced by the Sobolev space $(H^1[\mathbb{R}^3])^3$, so that:

$$\|u\|_V \equiv \left\|A^{1/2}u\right\|_H \quad \text{and} \quad \|u\|_{V^{-1}} \equiv \left\|A^{-1/2}u\right\|_H.$$

In addition, $A$ is an isomorphism from $D(A) \xrightarrow{\text{onto}} D(A^{-1})$. Furthermore, the embeddings $V \rightarrow H \rightarrow V^{-1}$ are continuous, and it is easy to see that $A^{-1}$ is the projection of an operator represented by the Riesz potential, mapping $D(A^{-1})$...
onto $D(A)$ (see Stein [S]). Applying the Leray projection to equation (1), with $C(u, u) = P(u \cdot \nabla)u$, we can recast equation (1) in the standard form:

$$
\begin{align*}
\partial_t u &= -\nu Au - C(u, u) + Pf(t) \quad \text{in } (0, T) \times \mathbb{R}^3, \\
\nabla \cdot u &= 0 \quad \text{in } (0, T) \times \mathbb{R}^3, \\
\lim_{\|x\| \to \infty} u(t, x) &= 0 \quad \text{on } (0, T) \times \mathbb{R}^3, \\
u(t, x) &= u_0(x) \quad \text{in } \mathbb{R}^3,
\end{align*}
$$

(3)

where we have used the fact that the orthogonal complement of $H[\mathbb{R}^3]$ relative to $(L^2[\mathbb{R}^3])^3$ is $\{ v : v = \nabla q, q \in (H^1[\mathbb{R}^3])^3 \}$ to eliminate the pressure term (see Galdi [GA] or [SY, T1, T2]). Theorem 1 below will be used to get our basic estimate in Theorem 3. This result is a simple extension of the bounded domain case first proved by Constantin and Foias [CF].

**Theorem 1.** Let $\alpha_i, 1 \leq i \leq 3$, satisfy $0 \leq \alpha_1 \leq 3$, $0 \leq \alpha_2 \leq 2$, $0 \leq \alpha_3 \leq 3$, with $\alpha_1 + \alpha_2 + \alpha_3 \geq 3/2$ and

$$(\alpha_1, \alpha_2, \alpha_3) \notin \{(3/2, 0, 0), (0, 3/2, 0), (0, 0, 3/2)\}.$$

Then there is a positive constant $c = c(\alpha_i)$ such that

$$
|\langle C(u, v), w \rangle_{\mathbb{H}}| \leq c \left\| A^{\alpha_1/2}u \right\|_\mathbb{H} \left\| A^{(1+\alpha_2)/2}v \right\|_\mathbb{H} \left\| A^{\alpha_3/2}w \right\|_\mathbb{H}.
$$

We shall make use of the following interpolation inequality: (see Sell and You [SY], page 363)

$$
\| A^\gamma u \|_\mathbb{H} \leq c \| A^\alpha u \|_\mathbb{H}^{\theta} \| A^\beta u \|_\mathbb{H}^{(1-\theta)}
$$

for all $u \in D(A^\alpha)$, where $\gamma = \theta \alpha + (1-\theta)\beta$, $\alpha, \beta, \gamma \in \mathbb{R}$, $0 \leq \theta \leq 1$ and $\beta \leq \alpha$. 
The Hermite-Stokes Operator

The operator $\hat{B} = 1/2(-\Delta + |x|^2)$ is the three-dimensional version of the standard harmonic oscillator operator, which generates the Hermite functions (products of the Hermite polynomials by $e^{-x^2/2}$) as eigenfunctions for the eigenvalue problem on $\mathbb{R}$, (see Hermite [HR], Appell and Kamé de Fériet [AK], and Magnus, Oberhettinger and Soni [MOS]). It is easy to show directly, by separation of variables, that the solution to the 3-dimensional problem is the product of the solutions to the 1-dimensional problem, while the eigenvalues for the 3-dimensional Hermite polynomials are the sums of those for the 1-dimensional polynomials. Furthermore, $\hat{B}$, and hence $B = \mathbb{F}\hat{B}$, is positive with a compact inverse, while $A$ has an unbounded inverse on $H_0^1(\mathbb{R}^3)$. It turns out that $\hat{B}$ is “natural” for $\mathbb{R}^3$ in the sense that it is the only positive self-adjoint (sectorial) operator of lowest degree that is invariant under both rotations and Fourier transformations. (This is actually true for $\mathbb{R}^n$, $n \geq 1$.)

We will have need of the fact that every function $h(t) \in H$ has an expansion in terms of the eigenfunctions of $B$ so that, for example, $B^{-\beta} h(t) = \sum_{k=1}^{\infty} \lambda_k^{-\beta} h_k(t) e_k(x)$ and, from here, it is easy to see that $\|B^{-\beta} h(t)\|_H \leq \lambda_1^{-\beta} \|h(t)\|_H$, where $\lambda_1^{-1}$ is the largest eigenvalue of $B^{-1}$. We also need the following result for our basic Theorem.

**Lemma 2.** $D(A) = D(B)$.

**Proof.** If we define a norm on $D(A)$ by $\|u\|_A = \|Au\|_H$, then $(D(A), \| \cdot \|_A)$ is a Hilbert space. Now note that the Fourier transform $\mathfrak{F}(\cdot)$ is an isometric isomorphism on $(D(A), \| \cdot \|_A)$ to $(D(P|x|^2), \| \cdot \|_A)$, since $\|Au\|_H = \|\mathfrak{F}(Au)\|_H =$...
\[ \|P \, x^2 \, \hat{u}\|_H. \] It is now easy to see that \( D(A) = D(P \, x^2). \) From this, it follows that \( D(A) = D(B). \) \qed

It follows from the above lemma that \((AB)^{-\delta}\) is bounded for \( \delta > 0. \) The following estimate is equation 61.24.1 on page 366 in Sell and You [SY]. If we set \( \alpha_1 = 1, \alpha_2 = 1/2, \) and \( \alpha_3 = 0 \) in Theorem 1, along with the interpolation inequality, we get that

\[ |\langle C(u, v), w \rangle_H| \leq c \|A^{1/2}u\|_H \|Av\|_H \|w\|_H. \] (4)

**Theorem 3.** Let \( u, v, w \in H, \) and let \( \varepsilon > 0 \) be arbitrary. Then, for \( \delta = 1/4 + \varepsilon/2, \) we have that:

\[ \left| \left\langle (AB)^{-(1+\delta)}C(u, v), w \right\rangle_H \right| \leq c \lambda_1^{-(1+\delta)} \|u\|_H \|v\|_H \|w\|_H. \] (5)

**Proof.** Using the self-adjoint property of \( A, \) and integration by parts, we have

\[ \langle A^{-\beta}C(u, v), h \rangle_H = \langle C(u, v), A^{-\beta}h \rangle_H = -\langle C(u, A^{-\beta}h), v \rangle_H. \]

It now follows from Theorem 1 that:

\[ \left| \langle A^{-\beta}C(u, v), h \rangle_H \right| \leq c \|A^{\alpha_1/2}u\|_H \|A^{-\beta+(1+\alpha_2)/2}h\|_H \|A^{\alpha_3/2}v\|_H. \]

If we set \( \beta = 1 + \delta, \) \( \alpha_1 = \alpha_3 = 0, \) we have

\[ \left| \langle A^{-(1+\delta)}C(u, v), h \rangle_H \right| \leq c \|u\|_H \|v\|_H \|A^{(\alpha_2-1-2\delta)/2}h\|_H. \]

With \( \delta = 1/4 + \varepsilon/2, \) we get that, for the last term to reduce to \( \|h\|_H, \) we can set \( \alpha_2 = 3/2 + \varepsilon. \) It follows that the conditions of Theorem 1 are satisfied if \( 3/2 + \varepsilon < 2. \) Thus, it suffices to assume that \( \varepsilon < 1/2, \) which we will do in the rest of the paper.
GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

without comment. Our proof is completed by taking $h = B^{-\beta}w$, and the fact that
\[ \|B^{-\beta}w\|_H \leq \lambda_1^{-\beta}\|w\|_H. \]
\[ \square \]

Example 4. If we use Theorem 1, with $\alpha_1 = 5/4$, $\alpha_2 = 1/4$, and $\alpha_3 = 0$, along with the interpolation inequality, and the fact that
\[ \|A^{1/2}u\|_H \leq \|Au\|_H \]
we have that, for all $u, v \in D(A)$,
\[ (6) \]
\[ \|C(u, v)\|_H \leq c \|A^{1/2}u\|^3_H \|Au\|^{1/4}_H \|A^{1/2}v\|^3_H \|Av\|^{1/4}_H \]
\[ \leq c \|Au\|_H \|Av\|_H. \]

A better estimate is possible, but for our use, equation (6) will suffice.

Definition 5. We say that the operator $J(\cdot, t)$ is (for each $t$)

(1) 0-Dissipative if \( \langle J(u, t), u \rangle_H \leq 0 \).

(2) Dissipative if \( \langle J(u, t) - J(v, t), u - v \rangle_H \leq 0 \).

(3) Strongly dissipative if there exists an $\alpha > 0$ such that
\[ \langle J(u, t) - J(v, t), u - v \rangle_H \leq -\alpha \|u - v\|_H^2. \]

(4) Uniformly dissipative if there exists a strictly monotone increasing function $a(t)$ with $a(0) = 0$, $\lim_{t \to \infty} a(t) = \infty$, and:
\[ \langle J(u, t) - J(v, t), u - v \rangle_H \leq -a (\|u - v\|_H) \|u - v\|_H. \]

Note that, if $J(\cdot, t)$ is a linear operator, definitions 1) and 2) coincide. Theorem 6 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35, page 887 in, Vol. IIB], while Theorem 7 is from Miyadera [M, p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]).
Theorem 6. Let \( B[\mathbb{R}^3] \) be a closed, bounded, convex subset of \( \mathbb{H}[\mathbb{R}^3] \). If \( J(\cdot, t) : B[\mathbb{R}^3] \to \mathbb{H}[\mathbb{R}^3] \) is closed and strongly dissipative for each fixed \( t \geq 0 \) then, for each \( b \in B[\mathbb{R}^3] \), there is a \( u \in B[\mathbb{R}^3] \) with \( J(u, t) = b \) (e.g., the range, \( \text{Ran}[J(\cdot, t)] \supset B[\mathbb{R}^3] \)).

Theorem 7. Let \( \{ A(t), t \in I = [0, \infty) \} \) be a family of operators defined on \( \mathbb{H}[\mathbb{R}^3] \) with domains \( D(A(t)) = D, \) independent of \( t \). We assume that \( D = D \cap B[\mathbb{R}^3] \) is a closed convex set (in an appropriate topology):

1. The operator \( A(t) \) is the generator of a contraction semigroup for each \( t \in I \).
2. The function \( A(t)u \) is continuous in both variables on \( I \times D \).

Then, for every \( u_0 \in D \), the problem \( \partial_t u(t, x) = A(t)u(t, x), u(0, x) = u_0(x) \), has a unique solution \( u(t, x) \in C^1(I; D) \).

M-Dissipative Conditions

Let us assume that \( f(t) \in L^\infty([0, \infty); \mathbb{H}] \) and is Lipschitz continuous in \( t \), with \( \|f(t) - f(\tau)\|_\mathbb{H} \leq d|t - \tau|^{\theta}, \ d > 0, \ 0 < \theta < 1. \) With \( \delta \) as in Theorem 3, we can rewrite equation (3) in the form:

\[
\partial_t u = \nu(AB)^{1+\delta} J(u, t) \text{ in } (0, T) \times \Omega,
\]

\[
J(u, t) = -B^{-(1+\delta)} A^{-\delta} u - \nu^{-1}(AB)^{-(1+\delta)} C(u, u) + \nu^{-1}(AB)^{-(1+\delta)} P f(t).
\]

Approach

We begin with a study of the operator \( J(\cdot, t) \), for fixed \( t \), and seek conditions depending on \( A, B, \nu, \) and \( f(t) \) which guarantee that \( J(\cdot, t) \) is m-dissipative for each \( t \). Clearly \( J(\cdot, t) : D[(AB)^{(1+\delta)}] \xrightarrow{onto} D[(AB)^{(1+\delta)}] \) and, since \( \nu(AB)^{(1+\delta)} \)
GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON \( \mathbb{R}^3 \)

is a closed positive (m-accretive) operator (so that \( -(AB)^{(1+\delta)} \) generates a linear contraction semigroup), we expect that \( \nu(AB)^{(1+\delta)}J(\cdot, t) \) will be m-dissipative for each \( t \).

Theorem 8. For \( t \in I = [0, \infty) \) and, for each fixed \( u \in H \), \( J(u, t) \) is Lipschitz continuous, with \( \|J(u, t) - J(u, \tau)\|_H \leq d' |t - \tau|^\theta \), where \( d' = d\nu^{-1}a^{-(1+\delta)} \), \( d \) is the Lipschitz constant for the function \( f(t) \) and \( a^{-(1+\delta)} = \left\| (AB)^{-1+\delta} \right\|_H \).

Proof. For fixed \( u \in H \),

\[
\|J(u, t) - J(u, \tau)\|_H = \nu^{-1}\left\| (AB)^{-1+\delta} [Pf(t) - Pf(\tau)] \right\|_H \\
\leq d\nu^{-1}a^{-(1+\delta)} |t - \tau|^\theta = d' |t - \tau|^\theta.
\]

□

Main Results

Theorem 9. Let \( f = \sup_{t \in \mathbb{R}^+} \|Pf(t)\|_H < \infty \), then there exists a positive constant \( u_+ \), depending only on \( f \), \( A \), \( B \) and \( \nu \) such that, for all \( u \) with \( \|u\|_H \leq u_+ \), \( J(\cdot, t) \) is strongly dissipative.

Proof. The proof of our first assertion has two parts. First, we require that the nonlinear operator \( J(\cdot, t) \) be 0-dissipative, which gives us an upper bound \( u_+ \) in terms of the norm (e.g., \( \|u\|_H \leq u_+ \)). We then use this part, and the fact that \( \|u\|_H \leq \|Au\|_H \), to show that \( J(\cdot, t) \) is strongly dissipative on the closed ball, \( B_+ = \{u \in H : \|Au\|_H \leq (1/2)u_+ \} \).
Part 1) From equation (5), we consider the expression

$$\langle J(u, t), (AB)^{-\delta}u \rangle_{H} = - \langle B^{-1}(AB)^{-\delta}u, (AB)^{-\delta}u \rangle_{H}$$

$$+ \nu^{-1} \left\langle - (AB)^{-\left(1+\delta\right)}C(u, u) + (AB)^{-\left(1+\delta\right)}Pf(t), (AB)^{-\delta}u \right\rangle_{H}$$

$$= - \left\| B^{-1/2}(AB)^{-\delta}u \right\|_{H}^{2} - \nu^{-1} \left\langle (AB)^{-\left(1+\delta\right)}C(u, u), (AB)^{-\delta}u \right\rangle + \nu^{-1} \left\langle (AB)^{-\left(1+\delta\right)}Pf(t), (AB)^{-\delta}u \right\rangle_{H}$$

$$= - \left\| B^{-1/2}(AB)^{-\delta}u \right\|_{H}^{2} - \nu^{-1} \left\langle C((AB)^{-\left(1+\delta\right)}u, u), (AB)^{-\delta}u \right\rangle + \nu^{-1} \left\langle (AB)^{-\left(1+\delta\right)}Pf(t), (AB)^{-\delta}u \right\rangle_{H}.$$
GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

Solving, we get that

$$u_\pm = \frac{\nu \lambda_1^{1+\delta}}{2c\lambda_0 a^2} \left\{ 1 \pm \sqrt{1 - \left(4c\lambda_0^2 f / (\nu^2 a^{1-\delta} \lambda_1^{1+\delta}) \right)} \right\} = \frac{\nu \lambda_1^{1+\delta}}{2c\lambda_0 a^2} \left\{ 1 \pm \sqrt{1 - \gamma} \right\},$$

where $\gamma = (4c\lambda_0^2 f) / (\nu^2 a^{1-\delta} \lambda_1^{1+\delta})$. Since we want real distinct solutions, we must require that

$$\gamma = (4c\lambda_0^2 f) / (\nu^2 a^{1-\delta} \lambda_1^{1+\delta}) < 1 \Rightarrow \nu^2 a^{1-\delta} \lambda_1^{1+\delta} > 4c\lambda_0^2 f \Rightarrow \nu > 2\lambda_0 a^{-(1-\delta)/2}\lambda_1^{-(1+\delta)/2}(cf)^{1/2}.$$

It follows that, if $Pf \neq 0$, then $u_- < u_+$, and our requirement that $J$ is 0-dissipative implies that, since our solution factors as $(\|u\|_H - u_+)(\|u\|_H - u_-) \leq 0$, we must have that:

$$\|u\|_H - u_+ \leq 0, \quad \|u\|_H - u_- \geq 0.$$  

First observe that terms of the form $(AB)^{-\delta}u$ are dense. Then note that $J(u, t)$ is closed, and the dissipative nature of an operator is determined on a dense set.

It follows that, for $u_- \leq \|u\|_H \leq u_+$, $(J(u, t), u)_H \leq 0$. (It is clear that, when $Pf(t) = 0, u_- = 0, \quad u_+ = \nu(c\lambda a^{\delta})^{-1}\lambda_1^{(1+\delta)}).$
Part 2: Now, for any $\mathbf{u}, \mathbf{v} \in \mathbb{H}$ with $\max(\|A\mathbf{u}\|_\mathbb{H},\|A\mathbf{v}\|_\mathbb{H}) \leq (1/2)\mathbf{u}_+$, we have that

\[
\langle \mathbf{J}(\mathbf{u}, t) - \mathbf{J}(\mathbf{v}, t), (AB)^{-\delta}(\mathbf{u} - \mathbf{v}) \rangle_\mathbb{H} = -\|B^{-1/2}(AB)^{-\delta}(\mathbf{u} - \mathbf{v})\|_\mathbb{H}^2 \\
-\nu^{-1} \left( (AB)^{-(1+\delta)}[C(\mathbf{u}, \mathbf{u} - \mathbf{v}) + C(\mathbf{v}, \mathbf{u} - \mathbf{v})], (AB)^{-\delta}(\mathbf{u} - \mathbf{v}) \right)_\mathbb{H} \\
\leq -\lambda_0^{-1}a^{-2\delta}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 + ca^{-\delta}\nu^{-1}\lambda_1^{-(1+\delta)}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 (\|\mathbf{u}\|_\mathbb{H} + \|\mathbf{v}\|_\mathbb{H}) \\
\leq -\lambda_0^{-1}a^{-2\delta}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 + ca^{-\delta}\nu^{-1}\lambda_1^{-(1+\delta)}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 \mathbf{u}_+ \\
= -\lambda_0^{-1}a^{-2\delta}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 + ca^{-\delta}\nu^{-1}\lambda_1^{-(1+\delta)}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 \left( \frac{1}{2}\nu\lambda_1^{(1+\delta)}(c^{-1}a^{-\delta}\lambda_0^{-1}) \left( 1 + \sqrt{1 - \gamma} \right) \right) \\
= -\frac{1}{2}\lambda_0^{-1}a^{-2\delta}\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2 \left( 1 - \sqrt{1 - \gamma} \right) \\
= -\alpha\|\mathbf{u} - \mathbf{v}\|_\mathbb{H}^2, \quad \alpha = \frac{1}{2}\lambda_0^{-1}a^{-2\delta} \left( 1 - \sqrt{1 - \gamma} \right).
\]

□

**Theorem 10.** The operator $A(t) = \nu A^{(1+\delta)}J(\cdot, t)$ is closed, uniformly dissipative and jointly continuous in $\mathbf{u}$ and $t$. Furthermore, for each $t \in \mathbb{R}^+$ and $\beta > 0$, $\text{Ran}[I - \beta A(t)] \supset \mathbb{B}[\Omega]$, so that $A(t)$ is $m$-dissipative on $\Omega$.

**Proof.** Since $J(\cdot, t)$ is strongly dissipative and closed on $\mathbb{B}$, it follows from Theorem 6 that $\text{Ran}[J(\cdot, t)] \supset \mathbb{B}$.

To show that $A(t) = \nu (AB)^{(1+\delta)}J(\cdot, t)$ is uniformly dissipative for $\mathbf{u}, \mathbf{v} \in \mathbb{B}_+$, we have

\[
\langle A(t)\mathbf{u} - A(t)\mathbf{v}, (\mathbf{u} - \mathbf{v}) \rangle_\mathbb{H} = -\nu \left\| A^{1/2}(\mathbf{u} - \mathbf{v}) \right\|_\mathbb{H}^2 \\
- \langle (1/2)[C(\mathbf{u} - \mathbf{v}) + C(\mathbf{u} - \mathbf{v})], (\mathbf{u} - \mathbf{v}) \rangle_\mathbb{H}.
\]
Now, from equation (4),

\[
\|([C(u - v, u) + C(u - v, v)], (u - v))_{\mathbb{H}}\| \\
\leq c \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \| (u - v) \|_{\mathbb{H}} \{ \| Au \|_{\mathbb{H}} + \| Av \|_{\mathbb{H}} \}.
\]

We now use \(-\lambda_0^{-1} a^{-\delta} \| (u - v) \|_{\mathbb{H}} \geq - \| A^{1/2}(u - v) \|_{\mathbb{H}}\), and the fact that the first eigenvalue of \( B \) is 1/2, so that \( \lambda_1^{1+\delta} < 1 \), to get:

\[
\langle A(t)u - A(t)v, u - v \rangle_{\mathbb{H}} \leq -\nu \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}}^2 + \frac{1}{2} c \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \| (u - v) \|_{\mathbb{H}} \{ \| Au \|_{\mathbb{H}} + \| Av \|_{\mathbb{H}} \}
\]

\[
= \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \left\{ -\nu \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} + \frac{1}{2} c \| u - v \|_{\mathbb{H}} \{ \| Au \|_{\mathbb{H}} + \| Av \|_{\mathbb{H}} \} \right\}
\]

\[
\leq \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \| u - v \|_{\mathbb{H}} \left\{ -\nu \lambda_0^{-1} a^{-\delta} + c u_+ \right\}
\]

\[
\leq \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \| u - v \|_{\mathbb{H}} \left\{ -\nu \lambda_0^{-1} a^{-\delta} + \frac{1}{2} \nu \lambda_1^{1+\delta} \lambda_0^{-1} a^{-\delta} \left[ 1 + \sqrt{1 - \gamma} \right] \right\}
\]

\[
< \frac{1}{2} \nu \lambda_0^{-1} a^{-\delta} \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \| u - v \|_{\mathbb{H}} \left\{ -1 + \sqrt{1 - \gamma} \right\} < 0.
\]

If we set \( a (\| (u - v) \|_{\mathbb{H}}) = \frac{1}{2} \nu \lambda_0^{-1} a^{-\delta} \left[ -1 + \sqrt{1 - \gamma} \right] \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \), we have that:

\[
\langle A(t)u - A(t)v, u - v \rangle_{\mathbb{H}} \leq -a (\| (u - v) \|_{\mathbb{H}}) \| (u - v) \|_{\mathbb{H}}.
\]

It follows that \( A(t) \) is uniformly dissipative. Since \( -A^{(1+\delta)} \) is \( m \)-dissipative, for \( \beta > 0 \), \( \text{Ran}(I + \beta(A)B^{(1+\delta)}) = \mathbb{H} \). As \( J \) is strongly dissipative (in the ball of radius \( \frac{1}{2} u_+ \)) and closed, with \( \text{Ran}[J] \supset \mathbb{B} \), and \( J(\cdot, t) : \mathbb{D} \xrightarrow{\text{onto}} \mathbb{D} \), \( A(t) \) is maximal dissipative (in the ball of radius \( \frac{1}{2} u_+ \)), and also closed, so that \( \text{Ran}[I - \beta A(t)] \supset \mathbb{B} \). It follows that \( A(t) \) is \( m \)-dissipative on \( \mathbb{B} \) for each \( t \in \mathbb{R}^+ \) (since \( \mathbb{H} \) is a Hilbert space). To see that \( A(t)u \) is continuous in both variables, let \( u_n, u \in \mathbb{B}_+ \), \( \| A(u_n - u) \|_{\mathbb{H}} \to 0 \),
with \( t_n, t \in I \) and \( t_n \to t \). Then (see equation (6))

\[
\|A(t_n)u_n - A(t)u\|_H \leq \|A(t_n)u - A(t)u\|_H + \|A(t_n)u_n - A(t_n)u\|_H
\]

\[
= \|(Pf(t_n) - Pf(t))\|_H + \|\nu A(u_n - u) + C(u_n - u, u_n) + C(u_n - u)\|_H
\]

\[
\leq d|t_n - t|^{\theta} + \|\nu A(u_n - u)\|_H + \|C(u_n - u, u_n) + C(u_n - u)\|_H
\]

\[
\leq d|t_n - t|^{\theta} + \|\nu A(u_n - u)\|_H + c\|A(u_n - u)\|_H \left\{ \|Au_n\|_H + \|Au\|_H \right\}
\]

\[
\leq d|t_n - t|^{\theta} + \|\nu A(u_n - u)\|_H + 2c\|A(u_n - u)\|_H u_+.
\]

It follows that \( A(t)u \) is continuous in both variables. \( \square \)

Since \( \mathbb{B}_+ \) is the closure of \( \mathbb{D} = D(A) \cap \mathbb{B} \) equipped with the restriction of the graph norm of \( A \) induced on \( D(A) \), it follows that \( \mathbb{B}_+ \) is a closed, bounded, convex set. We now have:

**Theorem 11.** For each \( T \in \mathbb{R}^+ \), \( t \in (0, T) \) and \( u_0 \in \mathbb{D} \subset \mathbb{B} \), the global in time Navier-Stokes initial-value problem in \( \mathbb{R}^3 \):

\[
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t) \text{ in } (0, T) \times \mathbb{R}^3,
\]

\[
\nabla \cdot u = 0 \text{ in } (0, T) \times \mathbb{R}^3,
\]

(9)

\[
\lim_{\|x\| \to \infty} u(t, x) = 0 \text{ on } (0, T) \times \mathbb{R}^3,
\]

\[
u u(0, x) = u_0(x) \text{ in } \mathbb{R}^3,
\]

has a unique strong solution \( u(t, x) \), which is in \( L^2_{\text{local}}([0, \infty); H^2) \) and in \( L^\infty_{\text{local}}([0, \infty); V] \cap C^1([0, \infty); H] \).

**Proof.** Theorem 7 allows us to conclude that, when \( u_0 \in \mathbb{D} \), the initial value problem is solved and the solution \( u(t, x) \) is in \( C^1([0, \infty); \mathbb{D}] \). Since \( \mathbb{D} \subset H^2 \), it follows that
GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

$u(t, x)$ is also in $\mathcal{V}$, for each $t > 0$. It is now clear that, for any $T > 0$,

$$\int_0^T \|u(t, x)\|_{H^2}^2 \, dt < \infty, \quad \text{and} \quad \sup_{0 < t < T} \|u(t, x)\|_{\mathcal{V}}^2 < \infty.$$ 

This gives our conclusion. \[\Box\]

DISCUSSION

It is known that, if $u_0 \in \mathcal{V}$, and $f(t)$ is $L^\infty([0, \infty); \mathbb{H}]$ then there is a time $T > 0$ such that a weak solution with this data is uniquely determined on any subinterval of $[0, T)$ (see Sell and You, page 396, [SY]). Thus, we also have that:

**Corollary 12.** For each $t \in \mathbb{R}^+$ and $u_0 \in \mathcal{D}$ the Navier-Stokes initial-value problem on $\mathbb{R}^3$:

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f(t) \text{ in } (0, T) \times \mathbb{R}^3,$$

$$\nabla \cdot u = 0 \text{ in } (0, T) \times \mathbb{R}^3,$$

$$\lim_{\|x\| \to \infty} u(t, x) = 0 \text{ on } (0, T) \times \mathbb{R}^3,$$

$$u(0, x) = u_0(x) \text{ in } \mathbb{R}^3.$$ 

(10)

has a unique weak solution $u(t, x)$, which is in $L^2_{\text{loc}}([0, \infty); \mathbb{H}^2]$ and in $L^\infty_{\text{loc}}([0, \infty); \mathcal{V}) \cap C^1([0, \infty); \mathbb{H}]$.

Since we require that our initial data be in $\mathbb{H}^2$, the conditions for the Leray-Hopf weak solutions are not satisfied. However, it was an open question as to whether these solutions developed singularities, even if $u_0 \in C^\infty_0$ (see Giga [G] and references therein). The above Corollary shows that it suffices that $u_0(x) \in \mathbb{H}^2$ to insure that the solutions develop no singularities.
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References


GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON \( \mathbb{R}^3 \)


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